Textbook & Full Exam

Mechanical FE

Technical Study Guide & 250+ Practice Problems

2025 CBT Exam

Learn the key concepts and skills necessary to pass the FE Exam

Engineering Pro Guides

by Justin Kauwale, P.E.

Mechanical FE Exam: Textbook & Full Exam

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Mechanical FE Textbook & Full Exam

How to pass the Mechanical FE exam

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0 - Introduction

How to Study for and Pass the FE Exam

MECHANICAL FE TEXTBOOK
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Section 0.0 - Introduction

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1.0 INTRODUCTION

One of the most important steps in an engineer's career is obtaining the professional engineering (P.E.) license. It allows an individual to legally practice engineering in the state of licensure. This credential can also help to obtain higher compensation and develop a credible reputation. The first step towards obtaining your P.E. is passing the Fundamentals of Engineering (F.E.) Exam. Both tests are administered by the National Council of Examiners for Engineering and Surveying (NCEES). The FE Exam is a year round computer based test that can be taken as early as your senior year in college or with at least 3 years of engineering-related work experience. Once passed, the FE Exam will certify you as an Engineering in Training (EIT). With enough experience after passing the EIT, you will become eligible for the PE Exam. Engineering Pro Guides focuses on helping engineers pass the NCEES exam through the use of free content on the website, http://www.engproguides.com and through the creation of books like sample exams and guides that outline how to pass the PE exam.

In the FE exam you will not be able to bring in any outside reference material. You will be given the *NCEES FE Exam Reference Handbook*, which contains all the necessary equations, tables, and graphs that you will need to solve each problem. The *NCEES FE Exam Reference Handbook* will be provided as a searchable electronic pdf during the test. The key to passing the FE exam is understanding the key concepts and skills that are tested on the exam and becoming familiar with using this handbook to solve each problems in approximately 2-3 minutes. Although the NCEES handbook provides the necessary equations for the exam, knowing how to apply them and which equations to use requires an understanding of the concepts and practice of the skills. The FE Exam is available for 6 disciplines plus a generic engineering discipline. This technical guide teaches you the key concepts and skills required to pass the Mechanical F.E. Exam in a single document.

1.1 EXAM FORMAT

How is the exam formatted?

The FE exam format and additional exam day information can be found on the NCEES Examinee Guide (https://ncees.org/exams/examinee-guide/). The entire exam period is about 6 hours, with 2 minutes for signing agreements, 8 minutes for tutorials, and one break up to 25 minutes. You will have a total of 5 hours 20 minutes of actual exam time to solve 110 problems, which equates to about 2.9 minutes per problem if spread out evenly. The test is broken up into two sessions. The length of each session is determined by the number of problems, 55 problems per session, and not the time. So, you could spend more or less than half the time on the first session, and the remaining 5 hours 20 minutes will be allotted for the second session. Since the first session doesn't have a halfway time limit, it is very important to keep watch of the clock to make sure you have enough time for the second session. Before each session is completed, you are allowed to go back to problems that you may have skipped or want to check in that session. However, once the first session is completed and submitted, you are no longer allowed to revisit the questions in that session. There is a 25 minute break in between the sessions. You are allowed to take less than a 25 minute break or no break at all, but this does not increase the time you have to answer exam questions. No points are deducted for incorrect answers, so be sure to provide an answer for all questions, even if it is a guess. The final results are scaled based on the exam difficulty. There are five types of question formats that could be presented on the exam.

1. Multiple Choice (4 choices) – Select one option, *majority of questions in the exam*
2. Multiple Answers – Select multiple answers that are correct
3. Select by Clicking – Click on a point on a graph, etc
4. Drag and Drop – Matching, sorting, labeling, etc
5. Fill in the Blank – Type in the answer

The types of questions and number of questions per topic will be based on the outline provided by NCEES, discussed in the next section. These topics will not be labeled on the test. Finally, the NCEES Examinee Guide states that there will be some questions that will not be scored in the exam. These are questions that are tested for their quality and possible use in future exams. Your final results will be given to you 7-10 days after you take the exam.

1.2 KEY CONCEPTS AND SKILLS

How are the key concepts and skills determined?

The key concepts and skills tested in the sample exams and taught in this technical study guide were first developed through an analysis of the topics and information presented by NCEES. The above factors related to timing is considered. The Mechanical FE exam will focus on the following topics as indicated by NCEES. (https://ncees.org/engineering/fe/):

1 **Mathematics - (6-9 questions)**
  i) *Analytic Geometry*
  ii) *Calculus*
  iii) *Ordinary Differential Equations*
  iv) *Linear Algebra*
  v) *Numerical Methods*
  vi) *Algorithm and Logic Development*
2 **Probability and Statistics - *(4-6 questions)***
  i) *Probability Distributions*
  ii) *Measures of Central Tendencies and Dispersions*
  iii) *Expected Value*
  iv) *Regression, Curve Fitting, Goodness of fit*
3 **Ethics and Professional Practice - *(4-6 questions)***
  i) *Codes of Ethics*
  ii) *Public Health, Safety, and Welfare*
  iii) *Intellectual Property*
  iv) *Societal Considerations*
4 **Engineering Economics - *(4-6 questions)***
  i) *Time Value of Money*
  ii) *Cost: incremental, average, sunk, estimating*
  iii) *Economic Analyses*

5. **Electricity and Magnetism *(5-8 questions)***
    - i) *Electrical Fundamentals*
    - ii) *DC Circuit Analysis*
    - iii) *AC Circuit Analysis*
    - iv) *Motors and Generators*
6. **Statics *(9-14 questions)***
    - i) *Resultants of Force Systems*
    - ii) *Concurrent Force Systems*
    - iii) *Equilibrium of Rigid Bodies*
    - iv) *Frames and Trusses*
    - v) *Centroids and Moments of Inertia*
    - vi) *Static friction*
7. **Dynamics, Kinematics, and Vibrations *(10-15 questions)***
    - i) *Kinematics of Particles*
    - ii) *Kinetic Friction*
    - iii) *Newton's Second Law for Particles*
    - iv) *Work-Energy of Particles*
    - v) *Impulse-Momentum of Particles*
    - vi) *Kinematics of Rigid Bodies*
    - vii) *Kinematics of Mechanisms*
    - viii) *Newton's Second Law for Rigid Bodies*
    - ix) *Work-Energy of Rigid Bodies*
    - x) *Impulse-Momentum of Rigid Bodies*
    - xi) *Free and Forced Vibrations*
8. **Mechanics of Materials *(9-14 questions)***
    - i) *Shear and Moment Diagrams*
    - ii) *Stress Transformations and Mohr's Circle*
    - iii) *Stress and Strain caused by Axial Loads*
    - iv) *Stress and Strain caused by Bending Loads*
    - v) *Stress and Strain caused by Torsion*
    - vi) *Stress and Strain caused by Shear*
    - vii) *Stress and Strain caused by Temperature Changes*
    - viii) *Combined Loading*
    - ix) *Deformations*
    - x) *Column Buckling*
    - xi) *Statically Indeterminate Systems*
9. **Material Properties and Processing *(7-11 questions)***
    - i) *Properties: chemical, electrical, mechanical, physical, thermal*
    - ii) *Stress-Strain Diagrams*
    - iii) *Ferrous Metals*
    - iv) *Nonferrous Metals*
    - v) *Engineered Materials*
    - vi) *Manufacturing Processes*
    - vii) *Phase Diagrams, Phase Transformation, and Heat Treating*
    - viii) *Materials Selection*
    - ix) *Corrosion Mechanisms and Control*

    x) *Failure Mechanisms*

10 **Fluid Mechanics *(10-15 questions)***
    i) *Fluid Properties*
    ii) *Fluid Statics*
    iii) *Energy, Impulse, Momentum*
    iv) *Internal Flow*
    v) *External Flow*
    vi) *Compressible Flow*
    vii) *Power and Efficiency*
    viii) *Performance Curves*
    ix) *Scaling Laws for Fans, Pumps, and Compressors*

11 **Thermodynamics *(10-15 questions)***
    i) *Properties of Ideal Gases and Pure Substances*
    ii) *Energy Transfers*
    iii) *Laws of Thermodynamics*
    iv) *Processes*
    v) *Performance of Components*
    vi) *Power Cycles*
    vii) *Refrigeration and Heat Pump Cycles*
    viii) *Non-reacting Mixtures of Gases*
    ix) *Psychrometrics*
    x) *Heating, Ventilation, and Air Conditioning (HVAC) processes*
    xi) *Combustion and Combustion Products*

12 **Heat Transfer *(7-11 questions)***
    i) *Conduction*
    ii) *Convection*
    iii) *Radiation*
    iv) *Transient Processes*
    v) *Heat Exchangers*

13 **Measurement, Instrumentation, and Controls *(5-8 questions)***
    i) *Sensors*
    ii) *Control Diagrams*
    iii) *Dynamic System Response*
    iv) *Measurement Uncertainty*

14 **Mechanical Design and Analysis *(10-15 questions)***
    i) *Stress Analysis of Machine Elements*
    ii) *Failure Theories and Analysis*
    iii) *Deformation and Stiffness*
    iv) *Springs*
    v) *Pressure Vessels and Piping*
    vi) *Bearings*
    vii) *Power Screws*
    viii) *Power Transmission*
    ix) *Joining Methods*
    x) *Manufacturability*
    xi) *Quality and Reliability*

xii) *Components (Hydraulic, Pneumatic, Electrochemical)*
xiii) *Engineering Drawing Interpretations and Geometric Dimensioning and Tolerancing*

Each of these broad topics were investigated and filtered for concepts and skills that met the following criteria:

(1) First, the concept and skill must be fundamental principles taught in college. The test is intended for the engineer right out of college without work or practical experience. The exam will focus on fundamental engineering principles you will need during your career. However, since the Mechanical Engineering discipline is broad, the exam will be based on the general knowledge that each Mechanical Engineer develops in school and will not include an in-depth, higher level analysis of a specific topic. The subjects listed above are the basic curriculum that Mechanical Engineers should encounter before they graduate.

(2) Second, the skill and concept must be testable in roughly 2.9 minutes per problem. There are (110) questions on the Mechanical FE exam and you will be provided with 5 hours 20 minutes to complete the exam. This results in an average of 2.9 minutes per problem. This criterion limits the complexity of the exam problems and the resulting solutions. For example, pressure drop calculations are common in Fluids, but the calculation is often very lengthy because of the number of steps involved, especially if a unique fluid and flow condition is used. Thus, common fluids like water/air and common pipe/duct materials are used.

(3) Third, the information and equations required to solve the problems should be in the *NCEES FE Reference Handbook.* Since you are not allowed to bring in outside resource, the Handbook and along with any information given to you in the problem should provide you with sufficient information needed to solve the problems. It is extremely unlikely that you will need an equation that is not given to you in the reference handbook. Thus, the handbook is an additional resource for understanding the types of questions that could be asked. Note that the *NCEES FE Reference Handbook* contains extraneous information for the Mechanical FE exam, since the same resource is used across all tested disciplines. To narrow down the relevant topics, the handbook was cross referenced with the NCEES Mechanical FE outline mentioned in the last section. Lastly, the solution may still require a variation of the equations in the reference handbook. Therefore it is very important to understand how to use these equations, as well as the variables and the units that the equations require.

(4) The F.E. Exam tests the background engineering concepts and skills for a practicing Mechanical Engineer and not the derivation of the topic or concept. The exam is intended to prove that the test taker is minimally competent to practice as an engineer in training and has the basic understanding of Mechanical Engineering principles. This background knowledge is necessary for the practicing engineer to understand how engineering concepts and skills are applied in the field. Therefore, the exam is less concerned with theory and more with how these concepts and skills can be applied. For example, the F.E. exam is less interested with the derivation of angular momentum equations and more with how to solve for resultant forces or final velocity conditions.

In summary, this book is intended to teach the necessary skills and concepts to develop a minimally competent, practicing Mechanical Engineer in Training, capable of passing the F.E. exam. This book and the sample exam do this through the following means:

(1) Teaching common skills, principles, and concepts in the Mechanical field.

(2) Providing sample problems that can be completed in roughly 2-3 minutes per problem.

(3) Teaching how to use and apply the equations in the *NCEES FE Reference Handbook*.

(4) Teaching the application of the skill and concept for an engineer in training.

1.3 UNITS

The units that are used in the F.E. Exam are the International System of Units (SI) and the United States Customary System Units (USCS). The equations in the *NCEES FE Reference Handbook* are more generic and does not necessarily differentiate between SI or USCS units. Therefore, it is very important, especially with the USCS problems, to make sure all necessary conversion factors are used and that the units cancel out to the unit of your desired final answer. Be aware of the use of the g_c conversion factor in USCS problems. See the fluids section for an explanation of the g_c term.

2.0 DISCLAIMER

In no event will Engineering Pro Guides be liable for any incidental, indirect, consequential, punitive or special damages of any kind, or any other damages whatsoever, including, without limitation, those resulting from loss of profit, loss of contracts, loss of reputation, goodwill, data, information, income, anticipated savings or business relationships, whether or not Engineering Pro Guides has been advised of the possibility of such damage, arising out of or in connection with the use of this document or any referenced documents and/or websites.

This book was created on the basis of determining an independent interpretation of the minimum required knowledge and skills of an engineer in training. In no way does this document represent the National Council of Examiners for Engineers and Surveying views or the views of any other professional engineering society.

3.0 HOW TO USE THIS BOOK

This book is organized into the topics as designated by the NCEES. These topics include:

- Section 0.0: Introduction
- Section 1.0: Mathematics
- Section 2.0: Probability and Statistics
- Section 3.0: Ethics and Professional Practice
- Section 4.0: Engineering Economics

- Section 5.0: Electricity and Magnetism
- Section 6.0: Statics
- Section 7.0: Dynamics, Kinematics, and Vibrations
- Section 8.0: Mechanics of Materials
- Section 9.0: Material Properties and Processing
- Section 10.0: Fluid Mechanics
- Section 11.0: Thermodynamics
- Section 12.0: Heat Transfer
- Section 13.0: Measurement, Instrumentation, and Controls
- Section 14.0: Mechanical Design and Analysis

First, it is recommended that the engineer in training gather the *NCEES FE Reference Handbook*.

Second, proceed through the book in the order designated. Go through and first read the material of the section, then complete the practice problems designated for that section. If you have trouble with the practice problems, review the material and then read the solutions. These problems are meant to practice the application of the skill or concept presented in the section. The problems are exam difficulty level.

Following the completion of each of the sections, it is recommended that you go through the checklists presented on the Engineering Pro Guides website. These checklists pose vital questions to the engineer in training about their understanding of all the skills and concepts presented in this book. If you are not confident with any of the items, please go back and revisit the section.

Finally, set aside a five-hour twenty-minute block of uninterrupted time to complete a sample exam. Gather your references and calculator and create a test-like environment. Set a timer and proceed to take the sample exam, which can be purchased separately. Remember that the exam is only 55 problems each for the first and second sessions and does not encompass all the possible items that can appear on an exam, but it should give you an idea of your level of readiness for the exam.

4.0 PRACTICE EXAM TIPS

Engineering Pro Guides practice exam problems can be used in multiple ways, depending on where you are in your study process. If you are at the beginning or middle, it can be used to test your competency, gain an understanding and feel for the test format, and help to highlight target areas to study. If you are at the end, it can be used to determine your preparedness for the real exam. Remember that the questions are a sample of the many topics that may be tested and are limited to fit a full exam length and therefore is not comprehensive of all concepts. Also the practice exam problems are split up throughout the entire book.

Because the exam is written to be similar to the difficulty and format of the NCEES exam, it is recommended that the test be completed in one sitting and timed for two hours forty minutes to simulate half of the real exam. This will give you a better indication of your status of preparation for the exam. If you are at the ending of your studying, it is recommended to couple this exam with the second section to simulate the full exam test day.

Review the exam day rules and replicate the environment for the real test as much as possible, including the type of calculator you may use and the acceptable references. Keep a watch or clock next to you to gauge your pace for 55 questions in 2 hours 40 minutes.

Based on the NCEES website, the following are general rules for exam day.

Allowed:

1. ID used for admission
2. Approved calculator (2 recommended for backup. The backup will be stored with your personal items)
3. Eyeglasses
4. Light sweater or jacket
5. Test center locker key
6. Test center provided booklet and marker
7. Test center comfort aids, approved upon visual inspection. See the *Pearson VUE Comfort Aid List* on the Pearson VUE website (includes medicine – inhaler, aspirin not in bottle, eye drops, cough drops, etc and mobility devices – crutches, wheelchairs, etc. Tissues and earplugs must be provided by the test center.)
8. Religious head coverings

Prohibited:

1. Cell phones
2. Electronic Devices (other than approved calculator)
3. Watches
4. Wallets and Purses
5. Hats and hoods
6. Bags
7. Coats
8. Books
9. Pens, Pencils, Erasers
10. Food, Drinks
11. Weapons
12. Tobacco
13. Eyeglass cases
14. Scratch Paper (all writing devices are provided)

Most test centers will have lockers for you to store your personal items. For additional references on exam day policies, exam day processes, and items to bring on your exam day, review the NCEES Examinee Guide:

http://ncees.org/exams/examinee-guide/

For best use of your time, answer the questions that you know first and return to the questions that you are unfamiliar with later. On the computer based test, you are able to bookmark the answers you may want to come back to later. Once all the known questions are answered, go through the test again and attempt to answer the remaining questions by level of difficulty. If time allots, review your answers.

If you are stuck on a question, seek the following avenues.

1. *NCEES FE Reference Handbook*: It is important to understand the *NCEES FE Reference Handbook*. During times of uncertainty, this will likely lead you to your answers. Determine the key concept that is being asked in the question and refer to this reference book. Remember that the reference is searchable, so you will able to do a search by keyword (Ctrl+F). Additional tips on this resource is discussed in the next section.

2. Process of Elimination: In most questions, there are only four possible choices for each question. Ask yourself if there is an answer that does not make sense and eliminate it. Further narrow down the answer that are derived from equations or concepts that you know are not right and are instead meant to deceive the test taker. See if there are answers that are similar or separated by something like a conversion error. This may be an indication that the correct equation was used.

3. Educated Guess: Remember that there is no penalty for wrong answers. Hopefully with the process of elimination you are able to narrow down as many answers as possible and are able to create an educated guess.

4. If the time is almost up and there are still unanswered questions remaining, determine whether it makes sense to check for mistakes on the problems you do know how to solve, or to tackle the unanswered problems.

Typical Exam Verbiage/Design:

1. Most Nearly: Due to rounding differences, the exam answers may not match yours exactly and in fact may not even closely resemble your answer. NCEES uses the term “most nearly” to test your confidence in your solution. When the question prompts you with “most nearly”, choose the answer that most closely matches yours, whether it be greater than or lesser to your value.

2. Irrelevant Information: The exam is intended to test your overall understanding of concepts. At times the question will include unnecessary information that is meant to misdirect you.

3. Deceiving Answers: NCEES wants to know that you are able to determine the appropriate methods for the solutions. There are answers that were intentionally produced from wrong equations to mislead the test taker. For example, you may forget

a 1/2 in the formula, KE = (1/2)MV^2 and there would be two answers each off by a factor of 1/2.

4. Do Not Overanalyze: The exam questions are meant to be completed in less than 3 minutes. Therefore, they are intended to be written as straight forward as possible. Do not be tempted to overanalyze the meaning of a question. This will only lead you down the wrong path.

Review the Solutions:

Once the sample test is completed, grade your results. Measure your aptitude in speed, concept comprehension, and overall score. If you score is above the 75% range then you are in good shape. This 75% score is only applicable if you have prepared completely for the exam. If you are just starting out, then do not be worried about a low score. This is number is also just a range; there is no finite score to determine passing the test. Instead, NCEES calibrates the results against practicing professional engineers. See this page http://ncees.org/exams/scoring-process/ for a better understanding of how NCEES grades the scores.

Review the answers that you got wrong and use the solutions as a learning tool on how to address these types of problems. Compare the types of questions you are missing with the NCEES outline of topics and determine where you should focus your studying. Finally repeat as many practice problems as you can to get a better grasp of the test and to continually improve your score.

5.0 NCEES FE REFERENCE HANDBOOK

As previously mentioned, the *NCEES FE Reference Handbook* is the only reference material you will have during the exam. Therefore, it is very important to use this reference book when doing practice problems. You should become familiar with the layout of the book, how to apply the equations, what the variables mean, what units the equations are in, and where to find common constants, tables, and graphs. The *NCEES FE Reference Handbook* can be purchased as a hardcopy on the NCEES website or downloaded a free pdf of the latest version from your *MyNCEES* account. I would recommend studying from the pdf to become familiar with using the reference book electronically with the search (Ctrl+F) options. The index will not be provided during the real exam. When studying, notice how the Handbook is organized and how it is broken out by subject, then by discipline. Take some time go browse through the entire reference handbook to see where different equations are located. Realize that some of the Mechanical FE questions may overlap with other disciplines, like Civil and Electrical.

5.1 UNIT CONVERSION

The first section of the *NCEES FE Reference Handbook* has a list of typical unit conversions as well as common constants, such as the universal gas constant, gravity, Stefan-Boltzmann constant.

5.2 TABLES AND GRAPHS

It is important to be able to quickly navigate through the *NCEES FE Reference Handbook* and know where the common tables are used across multiple subjects.

The following are examples of common tables or graphs that you should be aware of.

- Area Moment of Inertia & Centroid – *Statics*
- Mass Moment of Inertia & Centroid – *Dynamics*
- Modulus of Elasticity – *Material Properties*
- Metal Densities – *Material Properties*
- Coefficient of Thermal Expansion – *Material Properties*
- Beam Deflection – *Mechanics of Materials*
- Steam Tables – *Thermodynamics*
- Specific Heat Capacities – *Thermodynamics*
- Water Properties – *Fluid Mechanics*
- Moody Diagram/Roughness Coefficients – *Fluid Mechanics*
- Drag Coefficients – *Fluid Mechanics*
- Hazen-Williams Coefficients – *Civil Engineering*
- Beam Analysis (additional) – *Civil Engineering*
- Buckling – *Civil Engineering*
- Mechanical Application – *Mechanical Engineering*

6.0 PAST EXAMS

6.1 PASS RATES ON SURVEY VS. NCEES

The NCEES website indicates that 77% of Mechanical FE test takers pass the exam. These pass rates only include first time test takers that have attended an accredited engineering program and took the test within 12 months of graduation.

6.2 ESTIMATED CUT SCORE

Since the exam is weighted, the cut score is not clearly defined and is never posted by NCEES. The general online consensus is that the passing rate is about 50-55% correct. Our goal with this book is to get you to a score of at least 70%. You should be able to obtain at least this amount to increase your confidence of passing.

1 - Mathematics

MECHANICAL FE TEXTBOOK
www.engproguides.com

Section 1.0 – Mathematics

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1.0 INTRODUCTION

Mathematics accounts for approximately 6 to 9 questions on the Mechanical FE exam. The topics covered in this section include Analytic Geometry, Calculus, Ordinary Differential Equations, Linear Algebra, Numerical Methods, and Algorithm and Logic Development. At first glance, these topics may seem broad and daunting, but it can easily be narrowed down to the *NCEES FE Reference Handbook*, as they relate to Mechanical Engineering. Although a lot is covered in this chapter, keep in mind that mathematics will only account for about 8% of the exam and only two minutes are allotted per problem. Mathematics is a foundation for the other mechanical engineering topics, so unless you have been away from school for very long, you should be able to breeze through this chapter.

Section 1.0 Mathematics (6 to 9 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
1A	Section 2.0	Analytic Geometry
1B	Section 3.0	Calculus
1C	Section 4.0	Ordinary Differential Equations
1D	Section 5.0	Linear Algebra
1E	Section 6.0	Numerical Methods
1F	Section 7.0	Algorithm and Logic Development
	Section 8.0	Practice Exam Problems

2.0 ANALYTIC GEOMETRY

Analytic geometry uses algebra to characterize various geometric objects such as shapes, lines and points.

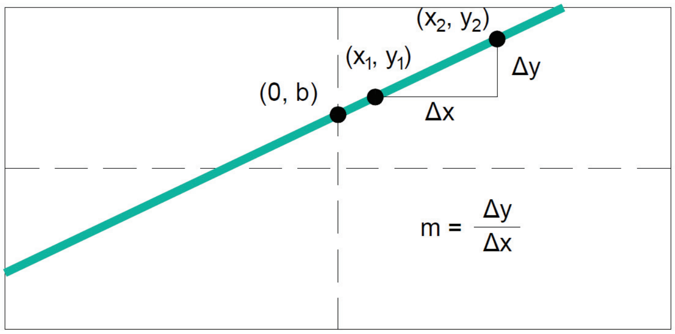


Figure 1: The slope of a line can be found with the difference between the y-values and x-values of two points.

2.1 FIND LINE EQUATION GIVEN TWO POINTS

Given two points, a line connecting the two points can be found with the process below.

First solve for slope, “m”, which is equal to the change in y over the change in x.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Next, solve for the y-intercept, “b”, using the following equation for a line. First, plug in the value of the slope, “m”, and one of the (x, y) points along the line. Then, solve for the y-intercept, “b”.

$$y - intercept\ equation\ for\ line \rightarrow\ y = mx + b$$

$$y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) * x_1 + b$$

Finally, replace “b” in the equation and you will have found the equation of the line.

2.2 PARALLEL AND PERPENDICULAR LINES

Another important skill is being able to calculate the equation for lines that are parallel or perpendicular to each other.

Parallel lines share the same slope.

$$Parallel\ Lines \rightarrow m_1 = m_2$$

The equation of a parallel line can then be determined by finding the vertical offset between the two parallel lines. Add the offset to the y-intercept of one line to find the equation of the other.

$$Line\ 1 \rightarrow y = m_1 x + b$$

$$Line\ 2 \rightarrow y = m_2 x + (b + offset); where\ m_1 = m_2$$

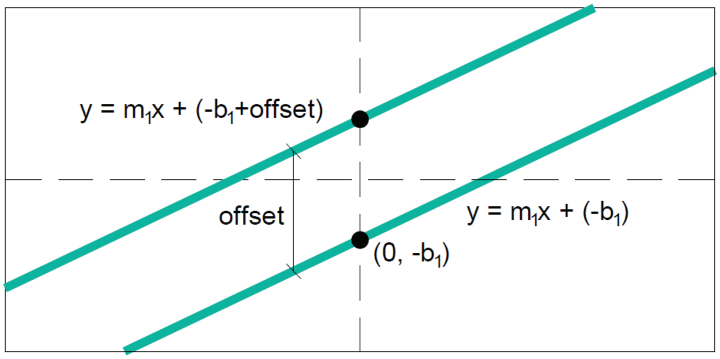


Figure 2: Parallel lines have the same slope and never intersect.

Perpendicular lines have inverse, negative slopes.

$$Perpendicual\ Lines \rightarrow -\frac{1}{m_1} = m_2$$

Find the slope of the perpendicular line, then solve for the new y-intercept, “b_2”, by substituting one of the (x, y) coordinates on the line.

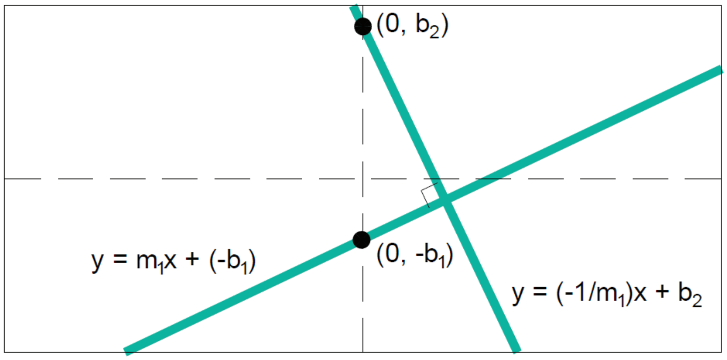


Figure 3: This figure shows the perpendicular intersection of two lines.

2.3 DISTANCE BETWEEN TWO POINTS

The distance between two points can be calculated with the Pythagorean Theorem.

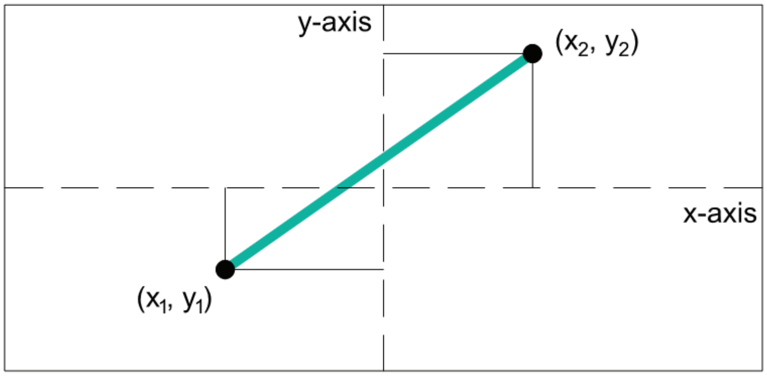


Figure 4: This figure sets up the calculation for the distance between two points.

First, find the total x-distance, $|x_1 - x_2|$, and the total y-distance, $|y_1 - y_2|$. Then, use those values to solve for the distance of the line with the Pythagorean Theorem.

$$Pythagorean\ Theorem: c^2 = a^2 + b^2\ ;$$

where a, b, and c are sides of a right triangle, and c is the hypotenuse.

$$Distance^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$$

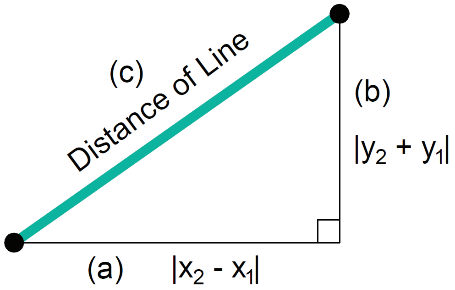


Figure 5: This figure isolates the previous graph and shows how the distance between two points can be shown as a right triangle.

2.4 LOGARITHMIC EQUATIONS

Logarithmic equations are be used in mechanical engineering to describe situations where one variable changes exponentially compared to another. Logarithmic equations have the following relationship with exponential functions.

$$Log_{base}(x) = y;\ Base^y = x$$

The most important logarithmic bases are “10” and “e”. The term “e” will be described in the next topic under natural logarithms. If a log function does not have a base specified, then you can assume that the base is “10”. The following table shows the result of taking the log of various values of “x”.

x	x (scientific)	y = LOG(x)
0	0.E+00	N/A
1	1.E+00	0
10	1.E+01	1
100	1.E+02	2
1,000	1.E+03	3
100,000	1.E+05	5
1,000,000	1.E+06	6
10,000,000	1.E+07	7
100,000,000	1.E+08	8
1,000,000,000	1.E+09	9
10,000,000,000	1.E+10	10

The following graph plots "y = log(x)". It shows that as "x" gets closer to 0, the value of "y" approaches a very large negative number. As "x" gets larger, "y" increases slowly. As you can see from the previous table, the difference between each x-value gets increasingly large, while each y-value only increases by one.

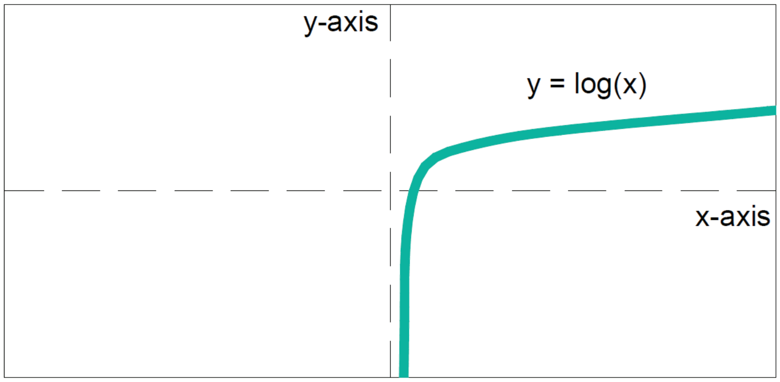


Figure 6: This figure plots the function y = log(x).

The *NCEES FE Reference Handbook* shows all the applicable logarithmic laws that are used to solve equations with log functions. You should be familiar with how to use those laws on the FE exam. An example of one of those laws is the change of base law, which is shown below.

Change Base: In order to change the original base "b" of a log function, take the log of the value "a" with the new base "n" divided by the log of the original base "b" with the new base "n".

$$Change\ Base\colon \log_b a \rightarrow \frac{\log_n a}{\log_n b}$$

$$Example\colon \log_{100} 1{,}000 \rightarrow \frac{\log_{10} 1{,}000}{\log_{10} 100} = \frac{3}{2}$$

$$\log_{100} 1{,}000 = 1.5$$

2.5 NATURAL LOGARITHMIC EQUATIONS

Natural logarithmic equations follow the same rules and laws as logarithmic equations, except that the base is “e”. The term “e” is a mathematical constant, founded by Euler. It is approximated as “2.71828”. Natural logarithmic equations are used in the rotational sections of mechanics and with pulleys. The natural log is not written as log base e, but written as “ln.” The base “e” is implied.

$$Natural\ Log\colon \cancel{\log_e x = y}$$

$$Natural\ Log\colon \ln(x) = y$$

$$\ln(x) = y \quad \Rightarrow \quad e^y = x$$

2.6 TRIGONOMETRY

The main skills you need for trigonometry are knowing how to use the right angle equations and the law of cosines & sines for all other triangles. These skills will help you to determine the angles and lengths of triangles for any situation.

2.6.1 Right Angle Equations

The right angle equations can be used to solve for unknown angles or lengths of a right angle (90 degree) triangle, when other angles or lengths are given. The pneumonic used to remember these equations is SOH-CAH-TOA, which stands for sine-opposite over hypotenuse, cosine-adjacent over hypotenuse, and tangent-opposite over adjacent.

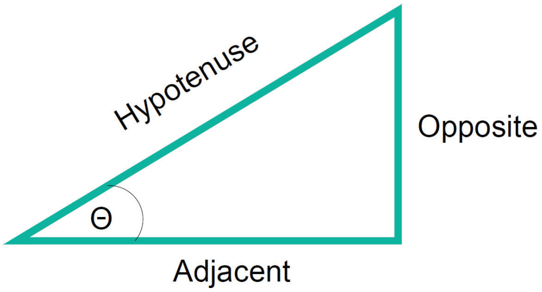


Figure 7: For all right angle triangles, you can use the equations below to solve for the relationships between the angles and lengths of the triangle.

$$SOH \rightarrow \sin(\theta) = \frac{opposite}{hypotenuse};\ CAH \rightarrow \cos(\theta) = \frac{adjacent}{hypotenuse};\ TOA \rightarrow \tan(\theta) = \frac{opposite}{adjacent}$$

2.6.2 Law of Sines & Cosines Equations

The equations for the law of sines and cosines can be used to solve for unknown angles or lengths of non-right angle triangles, when other lengths or angles are given.

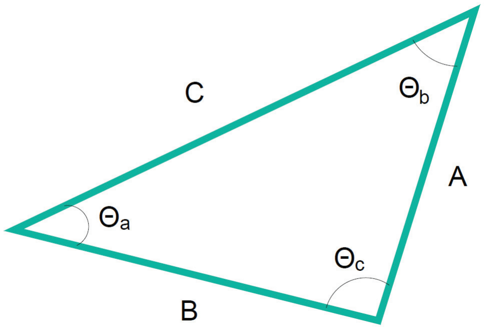


Figure 8: The law of cosines and sines is necessary to solve some dynamics questions.

The law of sines says that the length of one leg of a triangle divided by the sine of angle opposite the leg is equal to the length of any other leg of that same triangle divided by the sine of the angle opposite that leg.

$$\frac{A}{\sin(\theta_a)} = \frac{B}{\sin(\theta_b)} = \frac{C}{\sin(\theta_c)} \quad [Law\ of\ Sines]$$

The law of cosines follow the relationships below, using the same triangle figure above as reference.

$$[Law\ of\ Cosines]$$

$$A^2 = B^2 + C^2 - 2BC\cos(\theta_a)$$

$$B^2 = A^2 + C^2 - 2AC\cos(\theta_b)$$

$$C^2 = A^2 + B^2 - 2AB\cos(\theta_c)$$

2.6.3 Trigonomic Identities

Trigonomic relationships are useful for simplifying and solving equations. A few of the most common identities are provided below. It is not necessary to memorize these. Instead, a comprehensive list of trigonomic identities can be found in the *NCEES FE Reference Handbook* under the Mathematics chapter.

$$\sin^2\theta + \cos^2\theta = 1$$

$$tan\,\theta = \frac{\sin\theta}{cos\theta}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

3.0 CALCULUS

The calculus required for mechanical engineering includes integrals and derivatives. For example, you may need to integrate the area under a probability curve or find the derivative of a particle position equation to calculate the velocity. You should be familiar with the integral and derivative relationships of various functions.

3.1 DERIVATIVES

Derivatives are used to find the slope of a curve. One application is in dynamics, where it is used to find the speed of a component that has a certain displacement function or to find the acceleration of a component that has a certain velocity function.

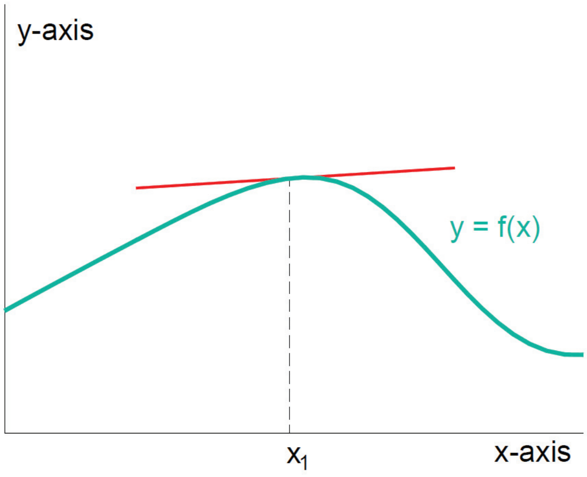


Figure 9: The derivative is the slope at a specific point along a function. This line is tangent to the function curve.

$$Derivative \rightarrow \frac{df(x)}{dx} \; evaluated\ at\ x_1$$

You should be familiar with calculating the derivative for all the functions shown in the *NCEES FE Reference Handbook* like exponential equations, logarithmic equations, natural log equations, trigonometry equations, etc.

3.1.1 First Order Partial Derivatives

When an equation is a function of multiple variables, you may be asked to calculate partial derivatives. As an example, consider the equation that is a function of variables, “x”, “y” and “z”.

Example 1:

$$f(x,y,z) = x^2 + 3xy + z^2 + yzx$$

First find the derivative of the function with respect to “x”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook.

$$\frac{\partial f}{\partial x} = 2x + 3y + yz$$

Next, find the derivative of the function with respect to “y”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook.

$$\frac{\partial f}{\partial y} = 3x + xz$$

Finally, find the derivative of the function with respect to “z”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook.

$$\frac{\partial f}{\partial z} = 2z + yx$$

3.1.2 Second Order Partial Derivatives

In the previous section, you calculate the first order partial derivative. You may be asked to calculate the second order partial derivatives too. As an example, consider the equation that is a function of variables, “x” and “y”. The second order partial derivatives are basically taking the derivative once and then a second time. Acceleration is the second order derivative of displacement. Since they are two variables, you will actually have four different partial derivatives.

$Deritiave\ of\ the\ function\ with\ respect\ to\ x,$ $then\ derivative\ again$ $with\ respect\ to\ x$ $\frac{\partial^2 f}{\partial x^2} = f_{xx}$	$Deritiave\ of\ the\ function\ with\ respect\ to\ y,$ $then\ derivative\ again$ $with\ respect\ to\ y$ $\frac{\partial^2 f}{\partial y^2} = f_{yy}$
$Deritiave\ of\ the\ function\ with\ respect\ to\ x,$ $then\ derivative\ again$ $with\ respect\ to\ y$ $\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial yx} = (f_x)_y = f_{xy}$	$Deritiave\ of\ the\ function\ with\ respect\ to\ y,$ $then\ derivative\ again$ $with\ respect\ to\ x$ $\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial xy} = (f_y)_x = f_{yx}$

Example 1:

$$f(x,y) = x^4y^3 + 3x^2y^2 + y^4$$

f_{xx}: First find the derivative of the function with respect to “x”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook. Then find the derivative again with respect to “x”.

$$f_x = 4x^3y^3 + 6xy^2$$

$$f_{xx} = 12x^2y^3 + 6y^2$$

f_{xy}: First find the derivative of the function with respect to “x”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook. Then find the derivative again with respect to “y”.

$$f_x = 4x^3y^3 + 6xy^2$$

$$f_{xy} = 12x^3y^2 + 12xy$$

f_{yy}: First find the derivative of the function with respect to “y”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook. Then find the derivative again with respect to “y”.

$$f_y = 3x^4y^2 + 6x^2y + 4y^3$$

$$f_{yy} = 6x^4y + 6x^2 + 12y^2$$

f_{yx}: First find the derivative of the function with respect to “y”. Treat all other variables as a constant and then use the same derivative properties from the FE Reference Handbook. Then find the derivative again with respect to “x”.

$$f_y = 3x^4y^2 + 6x^2y + 4y^3$$

$$f_{yx} = 12x^3y^2 + 12xy$$

3.2 INTEGRALS

Integrals are used to find the area under a curve. One application is in mechanics, where it is used to find the centroid or area of a shape.

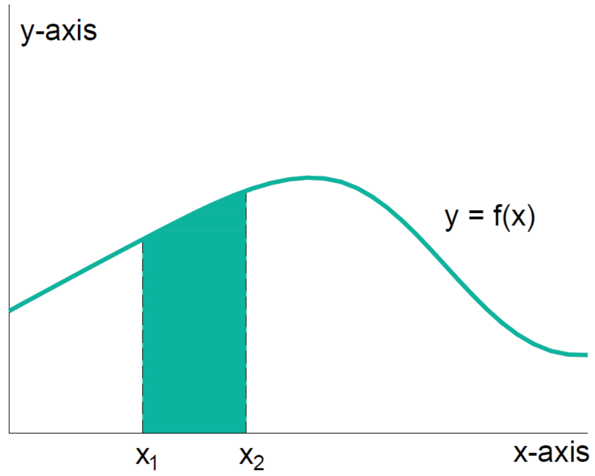


Figure 10: The integral is the area under a curve between two specific points on a function.

$$Integral \rightarrow \int_{x_1}^{x_2} f(x)$$

You should be familiar with calculating the integral for all the functions shown in the *NCEES FE Reference Handbook* like exponential equations, logarithmic equations, natural log equations, trigonometry equations, etc.

4.0 ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equations are commonly used in the dynamics topic of the mechanical engineering field, specifically for vibration where there is decay of an oscillating object over time. A differential equation is an equation that is made up of an unknown function and derivatives of that function. A differential equation is *ordinary* if there is only one independent variable for the unknown function. For example, there is one independent variable, x, for the unknown function y(x) in the differential equation below. The objective is to solve for the unknown function, y(x).

$$a * y'(x) + b * y(x) = 0$$

$$Example\ of\ first\ order\ differential\ equation, where\ a\ and\ b\ are\ constants$$

The *order* of the differential equation is based on the number of the highest derivative. A first order ordinary differential equation has a single derivative, as illustrated above. A second order differential equation will have a second derivative as shown below, and so on.

$$a * y''(x) + b * y'(x) + c * y(x) = d * f(x)$$

$$Example\ of\ second\ order\ differential\ equation$$

4.1 SEPARABLE DIFFERENTIAL EQUATIONS

First Order Differential Equation: Separation of Variables

One of the simplest types of differential equations are first order types, where the y's and x's may be separated to either side of the equation. Such equation will have the following format.

$g(y) * y'(x) = f(x)$, which can be rewritten below.

$$g(y)\frac{dy}{dx} = f(x)$$

If the equation is separable, then both sides can be integrated to solve for y(x).

$$\int g(y)dy = \int f(x)dx$$

Example: Solve the differential equation $y' = 4x^2y^{-1}$, where y(0) = 1.

Solution: Separate the x's and y's to either side of the equation.

$$\frac{dy}{dx} = \frac{4x^2}{y} \quad \Rightarrow \quad y * dy = 4x^2 * dx$$

Integrate both sides of the equation and solve for y.

$$\int y * dy = \int 4x^2 * dx$$

$$\frac{y^2}{2} = \frac{4}{3}x^3 + C_1$$

$$y = \sqrt{\frac{8}{3}x^3 + C}$$

Solve for C at the given initial condition.

$$y(0) = 1 = \sqrt{0 + C} \quad \Rightarrow \quad C = 1$$

Therefore,

$$y = \sqrt{\frac{8}{3}x^3 + 1}$$

If the equation is not easily separable, another method must be used. Review the characteristics of the differential equation to determine the appropriate method.

4.2 LINEAR DIFFERENTIAL EQUATIONS

The difference between a *linear* and *non-linear* differential equation is shown below. In order to be classified as a linear equation, the (y), (y'), (y'') functions must not be raised to any power.

$$Linear\ Differential\ Equation \rightarrow y'' + 2y' + y = 0$$

$$Non-Linear\ Differential\ Equation \rightarrow (y'')^3 + 2(y')^2 + y = 0$$

The difference between a *homogeneous* and *non-homogeneous* differential equation is shown below. Homogenous differential equations only involve "y" and its derivatives. Non-homogeneous will also involve "x".

$$Linear, Homogeneous\ Differential\ Equation \rightarrow y'' + 2y' + y = 0$$

$$Linear, Non-homogeneous\ Differential\ Equation \rightarrow x^2y'' + 2y' + yx = 0$$

A 2nd order differential equation will have its highest derivative as the 2nd derivative. A 1st order differential equation will have its highest derivative as the 1st derivative.

4.2.1 Homogeneous

A linear ordinary differential equation is homogeneous if f(x) in the previous equation is equal to zero.

$$a_n y^n(x) + \cdots + a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

The *NCEES FE Reference Handbook* provides simplified solutions for certain types of homogeneous and nonhomogeneous differential equations.

First Order, Linear Homogeneous Differential Equation, Constant Coefficients:

If the equation is a first order linear homogeneous equation, and the coefficients a_0 and a_1 are constants, then the solution for y(x) is simple. First convert the differential equation to the following format. Ensure that the factor of the derivative, y'(x), is one.

$$y'(x) + ay(x) = 0$$

The solution for y(x) is given below, where C is a constant, found by the initial condition, y(0).

$$y(x) = Ce^{-ax}$$

Example: Solve the differential equation 5y' + 2y = 0, where y(0) = 10.

Solution: Rewrite the equation in the appropriate format and solve for y.

$$y' + \frac{2}{5}y = 0$$

$$y(x) = Ce^{-\frac{2}{5}x}$$

Next, solve for the constant C, using the initial conditions.

$$y(0) = 10 = Ce^{-\frac{2}{5}*0}$$

$$\Longrightarrow C = 10$$

Therefore, the solution is:

$$y(x) = 10e^{-\frac{2}{5}x}$$

Second Order, Linear Homogeneous Differential Equation, Constant Coefficients:

If the equation is a second order linear homogeneous equation, and the coefficients are constants, then the solution for y(x) is still simple, but involves a couple more steps. First convert the differential equation to the following format. Ensure that the factor of the second order derivative, y''(x), is one.

$$y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

The solution is given by the three scenarios. For each scenario, r_1 and r_2 are found with the following equation.

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

Scenario 1: When $a_1^2 - 4a_0 < 0$, the value under the square root of the $r_{1,2}$ equation is negative, thus there are two imaginary solutions for r. The function is said to be *underdamped*. It will oscillate until it reaches equilibrium, similar to a spring going up and down before it stops.

$$y = e^{-\frac{a_1}{2}x}\left[C_1 \cos\left(x\sqrt{a_0 - a_1^2/4}\right) + C_2 sin\left(x\sqrt{a_0 - a_1^2/4}\right)\right]$$

Scenario 2: When $a_1^2 - 4a_0 > 0$, there are two real solutions for r. The function is *overdamped*. It does not oscillate and reaches equilibrium slower than scenarios one and three.

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Scenario 3: When $a_1^2 - 4a_0 = 0$, the value under the square root of the $r_{1,2}$ equation is equal to zero. This means there is one real solution. This function reaches equilibrium the fastest.

$$y = (C_1 + C_2)e^{r_1 x}$$

Example: Solve the differential equation 2y''+8y' + 6y = 0, where y(0) = 0, y'(0)=2.

Solution: Rewrite the equation in the appropriate format and solve for y.

$$y'' + 4y' + 3y = 0$$

Test the function to see which scenario it falls under.

$$a_1^2 - 4a_0 = 4^2 - 4 * 3 = 4 > 0$$

The solution will use scenario 2, overdamped. Solve for r and substitute it into the solution for y.

$$r_{1,2} = \frac{-4 \pm \sqrt{4}}{2} = \{-1, -3\}$$

$$y = C_1 e^{-x} + C_2 e^{-3x}$$

Finally, solve for the constants using the initial conditions.

$$y(0) = C_1 e^{-0} + C_2 e^{-3*0} = C_1 + C_2 = 0$$

$$\Rightarrow C_1 = -C_2$$

$$y'(x) = -C_1 e^{-x} - 3C_2 e^{-3x}$$

$$y'(0) = -C_1 e^{-0} - 3C_2 e^{-3*0} = -C_1 - 3C_2 = 2$$

Substitute C_1 and solve.

$$-(-C_2) - 3C_2 = 2 \Rightarrow C_2 = -1$$

$$\Rightarrow C_1 = -C_2 = 1$$

Replace the constants for the final solution of y.

$$y = e^{-x} - e^{-3x}$$

Higher Order, Linear Homogeneous Differential Equation, Constant Coefficients:

For higher order differential equations with constant coefficients, the process is similar to the second order. The previous explanation skipped the steps on how r was solved, but it is briefly explained here. Since time is limited on the exam, it is unlikely that you will have to solve a higher order differential equation. If a question does appear, it will more likely be conceptual. A higher order homogeneous linear differential equation is given in the format below, where the coefficients, a_n, are constants.

$$a_n y^n(x) + \cdots + a_2 y^2(x) + a_1 y^1(x) + a_0 y(x) = 0$$

$$assume\ y(x) = e^{rx}, then\ y^n(x) = r^n e^{rx}$$

The differential equation becomes:

$$a_n r^n e^{rx} + \cdots + a_2 r^2 e^{rx} + a_1 r^1 e^{rx} + a_0 e^{rx} = 0$$

Factor out e^{rx}. Since e^{rx} cannot equal zero, the following equation results and r can be solved. In the second order differential equation, the highest exponent for r was two, thus there were a maximum of two solutions. In the following equation, there are n possible solutions for r.

$$a_n r^n + \cdots + a_2 r^2 + a_1 r^1 + a_0 = 0$$

The constant coefficients, a_0, a_1,…, a_n, are known, so the polynomial can be solved for r. After the roots of r are solved, it can be substituted into the solution for the function y. Finally, the constants, C_1, C_2,…, C_n may be solved with given the initial conditions.

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x}$$

4.2.2 Non-Homogeneous

A linear ordinary differential equation is non-homogeneous if f(x) in the following equation is not equal to zero.

$$a_n(x)y^n(x) + \cdots + a_2(x)y^2(x) + a_1(x)y^1(x) + a_0(x)y(x) = f(x)$$

First Order, Linear Differential Equation:
Integrating Factor Method

To solve a first order, linear differential equation, the integrating factor method may be used. This process applies for differential coefficients, a_0, a_1,…, a_n, that are constant or contain variables of x, as well as homogeneous or non-homogeneous differential equations. The method is not explicitly described in the *NCEES FE Reference Handbook*, but it uses the principles of integration by parts, listed under the Calculus equations. See the following example for the integrating factor method.

A first order linear non-homogeneous differential equation has the following generic format.

$$a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

Step 1: Rewrite the equation so that it is in the following format, where y' has a coefficient of one.

$$y'(x) + \frac{a_0(x)}{a_1(x)}y(x) = \frac{f(x)}{a_1(x)}$$

Substitute the coefficients with v(x) and g(x) to simplify the equation.

$$y'(x) + v(x)y(x) = g(x)$$

Step 2: Multiply both sides of the equation with the *integrating factor*, u(x).

$$u(x)y'(x) + u(x)v(x)y(x) = u(x)g(x)$$

$$where\ u(x) = e^{\int v(x)dx}$$

Step 3a: Set the derivative of this integrating factor to equal the following.

$$u'(x) = u(x)v(x)$$

Step 3b: Substitute u' into the equation.

$$u(x)y'(x) + u'(x)y(x) = u(x)g(x)$$

Step 3c: We know from the product rule that we can rewrite the equation into the following.

$$\big(u(x)y(x)\big)' = u(x)g(x)$$

Step 4: Solve for y(x).

$$u(x)y(x) + c = \int u(x)g(x)dx$$

$$y(x) = \frac{\int u(x)g(x)dx - c}{u(x)}$$

Example: Solve the differential equation 2y' + 5xy = 6x

Solution: Step 1 - Rewrite the equation in the appropriate format.

$$y' + \frac{5}{2}xy = 3x$$

Step 2- Multiply the equation by the integrating factor, u(x).

$$where\ u(x) = e^{\int v(x)dx} = e^{\int 5x/2\,dx} = e^{5x^2/4}$$

$$e^{5x^2/4}y' + e^{5x^2/4}\frac{5}{2}xy = 3xe^{5x^2/4}$$

Step 3 - Knowing that the above equation is the derivative of the product of u(x) and y(x), the equation can be rewritten below.

$$(e^{5x^2/4}y)' = 3xe^{5x^2/4}$$

Step 4 - Solve for y(x).

$$\int (e^{5x^2/4}y)'\,dx = \int 3xe^{5x^2/4}\,dx$$

$$e^{5x^2/4}y + c_1 = \frac{3*2}{5}e^{5x^2/4} + c_2$$

$$y = \frac{6}{5} + ce^{5x^2/4}$$

Higher Order, Linear Non-Homogeneous Differential Equation, Constant Coefficients: *Undetermined Coefficients Method*

A higher order non-homogeneous linear differential equation has the following format, where the coefficients, a_n, are constants and f(x) is not equal to zero.

$$a_n y^n(x) + \cdots + a_2 y^2(x) + a_1 y^1(x) + a_0 y(x) = f(x)$$

Using the undetermined coefficients method, the solution for y(x) will be the summation of two components. The first is the solution for the homogenous equation, the second is the particular solution for the non-homogeneous entity, f(x).

$$y(x) = y_{homogeneous} + y_{particular}$$

The homogenous solution, $y_{homogeneous}$, can be found with the methods mentioned in the homogeneous section. To solve for the non-homogeneous solution, $y_{particular}$, first find the format it should be in based on the format of f(x). For example, if $f(x) = 2\sin(\pi x)$, then $y_{particular}$ will have the format $C_2\sin(\pi x) + C_3\cos(\pi x)$, based on the table below, where C_2 and C_3 are unknown constants. Substitute $y_{particular}$ into the original equation, then solve for the constants.

f(x)	→	**$y_{particular}$**
C_1	→	C_2
$C_1 e^{\alpha x}$	→	$C_2 e^{\alpha x}$
$C_1\sin(\alpha x)$	→	$C_2\sin(\alpha x) + C_3\cos(\alpha x)$
$C_1\cos(\alpha x)$	→	$C_2\sin(\alpha x) + C_3\cos(\alpha x)$
$C_{1a}\sin(\alpha x) + C_{1b}\cos(\alpha x)$	→	$C_2\sin(\alpha x) + C_3\cos(\alpha x)$
$a_2x^2 + a_1x + a_0$	→	$b_2x^2 + b_1x + b_0$
where a, b, C, and α's are constants		

A summary of this table may be found in the Mathematics Chapter of the *NCEES FE Reference Handbook*. This method typically takes a few steps, so any questions that are asked will have to be simplified.

Example: Solve for the non-homogeneous equation 2y''+8y' + 6y = $5e^{2x}$.

Solution: Using the undetermined coefficients method, the solution for y will have the following format.

$$y(x) = y_{homogeneous} + y_{particular}$$

Rewrite the equation to the appropriate format to solve for $y_{\text{homogeneous}}$.

$$y'' + 4y' + 3y = \frac{5}{2}e^{2x}$$

From the previous second order linear homogeneous differential equation example, we found the solution for $y'' + 4y' + 3y = 0$ to be $y_{homogeneous} = C_1e^{-x} + C_2e^{-3x}$.

Solve for $y_{\text{particular}}$. Since $f(x) = (5/2)e^{2x}$, $y_{\text{particular}}$ will have the format C_2e^{2x}. Substitute $y_{\text{particular}}$ into the equation.

$$(C_2e^{2x})'' + 4(C_2e^{2x})' + 3(C_2e^{2x}) = \frac{5}{2}e^{2x}$$

Solve for C_2.

$$4C_2e^{2x} + 8C_2e^{2x} + 3C_2e^{2x} = \frac{5}{2}e^{2x}$$

$$C_2 = 1/6$$

Therefore,

$$y_{particular} = 1/6\ e^{2x}$$

The solution for y is:

$$y = C_1e^{-x} + C_2e^{-3x} + 1/6\ e^{2x}$$

4.2.3 Application

The main application for differential equations on the Mechanical FE exam is in the Dynamics topic. A second order linear differential equation is used to characterize vibrations for a spring with damper and for torsional vibration. However, the differential equation is already solved, based on the mass, spring, and damping coefficient, and listed in the *NCEES FE Reference Handbook*. See the Dynamics chapter of this book for further discussion.

4.3 LAPLACE TRANSFORMS

The Laplace transform will convert difficult differential equations to more manageable algebraic equations before it is solved. In engineering, the variable x is typically used for time, t, since properties like position, velocity, etc, vary with time. These equations are plotted on an x-y plane, where the x-axis is time. This is the t-domain. For Laplace transforms, the function in the t-domain is converted to a frequency s-domain, to simplify the equation. (The s-domain plots the function on a new s-plane. Knowing the intricacies of this is not important for the FE exam. The intent is to help you visualize one function being transformed onto a different graphical plane to be solved.) Once the equations are in the s-domain, they become much easier to solve. After solved, the s-domain equations can then be converted back to the time domain. The Laplace transform is used mostly in electrical engineering topics like controls, circuits and electronics.

The format for Laplace transform is as follows:

$$F(s) = \mathcal{L}\{f(t)\}$$

Where $F(s)$ is a function in the s-domain, $f(t)$ is a function in the t-domain, and $\mathcal{L}\{f(t)\}$ is the Laplace transform of the t-domain.

The following formula shows how you would convert from the time domain to the s-domain. This is the Laplace transform. Note that the constant resulting from the integral must be given to solve these problems.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

$$where\ t \geq 0$$

The following formula shows how you would convert from the s-domain back to the t-domain. This is called the inverse Laplace transform.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

$$s = \sigma + j\omega, where\ \sigma\ and\ \omega\ are\ real\ numbers$$

A simple example is finding the Laplace transform for the function $f(t) = 1, where\ t \geq 0$.

$$F(s) = \mathcal{L}\{1\} = \int_0^{\infty} 1 * e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

$$F(s) = \frac{1}{s}, for\ f(t) = 1$$

Step Function:

On the exam, the previous transformation is the same as the step function equation. A step function is shown in the following figure, where there is a jump in the function. The unit step function goes from 0 to 1, starting at t=0.

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases}, \qquad F(s) = \frac{1}{s}, for\ f(t) = u(t)$$

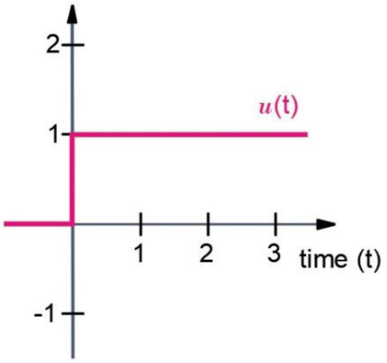


Figure 11: Unit step function for $u(t)$.

Impulse Function:

Impulse is when there is an immediate spike at a singular point in time. In this case, the impulse function, $\delta(t)$ occurs at time t=0. A unit impulse with a width equal to 0 and an infinite height has an area equal to 1.

$$\delta(t) = \begin{cases} 0, t \neq 0 \\ \infty, t = 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Then, because the function only has non-zero value at time t=0, the equation can be rewritten as the following.

$$F(s) = \mathcal{L}\{\delta(t)\} = \int_0^0 \delta(t) * e^{-st}dt = \int_0^0 \delta(t)dt * \left[-\frac{1}{s}e^{-st}\right]_0^0 = \int_0^0 \delta(t)dt * 1 = 1$$

$$F(s) = 1, for\ f(t) = \delta(t)$$

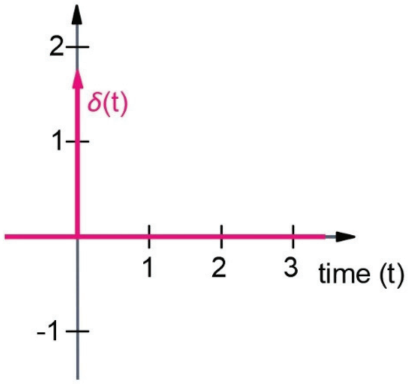


Figure 12: Impulse function for $\delta(t)$.

Ramp Function:

The ramp function is a straight line with slope. The unit ramp function has a slope of 1 beginning at t=0, written as $f(t) = t$. The unit step function, $u(t)$, can be multiplied to the straight line equation to shift the start point or adjust the slope. The two forms are given below. The step by step solution is not given, since these formulas are provided in tables of the *NCEES FE Reference Handbook.*

$$R(t) = \begin{cases} 0, t < 0 \\ t, t \geq 0 \end{cases}, \qquad f(s) = R(t) = t * u(t)$$

$$F(s) = \mathcal{L}\{t * u(t)\} = \int_0^\infty t * 1 * e^{-st} dt = \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2}, \qquad for\ f(t) = t * u(t)$$

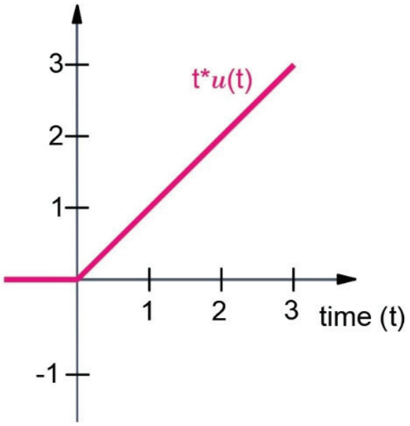


Figure 13: Unit ramp function for $f(s) = t * u(t)$

The majority of the Laplace transform equations from f(t) to F(s) require complicated calculations. On the FE exam, you should not have to perform the complicated transformations because it will take too long and in practice a computer is used. You will, however, have to be familiar with the various transforms in the table of the *NCEES FE Reference Handbook*.

Example - nth power:

One transformation that is not listed in the tables, but is commonly used is shown in this next example. This is provided for your reference. The equation would have to be given to you in the problem, due to time limitations.

The time domain function is as shown below. With the given transfer function equations, you can quickly determine the corresponding s-domain function. First you must rewrite the t-domain equation into the same format of the ones shown in the table or given equation.

$$f(t) = t^2 + 3t$$

The Laplace Transform is given as the following.

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

First, you would separate each component with the similar format $f(t) = t^n$, then convert it to the F(s) format. The separation is possible because of the linearity rule discussed later. The transformed equation in the s-domain becomes the following.

$$F(s) = \frac{2}{s^3} * +3 * \frac{1}{s^2};$$

Example - Derivative:

The following example shows how to solve for a linear second order differential equation with the Laplace transform. In the *NCEES FE Reference Handbook*, there is a formula for taking the Laplace of a derivative but it is difficult to follow. The process is easier to understand through an example. This example takes the Laplace transform of the following equation.

$$5 = 3y'' + y' - 4y; \quad y(0) = 0; \quad y'(0) = 0;$$

$$= L\{3y'' + y' - 4y\}$$

Break-up each component and solve for the Laplace transform of each component.

The constant may be moved to the outside of the Laplace transform. When taking the Laplace of a derivative, there will be an s-component for each # derivative. In this example, the first component is the second derivative, so there will be two s-components.

$$Component\ 1 \rightarrow L\{3y''\} = 3 * \left[s^2F(s) - s\big(y(0)\big) - y'(0)\right]$$

$$Component\ 2 \rightarrow L\{y'\} = 1 * [sF(s) - y(0)]$$

$$Component\ 3 \rightarrow L\{-4y\} = -4F(s)$$

$$Component\ 4 \rightarrow L\{5\} = \frac{5}{s}$$

Now, add up all the components and make sure to substitute the initial values from above.

$$\frac{5}{s} = 3 * [s^2F(s) - s * 0 - 0] + [sF(s) - 0] - 4 * F(s)$$

$$\frac{5}{s} = 3s^2F(s) + sF(s) - 4F(s)$$

After simplifying the problem, solve for F(s).

$$\frac{5}{s} = F(s)(3s^2 + s - 4)$$

$$F(s) = \frac{5}{s(3s+4)(s-1)}$$

Because time is limited on the exam you will probably be asked to stop here. The question may even be broken up into parts. However, to fully illustrate the complete process of using the Laplace transform, let's transform F(s) back into the time domain, y(t). First rewrite the equation into an easily transformable format, using A, B, and C with the $1/s$ and $1/(s+a)$ arrangements.

$$F(s) = \frac{5}{s(3s+4)(s-1)} = \frac{A}{s} + \frac{B}{3s+4} + \frac{C}{s-1}$$

$$\Rightarrow\ 5 = A(3s+4)(s-1) + Bs(s-1) + Cs(3s+4)$$

When s=0, $5 = A(4)(-1) \ \Rightarrow\ A = -5/4$

When s= –4/3, $5 = B(-4/3)(-4/3-1) \ \Rightarrow\ B = 45/28$

When s= 1, $5 = C(3+4) \ \Rightarrow\ C = 5/7$

Substitute A, B, and C, then transform each component back to the time domain.

$$F(s) = \frac{-5/4}{s} + \frac{45/28}{3(s+4/3)} + \frac{5/7}{s-1} = -\frac{5}{4}\left(\frac{1}{s}\right) + \frac{45}{84}\left(\frac{1}{s+\frac{4}{3}}\right) + \frac{5}{7}\left(\frac{1}{s-1}\right)$$

$$y(t) = -\frac{5}{4} + \frac{45}{84}e^{-\frac{4}{3}t} + \frac{5}{7}e^{t}$$

This is the solution for the second order differential equation using Laplace transforms. Again, remember that the problem has to be solvable within the 2 minute range, so the overall question should be simpler and solvable within a few steps.

4.3.1 Laplace Transform Pairs

This section will discuss the common Laplace pairs that you should know for the exam.

Name	Time Domain	S-Domain Laplace
Constant/ Unit Step	$u(t)@\ t = 0$	$\frac{1}{s}$

Unit Impulse	$\delta(t) @ \, t = 0$	1
Ramp	$t * \mathrm{u}(t) @ \, t = 0$	$\frac{1}{s^2}$

Name	**Time Domain**	**S-Domain Laplace**
Parabola	t^2	$\frac{2}{s^3}$
tⁿ	t^n	$\frac{n!}{s^{n-1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Time Exponential	te^{-at}	$\frac{1}{(s+a)^2}$

Name	**Time Domain**	**S-Domain Laplace**
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Decaying Sine	$e^{-at}\sin(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Decaying Cosine	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

4.3.2 Laplace Properties

The Laplace properties are used on the exam to simplify complex functions into easy to use functions. These easy to use functions should preferably be in a format listed in the Laplace transform table of the *NCEES FE Reference Handbook*.

4.3.2.1 LINEARITY

This property states that you can add two functions and multiply each of those functions by a scalar, then take the Laplace transform of that combined function and it will equal the Laplace transform of each individual function added together.

$$\mathit{Laplace\ Transform}\ \mathcal{L}\{Af(t)\} \rightarrow AF(s)$$

$$\mathcal{L}\{Bg(t)\} \rightarrow BG(s)$$

$$\mathcal{L}\{Af(t)+Bg(t)\} \rightarrow AF(s)+B(s)$$

The linear property is very useful on the exam because you can take a complex function and break it up into the sum of two or more functions. Then, you can take the Laplace transform of each individual function and add the Laplace transforms of each.

The following shows a simple example as to how to use this property.

$$\mathcal{L}(2e^{-at}+3e^{at})$$

The function can be broken up into two different functions and the Laplace transform of each individual function can be found.

$$\mathcal{L}(2e^{-at})+\mathcal{L}(3e^{at})$$

$$Laplace\ Transform\ Pairs \rightarrow \mathcal{L}(e^{-at})=\frac{1}{s+a};\ \mathcal{L}(e^{at})=\frac{1}{s-a};$$

$$\mathcal{L}(2e^{-at}+3e^{at})=2\frac{1}{s+a}+3\frac{1}{s-a}$$

4.3.2.2 FIRST, SECOND & NTH DERIVATIVE

The Laplace transform of the first derivative of a function is equal to the Laplace transform of the original function multiplied by s. There is also a term for the initial condition at time t = 0.

$$\mathcal{L}\left\{\frac{d\,f(t)}{dt}\right\}=sF(s)-f(0^-)$$

$$f(0^-)\ is\ the\ value\ at\ t=0$$

The second derivative is equal to the Laplace transform of the original function multiplied by s squared. There are now two terms that include the initial condition. The difference is that one of the terms is multiplied by s.

$$\mathcal{L}\left\{\frac{d^2\,f(t)}{dt^2}\right\}=s^2F(s)-s^1f(0^-)-f(0^-)$$

The ultimate pattern is shown on the *NCEES FE Reference Handbook*. It is labeled as the nth derivative property. Take the Laplace transform of the original function, then multiply that term by s raised to the nth power. Next, you subtract the initial condition multiplied by s raised to the n-1 power and continue until you reach s raised to the 0th power.

$$\mathcal{L}\left\{\frac{d^n\,f(t)}{dt^n}\right\}=s^nF(s)-\sum_{i=1}^{n}s^{n-1}f^{i-1}(0^-)$$

Example: This is one way that you can use this property to solve complex Laplace transforms.

$$\mathcal{L}\{t^4\} = ?$$

First, find the derivatives of this function and the conditions at t=0.

$$\frac{d}{dt}(t^4) = 4t^3, f(0) = 0$$

$$\frac{d^2}{dt^2}(t^4) = 12t^2;\ f'(0) = 0$$

$$\frac{d^3}{dt^3}(t^4) = 24t;\ f''(0) = 0$$

$$\frac{d4}{dt^4}(t^4) = 24;\ f'''(0) = 0$$

Now, use the formula for the derivative property.

$$\mathcal{L}\left\{\frac{d4\, f(t)}{dt^4}\right\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

Finally, solve for the Laplace transform of the original function.

$$\mathcal{L}\{24\} = s^4\mathcal{L}\{t^4\} - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

$$24\left(\frac{1}{s}\right) = s^4\mathcal{L}\{t^4\} - 0$$

$$\mathcal{L}\{t^4\} = \frac{24}{s^5}$$

4.3.2.3 INTEGRAL

The integral property is the inverse of the derivative property. Instead of multiplying by s, you multiply by 1/s. You can see that if you keep taking the integral of the integral that you will multiply by 1/s raised to the nth power.

$$\mathcal{L}\left\{\int_0^t f(\lambda)d\lambda\right\} = \frac{1}{s}F(s)$$

$$where\ F(s)\ is\ essentially\ \mathcal{L}\{f(\lambda)\}$$

Example: Find the Laplace transform of the following integral.

$$\mathcal{L}\left\{\int_0^t (\lambda^2 + 1)\, d\lambda\right\} = ?$$

First realize that $\mathcal{L}\{t^2 + 1\} = \mathcal{L}\{t^2\} + \mathcal{L}\{1\} = \frac{2}{s^3} + \frac{1}{s}$. Therefore, the solution is the following.

$$\mathcal{L}\left\{\int_0^t (\lambda^2 + 1)\, d\lambda\right\} = \frac{1}{s}\mathcal{L}\{t^2 + 1\} = \frac{1}{s}\left(\frac{2}{s^3} + \frac{1}{s}\right) = \frac{2}{s^4} + \frac{1}{s^2}$$

4.3.2.4 TIME MULTIPLICATION/DIVISION

The time multiplication property states that the Laplace transform of the original function multiplied by the time variable to the nth power is equal to the nth derivative of F(s) multiplied by negative one to the nth power.

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds}$$

The time division property simplifies the Laplace transform of function $f(t)$ divided by the time variable, t, to equal the integral of $f(x)$ from s to infinity. The intermediary function, $f(x)$, is the Laplace transform of only $f(t)$.

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(x)dx$$

$$where\ f(x) = \mathcal{L}\{f(t)\}$$

These properties are not listed in the *NCEES FE Reference Handbook*, but is shown here for your reference.

4.3.2.5 TIME DELAY/ADVANCEMENT

The time delay or advancement will shift the time function curve. Subtracting a value from t, will give you a time delay and adding a value from t will give you a time advancement. Basically, this will shift the function in the time domain. In the s-domain, the function will be multiplied by the exponential shown below.

$$\mathcal{L}\{f(t-a)\} = e^{-as}F(s)$$

Example: As an example, consider the following function.

$$\mathcal{L}\{f(t-5)^2\} =?$$

The original function is shown below, without the time delay. This function has a transform pair on the table.

$$\mathcal{L}\{f(t)^2\} = \frac{2}{s^3}$$

The Laplace transform of the function in equation will just be this previous transform multiplied by the exponential as shown below.

$$\mathcal{L}\{f(t-5)^2\} = e^{-5s}\frac{2}{s^3}$$

4.3.2.6 COMPLEX SHIFT

The complex shift occurs when you multiply a function by the exponential shown in the equation below. The Laplace transform of this function will equal the Laplace transform of the original function but with a complex shift.

$$\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$$

Example: For example, the following function is shifted with the exponential function.

$$\mathcal{L}\{e^{5t}\sin(2t)\}$$

Check the transform pairs table for the Laplace transform of the sine function.

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+2^2}$$

Next, apply the complex shift to the function. So wherever there is an s, you must replace it with the s + a term.

$$\mathcal{L}\{e^{5t}\sin(2t)\} = \frac{2}{(s-5)^2+2^2}$$

$$\mathcal{L}\{e^{5t}\sin(2t)) = \frac{2}{s^2-10s+25+4} = \frac{2}{s^2-10s+29}$$

4.3.2.7 SCALE

The scale property states that when a function is scaled in the time domain, it will be equal to the inverse of that scalar multiplied by the Laplace transform as shown in the equation below.

$$f\left(\frac{1}{a}t\right) = aF(as)$$

4.3.2.8 CONVOLUTION

The convolution property states that when two functions are convolved together they will be equal to the product of the Laplace transforms of each individual function. The convolution operation is denoted by the symbol, " * ".

$$f(t) * g(t) = F(s)G(s)$$

Convolution is a tool for signal processing. It is a more involved calculation process and thus the opinion of the author that it is more applicable to the Electrical FE exam.

5.0 LINEAR ALGEBRA

The linear algebra required for mechanical engineering includes being able to define lines with equations and matrices.

5.1 LINEAR EQUATIONS

Linear equations can be used in statics and dynamics, especially when you need to find the intersection of two functions. In these types of problems you will be given two equations and two unknowns. Then, you can use the skills in linear equations to find the unknown values that satisfy both equations. A very simple example is shown below. The actual questions will most likely be more difficult and will be in the context of mechanical engineering.

$$y = 2x + 5; y = -5x + 10;$$

First, substitute one of the y-values into the other equation, then solve for x.

$$(-5x + 10) = 2x + 5;$$

$$-7x = -5 \rightarrow x = \frac{5}{7}$$

Then plug in the x-value to either equation.

$$y = 2 * \left(\frac{5}{7}\right) + 5$$

$$y = \frac{45}{7}$$

The correct answer is, (x,y) = (5/7, 45/7).

Linear equations can extend to multiple unknowns and multiple equations. To simplify the process, matrices are used.

5.2 MATRICES

Matrices are another way to solve linear equations. Matrices are an array of values that can be used to represent multiple linear equations.

For example, the following matrices will represent the below equations.

$$y = x + z - 10; y = 2x + 80; y = -5x + 4z - 45;$$

Rearrange the equation so that the variables follow the same order, constant times x plus constant times y plus constant times z equal to a constant.

$$1st\ Equation \rightarrow x + (-1)y + z = 10;$$

$$2nd\ Equation \rightarrow 2x + (-1)y + (0)z = -80;$$

$$3rd\ Equation \rightarrow (-5)x + (-1)y + (4)z = 45;$$

The constants for each variable are placed in the first matrix and the variables are placed in the second matrix. The third matrix shows the other side of each equation. The constants in each same row all belong to the same equation and the constants in each column all belong to the same variable.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -5 & -1 & 4 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -80 \\ 45 \end{bmatrix}$$

The values for x, y and z that satisfy all three equations can be found with matrix functions.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -5 & -1 & 4 \end{bmatrix}^{-1} * \begin{bmatrix} 10 \\ -80 \\ 45 \end{bmatrix}$$

The rules for adding, multiplying, taking the transpose, and finding the inverse of a matrix is described in the *NCEES FE Reference Handbook.* For this test, you should be able to quickly solve for matrices on your calculator. The steps for a typical calculator are shown below. Read through your calculator instruction manual for any variations for your specific model.

Calculator Button	**Description**
2ND	Press 2^{nd} function
$Matrix$	Access matrix function
$A \rightarrow Edit$	Choose matrix A and edit.
Row/Column	Input 3 rows by 3 columns. Sets dimensions of matrix
Enter Values	You will be able to input the values for the matrix
$Enter/Clear$	Now the matrix is stored as matrix A.
2ND	Press 2^{nd} function
$Matrix$	Access matrix function
$B - Edit$	Choose matrix B and edit
Row/Column	Input 3 rows by 1 columns. Sets dimensions of matrix
Enter Values	You will be able to input the values for the matrix
$Enter/Clear$	Now the matrix is stored as matrix A.

Now that you have input and stored each matrix, the next step is to use the matrices in functions.

Calculator Button	Description
$2nd \rightarrow Matrix \rightarrow A$	Press 2nd function, matrix, then select matrix A.
$[A] \rightarrow Enter$	With matrix A selected (not edit), press enter and matrix A will appear on your screen.
$2nd \rightarrow Matrix$	Go back to the matrix menu
$Path \rightarrow Inverse$	Select the inverse function and press enter. This will apply the inverse function to matrix A.
$Multioly$	Insert the standard multiplication function.
$2nd \rightarrow Matrix$	Go back to the matrix menu
$[B] \rightarrow Enter$	With matrix B selected (not edit), press enter and matrix B will appear on your screen.
$Enter$	Complete the function, [A]$^{-1}$ * [B]

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -5 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4/3 & -1 & -1/3 \\ 8/3 & -3 & -2/3 \\ 7/3 & -2 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 4/3 & -1 & -1/3 \\ 8/3 & -3 & -2/3 \\ 7/3 & -2 & -1/3 \end{bmatrix} * \begin{bmatrix} 10 \\ -80 \\ 45 \end{bmatrix} = \begin{bmatrix} 235/3 \\ 710/3 \\ 505/3 \end{bmatrix}$$

The correct answer is shown below.

$$(x, y, z) = (78.33, 236.67, 168.33)$$

You should also be able to calculate determinants, dot products, and cross products of matrices on your calculator.

5.3 VECTOR ANALYSIS

Vector analysis is used heavily in analyzing forces in Section 7 Statics, Section 8 Dynamics, Kinematics and Vibrations, Section 9 Mechanics of Materials, and Section 15 Mechanical Design and Analysis. One of the main skills that you need is the ability to break down a vector into its x and y components.

Vectors are used to visualize forces and moments in Engineering Mechanics. Vectors describe the magnitude and direction of force or moment. On the FE exam you must be able to translate words or diagrams into force/moment vectors and you must be able to add/subtract vectors and multiply/divide vectors by scalars.

Vectors can either be represented in a rectangular form or a polar form. The rectangular form consists of an x-component and a y-component. These values are used to represent the magnitude in the x and y directions. In real applications, there will also be a z-component, but for the purposes of the exam you most likely will only need the x and y components. The polar form is shown as a magnitude and an angle. The magnitude describes the length of the vector while the angle determines the direction.

5.3.1 Rectangular Form

The rectangular form is shown as, x plus the y component. The y component is shown as j and the x component is shown as i.

$$Rectangular\ form:\ xi + yj$$

$$x = x-\ component; y = y-\ component$$

$$Example\ A\ \rightarrow 5i + 2j\ or\ Example\ B \rightarrow\ -1i - 5j$$

The rectangular form is used when adding and subtracting vectors and follows the same rules as normal addition and subtraction, where only like terms can be added and subtracted. For example, “Example A” plus “Example B”, is solved with the following process.

$$Example\ A + Example\ B = (5i + 2j) + (-1i - 5j)$$

$$(5i - 1i) + (2j - 5j) \rightarrow 4i - 3j$$

The rectangular form can also be understood via a graphical format, where the x-axis represents the real component and the y-axis represents the imaginary component.

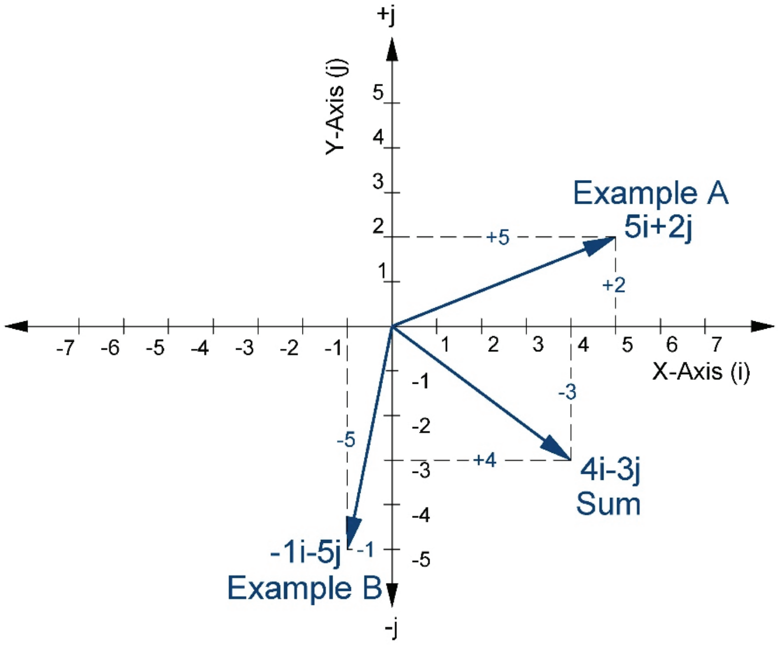


Figure 14: Example “A” vector, example “B” vector and the sum of the two vectors is shown in the above graph.

5.3.2 Polar Form

The polar form is best understood in its graphical format. The format consists of a phasor magnitude at a phasor angle relative to the x-axis.

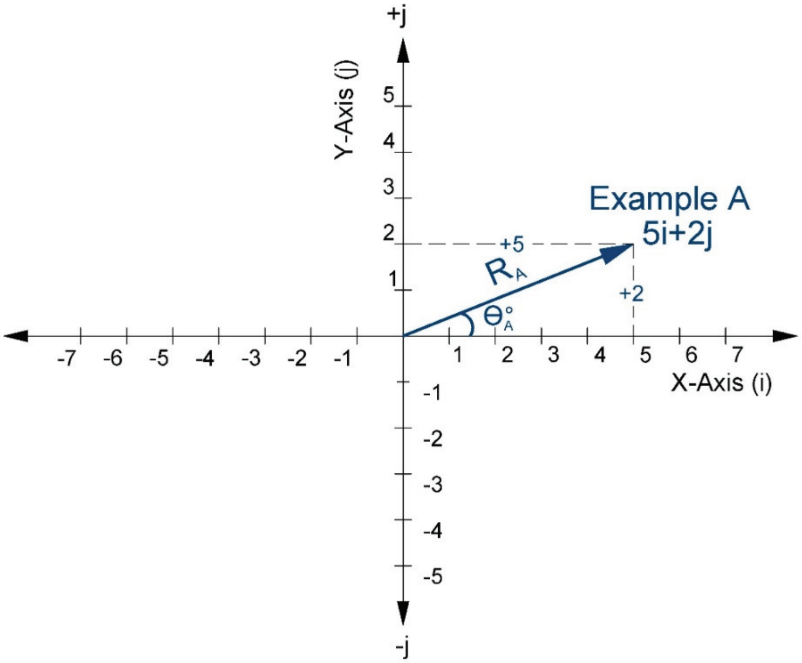


Figure 15: A phasor shown in both polar and rectangular form.

$$Polar\ form{:}\ R\angle\theta°\ ;\ R = phasor\ magnitude;\ \theta = phasor\ angle\ relative\ to\ x - axis$$

$$Example\ A{:}\ R_A = \sqrt{5^2 + 2^2} = 5.4;\ \ \theta_A = \tan^{-1}\left(\frac{2}{5}\right) = 21.8°$$

In the above example, the polar form is converted from the rectangular form by using the Pythagorean Theorem to find the radius (i.e. the magnitude) and the inverse tangent to find the angle. The polar form is not typically used for adding or subtracting, but it is used for multiplication and division. When multiplying or dividing two polar forms, you must multiple/divide the radiuses and add or subtract the angles. If the polar forms are being multiplied, then you must add the angles and if you are dividing one polar form from another then you subtract the divisor angle from the dividend angle.

$$5.5\angle 45° \ \div\ 1.1\angle 85° \ \rightarrow\ \left(\frac{5.5}{1.1}\right)\angle(45 - 85°) \ \rightarrow\ 5\angle - 40°$$

5.3.3 Converting Polar and Rectangular Forms - Calculator

During the exam, you will need to convert from polar form to rectangular form and vice versa. You will need to convert between the two forms in order to carry out multiplication/division or addition/subtraction. You should be able to quickly convert between the forms with your calculator. This will help to save you time for more difficult tasks during the exam.

You should use the most advanced calculator allowed by NCEES. The calculators allowed by the NCEES are shown below.

- Casio: All fx-115 and fx-991 models (Any Casio calculator must have “fx-115” or “fx-991” in its model name.)
- Hewlett Packard: The HP 33s and HP 35s models, but no others
- Texas Instruments: All TI-30X and TI-36X models (Any Texas Instruments calculator must have “TI-30X” or “TI-36X” in its model name.)

Casio FX-115, Casio FX-991, HP35 and TI-36X are all decent calculators that can do multiple calculation steps without having to use the memory function in the calculator or without having to write down numbers in between steps. However, the Casio and TI models are the cheapest. Past test takers seem to prefer the TI-36X Pro for its interface and its ability to save previous calculations when it is shut off and the Casio FX-991EX ClassWiz for its faster processor and its spreadsheet option. Whichever calculator you decide on using, make sure that you use the calculator during your studying so that you can be well versed in the use of your calculator prior to taking the exam.

The following are the basic steps in converting from polar form to rectangular form. The above calculators also have the option of adding/subtracting and multiplying/dividing multiple polar or rectangular numbers. Please be sure to learn this skill, because you will use the skill throughout the exam.

- Convert Polar $(R\angle\theta°)$ to Rectangular $xi + yj$

Calculator Button	Description
R	Press your value for “R”
$P \rightarrow R$	Press the Polar to Rectangular Button
θ	Press your value for “θ”
=	Press the equal sign
	The first value shown will be your x-component
$x \rightarrow y$	Press this button to show the y-component
	The second value shown will be your y-component

- Convert Rectangular $(xi + yj)$ to Polar $R\angle\theta°$

Calculator Button	Description
“x” value	Press your value for “x”

$R \rightarrow P$	Press the Rectangular to Polar Button
"y" value	Press your value for "y"
=	Press the equal sign
	The first value shown will be your R -component
$R \rightarrow \theta$	Press this button to show the y-component
	The second value shown will be your θ -component

The Casio FX-115ES Plus/Casio FX-991EX ClassWiz and TI-36X Pro also has a complex mode which allows you to add/multiply complex numbers in various forms and easily converting the results between rectangular and polar. The stored lines of calculations and the various functions, including matrices and integrals make this calculator one of the favored among test takers.

5.3.4 Vector Mathematical Operations

Addition & Subtraction of Vectors: Vectors can be added or subtracted from one another by simply adding or subtracting their real components together and doing the same for their imaginary components. It is very easy to do this when vectors are presented in their Rectangular coordinates, however it is more difficult to do this by hand when the vectors are in polar form. For example, if you are adding 3 and 3j, then the answer is simply 3 + 3j. Another example would be if you were to add the vectors "2+2j" and "4+4j", then the answer would be "6+6j".

1st Example: You can complete this problem by inputting the polar vectors and adding them. If you are doing the equation by hand, then you should follow the recommended method by converting polar to rectangular and then adding the like terms to one another (real + real and imaginary + imaginary).

$$Convert\ to\ Rectangular \rightarrow 3\angle 0° + 3\angle 90° \rightarrow 3 + 3j$$

$$Convert\ to\ Polar \rightarrow 3 + 3j \rightarrow 3\sqrt{2}\angle 45°$$

2nd Example: This example is a little less intuitive, but since both vectors have the same angle, they can be added directly or with your calculator. If you are doing the equation by hand, then you should follow the recommended method by converting polar to rectangular and then adding the like terms to one another (real + real and imaginary + imaginary).

$$Convert\ to\ Rectangular \rightarrow 2\angle 45° + 4\angle 45° \rightarrow 1.414 + 1.414j + 2.828 + 2.828j \rightarrow 4.24 + 4.24j$$

$$Convert\ to\ Polar \rightarrow 4.24 + 4.24j \rightarrow 6\angle 45°$$

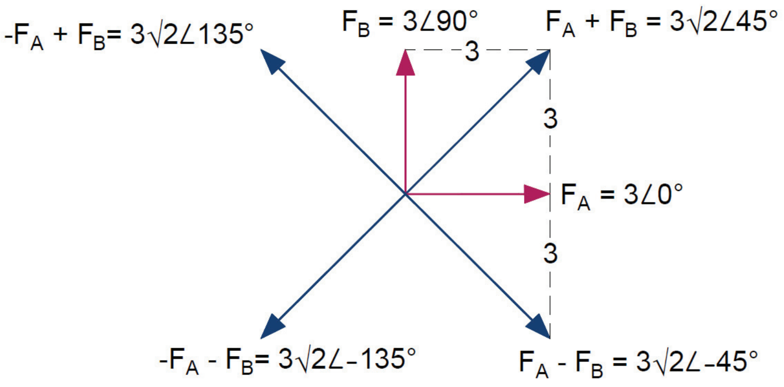
3rd Example: In this example, the answer cannot be seen as intuitively, but as long as you either use your calculator to add polar coordinates or convert to rectangular and then add the like terms, then you will be able to complete this problem.

$$3\angle 35° + 5\angle 60°$$

$$Convert\ to\ Rectangular \rightarrow 3\angle 35° + 5\angle 60° \rightarrow (2.46 + 1.72j) + (2.5 + 4.33j)$$

$$Convert\ Sum\ to\ Polar \rightarrow (4.96 + 6.05j) \rightarrow 7.82\angle 50.65°$$

This figure below shows how vectors can be added and subtracted. This figure also corresponds to the first example and it also shows how the answer would change if vectors A and B were subtracted or added.



$-F_A + F_B = 3\sqrt{2}\angle 135°$
$F_B = 3\angle 90°$
$F_A + F_B = 3\sqrt{2}\angle 45°$
3
3
$F_A = 3\angle 0°$
3
$-F_A - F_B = 3\sqrt{2}\angle -135°$
$F_A - F_B = 3\sqrt{2}\angle -45°$

Multiplication & Division of Vectors: The multiplication and division of vectors is easy for vectors in polar form but difficult for vectors in rectangular form. Your calculator should be able to divide and subtract in both rectangular form, but if you are not familiar with this function on your calculator, then you can also complete this by hand. In order to multiply or divide vectors by hand you must convert the vector to polar form and then multiply or divide the radius of the vectors and then add or subtract the angles. You add the vector angles for multiplication and subtract the vectors angles for division.

1st Example: In this example, the vectors are already in polar form, so you can simply multiply the radiuses and add the vector angles.

$$3\angle 0° \times 3\angle 0° = 9\angle 0°$$

2nd Example: Again, the vectors are already in polar form, so multiply the radiuses and add the vector angles.

$$3\angle 0° \times 3\angle 45° = 9\angle 45°$$

3rd Example: In this example, the vectors must first be converted to polar and then the radiuses can be divided and the vector angles can be subtracted.

$$(2 + 2j) \div (4 + 4j) \rightarrow 2.828\angle 45° \div 5.66\angle 45°$$

$$\frac{1}{2}\angle 0° \rightarrow 0.5 + 0j$$

5.4 DIVIDING POLYNOMIALS

Polynomial division will have three possible scenarios. You should look for the easiest one first, since the FE exam will typically test the quickest method possible.

Scenario 1: Polynomial divided by a monomial. In this scenario, you can just divide each term of the polynomial by the monomial.

$$0 = \frac{x^3 + 4x^2 + 10x + 1}{5x^2}$$

$$0 = \frac{1}{5}x + \frac{4}{5} + 2x^{-1} + \frac{1}{5}x^{-2}$$

Scenario 2: Polynomial divided by a polynomial with factoring. In this scenario, you can simplify the polynomials into similar factors.

$$0 = \frac{4x^3 + 16x^2}{x + 4}$$

$$0 = \frac{4x^2(x + 4)}{x + 4} = 4x^2$$

Scenario 3: Polynomial divided by a polynomial with long division. In this scenario, you must carr

$$0 = \frac{x^3 + 9x^2 + 26x + 24}{x + 4}$$

$$(x + 4)\sqrt{x^3 + 9x^2 + 26x + 24}$$

$$\mathit{First\ term} \rightarrow x^2$$

$$(x + 4)\sqrt{5x^2 + 26x + 24}$$

$$\mathit{Second\ term} \rightarrow 5x$$

$$(x + 4)\sqrt{6x + 24}$$

$$\mathit{Third\ term} \rightarrow 6$$

$$(x + 4)\sqrt{x^3 + 9x^2 + 26x + 24} = x^2 + 5x + 6$$

The roots/solutions to this equation are shown below.

$$(x + 3)(x + 2) = 0$$

$$x = -3; x = -2;$$

6.0 NUMERICAL METHODS

Numerical methods are used to create approximate solutions for equations that are difficult to solve, including differential equations, complex polynomial equations, etc. Instead of trying to solve a difficult equation head on, it is approached indirectly, for example, by taking a first guess to iterate a solution or by breaking up the function into fine segments to simplify the equation.

6.1 APPROXIMATIONS

Numerical methods by nature requires approximations. When the approximations are small enough, the solutions can be almost indistinguishable from a direct solution. The level of approximation depends on how many significant figures or decimal points the solution converges to. Typically the exam question will direct you on how much a solution should be approximated. Approximations are demonstrated in the Taylor series, Newton's method, and Numerical integration sections.

6.2 PRECISION LIMITS

Precision describes how close repeated measurements are to each other. Since much of engineering data depends on empirical measurements, precision is one way to describe how reliable that data is. Smaller variations in repeated measurements means higher precision, and thus lower uncertainty of the data. Precision can also depend on how many significant figures a measuring device can produce. If a ruler measures in increments of 0.1 mm, then the device has a precision of +/- 0.05mm. Any value to the second decimal place is an estimate and carries a level of uncertainty. Finally, the more measurements that are taken, the lower the uncertainty and the higher the precision.

Precision limits are not explicitly mentioned in the *NCEES FE Reference Handbook*, but the topic does have some overlap with probability and statistics. The uncertainty of the measurements can be expressed by standard deviations and curve fittings and the range of the uncertainty can be presented with a level of confidence, which are all topics of statistical analysis. See the Probability and Statistics Chapter for more discussion on these topics.

In this mathematics section, precision limits are likely meant to describe the range of uncertainty involved. For example, a number could be precise up to the third decimal place with +/- value of uncertainty. This uncertainty is then carried out in calculations with a propagation of error, as discussed in the next section.

6.3 ERROR PROPAGATION

When a measurement has an uncertainty (or a standard deviation), and these values are carried out in calculations, it is important to know that the uncertainty propagates at a different rate than the straight calculation. For example, if quantities are added, then the uncertainty is calculated as the square root of the squares.

The following table contains simplified error propagation equations for common mathematical operations. They are derived from the following formula, which is provided in the Engineering Probability and Statistics Chapter of the *NCEES FE Reference Handbook*.

$$\sigma_x^2 = \left(\frac{\delta x}{\delta a}\sigma_a\right)^2 + \left(\frac{\delta x}{\delta b}\sigma_b\right)^2 + \left(\frac{\delta x}{\delta c}\sigma_c\right)^2 + \cdots \left(\frac{\delta x}{\delta n}\sigma_n\right)^2 \quad [Generic\ formula]$$

	Quantity, x	**Uncertainty, σ_x**
Addition/Subtraction	$x = a + b - c + d \ldots - n$	$\sigma_x = \sqrt{\sigma_a^2 + \sigma_b^2 + \sigma_c^2 \ldots + \sigma_n^2}$
Multiplication/Division	$x = \frac{a * b \ldots * m}{c * d \ldots * n}$	$\sigma_x = x * \sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \cdots \left(\frac{\sigma_c}{c}\right)^2 + \left(\frac{\sigma_n}{n}\right)^2}$
Power	$x = a^\beta$	$\sigma_x = x\beta\frac{\sigma_a}{a}$
where a, b, c, etc are quantity values that include uncertainties σ_a, σ_b, σ_c, etc (or standard deviations). In a problem these variables would be presented as a ± σ_a, b ± σ_b, etc		

Example (-): Two measurements are taken at 5.8 ± 0.5 mm and 3.2 ± 0.3 mm. What is the difference between the two measurements?

Solution (-): First calculate the distance without uncertainties.

$$5.8mm - 3.2mm = 2.6mm$$

Next calculate the uncertainty with the propagation of error formula.

$$\sigma_x = \sqrt{0.5^2 + 0.3^2} = 0.6$$

The solution is 2.6 ± 0.6mm.

Example (*): The mass of an object is 4.02 ± 0.05 kg. What is the force exerted on the object that causes it to accelerate 10.2 ± 0.8 m/s^2?

Solution (*): First calculate the force without uncertainties.

$$F = ma = 4.02\ kg * 10.2m/s^2 = 41.0N$$

Next, calculate the uncertainty using the multiplication formula.

$$\sigma_x = 41.0 * \sqrt{\left(\frac{0.05}{4.02}\right)^2 + \left(\frac{0.8}{10.2}\right)^2} = 3.26N$$

The solution is 41 ± 3 N.

6.4 TAYLOR SERIES

Taylor series rewrites a continuous function into an infinite sum of simpler polynomial terms. For a function f(x), the Taylor series is defined by the following format, where the function is based around the point a.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

This can be rewritten as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

When a = 0, this is a specific Taylor series, known as the Maclaurin series. The best way to explain how to produce a Taylor series is by doing a couple examples. The following are very simple examples of the Taylor series that can be done within the timeframe of the exam.

Example: What is the Taylor series for e^x centered around a = 1?

Solution: Expand the function $f(x) = e^x$ into the Taylor series.

Find the derivatives of the function.

$$f'(x) = e^x$$

$$f''(x) = e^x$$

Notice that all derivatives will equal e^x, so it can be rewritten into the generic term for the nth order derivative.

$$f^n(x) = e^x$$

Solve for the derivatives at a = 1.

$$f^n(a) = f^n(1) = e^1 = e$$

Insert these values into the Taylor series equation.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

So, the function e^x around a=1 can be written out as the following.

$$e^x = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots + \frac{e}{n!}(x-1)^n + \cdots$$

Example: What is the Taylor series for $f(x) = \sin(x)$ around a = 0?

Solution: Compute the derivatives of the function at a=0.

$$f(x) = \sin(x);\;\; f(0) = 0$$

$$f'(x) = \cos(x);\;\; f'(0) = 1$$

$$f''(x) = -\sin(x);\;\; f''(0) = 0$$

$$f'''(x) = -\cos(x);\;\; f'''(0) = -1$$

$$f^4(x) = \sin(x)\,;\;\; f^4(0) = 0$$

$$f^5(x) = \cos(x)\,;\;\; f^5(0) = 1$$

Insert the derivatives at a=0 into the Taylor series.

$$f(x) = \sin(x) = \frac{1}{1!}x + \frac{-1}{3!}(x)^3 + \frac{1}{5!}(x)^5 + \cdots$$

Consolidate the pattern of the series into a simplified term. This is the Taylor series for sin(x).

$$f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Definite Integral Approximations:

One application of the Taylor series is to approximate a definite integral of a complex function. For example, it would normally take quite a few steps to compute the integral of the function e^{-x^2}. However, if we rewrite the function into a Taylor series about a = 0, the integral becomes much simpler. (*Side note: if we know the Taylor series for* e^y *about a = 0, we can substitute* $y = -x^2$ *to simply find the Taylor series of* $e^{-x\wedge 2}$.)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

Take the integral of the series from 0 to 1.

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots\right) dx$$

$$= x - \frac{x^3}{3*1!} + \frac{x^5}{5*2!} - \frac{x^7}{7*3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)*n!} + \cdots \Bigg|_0^1$$

$$= 1 - \frac{1}{3*1!} + \frac{1}{5*2!} - \frac{1}{7*3!} + \cdots + \frac{(-1)^n}{(2n+1)*n!} + \cdots$$

Notice as n increases, each term gets smaller. The terms also alternate between positive and negative, which means the function is converging. At n = 4, the term is equal to 0.0046. So, the series can be computed up to a certain value of n for an approximation. Taking the series up to n = 4 results in the following approximate solution.

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3*1!} + \frac{1}{5*2!} - \frac{1}{7*3!} + \frac{1}{9*4!} = 0.747$$

6.5 Newton’s Method

Newton’s method uses iterations to approximate the root of an equation. The root is the value of x, when f(x) is equal to zero. Graphically, this is the value of x when the function crosses the x-axis. To approximate the root, an initial guess near the solution is made, called x_0. Then, the line tangent to the function at x_0 can be approximated with the equation below.

$$tangent\ line, y_{t,0} \approx f(x) + f'(x) * (x_1 - x_0)$$

Set the tangent line equal to zero to find the intersection with the x-axis and solve for x.

$$0 = f(x) + f'(x) * (x_1 - x_0) \quad \Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This new value of x, called x_1, is a point closer to the root. Repeating this process brings the value closer and closer to the solution.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

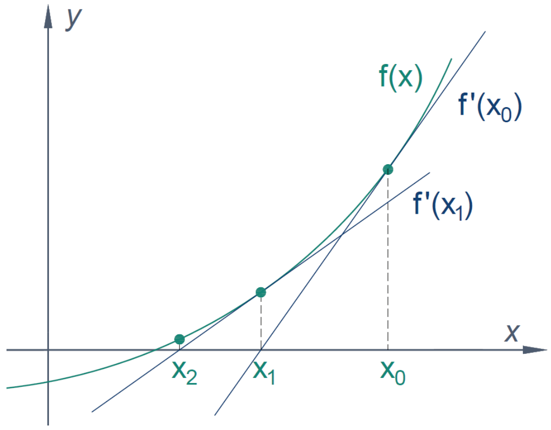
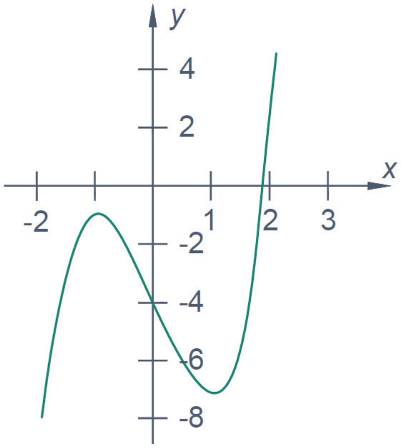


Figure 16: Newton's Method for approximating the root of a function.

The precision of the approximation can be determined by comparing the change between the last iterations of x. Difficulty with this method arises when there are multiple roots that can cause confusion if the initial guess is too far away. One approach is to first graph the function to determine how many roots there are and what value is a reasonable first guess.

Example: Find the roots of $y = 2x^3 - 5x - 4$ to the nearest 4 decimal places. The graph of the function is shown below.



Solution: The graph shows that there is one root near x=2. Solve for the derivative of the function and set the initial guess at $x_0 = 2$.

$$y' = 6x^2 - 5$$

The formula for the next x using Newton's method is written out below.

$$x_{n+1} = x_n - \frac{2x_n^3 - 5x - 4}{6x_n^2 - 5}$$

Solve for x until it converges to the fourth decimal place.

x_0	2
x_1	1.894737
x_2	1.886839
x_3	1.886796
x_4	1.886796

The root of the function, rounded to the nearest fourth decimal place is **x=1.8868**.

6.6 NUMERICAL INTEGRATION

Numerical integration provides a numerical process for calculating definite integrals that are otherwise difficult to calculate. The three methods included in the *NCEES FE Reference Handbook* are Forward Rectangular (Euler), Trapezoidal, and Simpson's Rule. Each method breaks up the function into tiny vertical strips. The area of the strips are calculated and added together to form an approximate integral, or area under the curve.

Forward Rectangular:

The simplest method is the rectangular rule. As suggested in the name, the function is broken up into rectangular segments, each with a width of Δx. In the forward rectangular rule, the upper left corner of the rectangle is located on the function. This point is the height of each rectangle.

For a function, f(x), the integral from x = a to b using the forward rectangular integration process can be graphed as shown below.

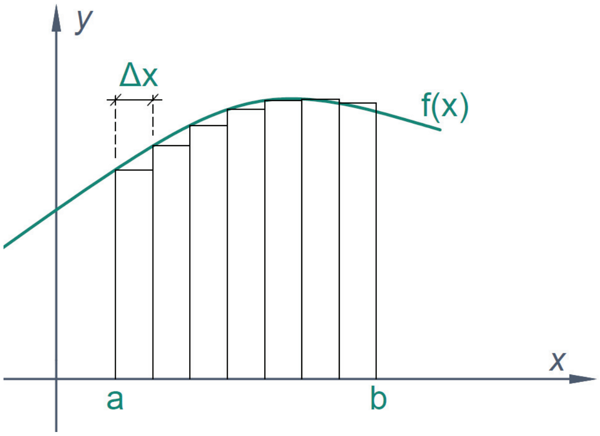


Figure 17: Forward Rectangular Rule for numerical integration

The area of the first rectangle is equal to *f(a)* Δx*. The area of the second is *f(a+ Δx)* Δx*. The third rectangle has area *f(a+2*Δx)* Δx*, and so on. Thus the total area from a to b can be written with the following equation.

$$\mathit{Forward\ Rectangular\ Rule}: \quad \int_a^b f(x)\,dx \approx \Delta x * \sum_{i=0}^{n-1} f(a+i\Delta x), \qquad where\ \Delta x = \frac{b-a}{n}$$

The greater the value for *n*, the smaller the interval for Δx, the better the approximation. The value of *n* can continuously be increased until it begins to converge within a specified amount of decimal places. Modifications of the forward rectangular method are the backward rectangular rule and the midpoint rule, where the height is instead taken at the upper right corner of the rectangle or the middle of the rectangle, respectively.

Trapezoidal:

An even better approximation is the trapezoidal rule. In this method, the top corners of the trapezoid are located on the function f(x), as illustrated below.

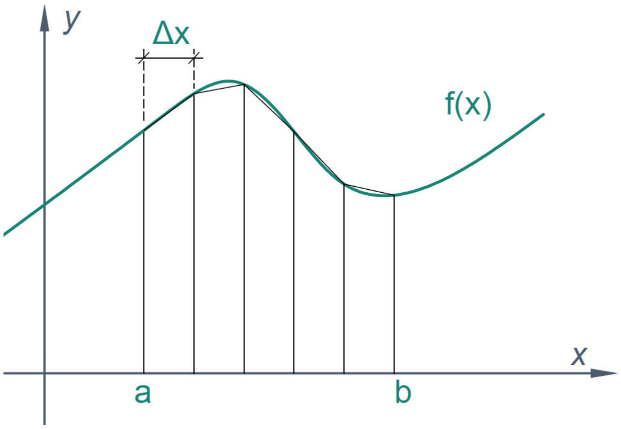


Figure 18: Numerical integration using the trapezoidal rule

The area of each trapezoid segment is equal to the width times the average height of the upper left and right corners. The area of each trapezoid is equal to Δx*[f(x_{left})+f(x_{right})]/2. Thus, the area of the first trapezoid is equal to Δx*[f(a)+f(a+Δx)]/2. The area of the second trapezoid is Δx*[f(a+Δx)+f(a+2Δx)]/2, the third is Δx*[f(a+2Δx)+f(a+3Δx)]/2, and so on. Sum all segments to find the total area from a to b, as written below.

$$Trapezoidal\ Rule\colon \int_a^b f(x)\,dx \approx \frac{\Delta x}{2} * \sum_{i=1}^{n-1} [f(a) + 2 * f(a + i * \Delta x) + f(b)], \quad where\ \Delta x = \frac{b-a}{n}$$

Simpson's:

Simpson's rule further improves the approximation by using a parabola to estimate the height of each segment. In this method, Δx must be even. The area of each segment is estimated as Δx/6*[f(x_{left})+ 4*f(x_{mid})+f(x_{right})]. The total area using Simpson's Rule can be found with the following general equation.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3} * [f(a) + 4 * \sum_{i=1,3,5\ldots}^{n-1} [f(a + i * \Delta x) + f(b)] + 2 * \sum_{i=2,4,6\ldots}^{n-2} [f(a + i * \Delta x)], \quad where\ \Delta x = \frac{b-a}{n},$$

Example: Use the trapezoidal rule to approximate the integral of f(x) = 5^{x+1} from x = 0 to 2. Use n = 3.

Solution: First solve for Δx.

$$\Delta x = \frac{2}{3}$$

Next, approximate the integral with the trapezoidal rule.

$$\int_0^2 5^{x+1}\,dx \ \approx \left(\frac{2}{3}\right) * \frac{1}{2} * \left[5^{0+1} + 2 * 5^{\frac{2}{3}+1} + 2 * 5^{\frac{4}{3}+1} + 5^{2+1}\right] = 81.6$$

The answer is 81.6.

7.0 ALGORITHM AND LOGIC DEVELOPMENT

Algorithm and logic development are the basis to developing decision making processes or constructing a computer program to simplify and optimize tasks for engineers. This is especially appropriate for numerical calculations that require lengthy iterations or calculations with a significant amount of data. This topic is broken up into flowcharts, pseudocodes, and spreadsheets, all of which are simple, testable applications of algorithm and logic development.

7.1 FLOW CHARTS

In mechanical engineering, flow charts are used in decision making processes and for determining the course of action for a design. Often times this will entail choosing one alternative instead of several other design alternatives, which can be based on the economics of the various alternatives. When you have selected a design, you can also optimize the design with proper project management. Other ways flow charts are used include system level charts that show how resources flow, program flow charts that are used in controls logic, data flow charts that show data movement, and document flow. Two techniques that can easily be tested are logic flow charts and the critical path method.

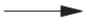
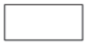
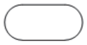
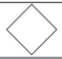
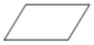
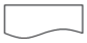
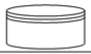
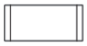
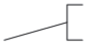
7.1.1 Logic Based Flow Charts

Flow charts can be represented with words to lay out a project workflow or be represented as mathematical and logical expressions for computer based algorithms. The FE will most likely test you on the latter, which is essentially a test on basic computer programming. The intent is to confirm that as an engineer, you are capable of following simple computer programming and flow chart logic. Remember that these questions will not be too complicated. In fact, you should be able to intuitively solve these problems. Since each programming language varies, the test will only use the very basic and commonly understood commands. Commands will be represented in line by line syntax format or diagrammatic flow charts.

7.1.1.1 FLOW CHART SYMBOLS

The following symbols are based on the American National Standards Institute (ANSI) and the International Organization for Standardization (ISO) standards for flow chart symbols. These can be found under the "Flow Chart Definition" section of the Electrical and Computer Engineering chapter in the *NCEES FE Reference Handbook*.

ANSI Flow Chart Symbols

	Flowline	Shows process flow
	Process	Operation performed
	Terminal	Start/stop of a program
	Decision	Point of multiple paths
	Input/Output	Data input and output
	Document	Readable data
	Database	Stored list of info
	Connector	Inspection point
	Predefined Process	Process defined elsewhere
	Annotation	Notes about a process/program

7.1.1.2 VARIABLES, CONSTANTS, AND FORMULAS

The first of the computer or logic based commands is defining a variable or a constant. A variable or constant is an identifier that is defined as a numerical value, typically variables or constants are only letters. For example, A=5 means that the variable A is assigned to be equal to the number 5. The difference between a variable and a constant is that a variable can be changed throughout the program, but a constant cannot. Once the variable is defined, it can be used within a "process" or a formula. A formula will use mathematical operations, such as addition, multiplication, square root, sine, cosine, logarithm, etc.

Example:

```
A = 5;
Constant pi = 3.14;
A=5 +1;
B=A^2*pi;
```

Solution: Initially A is defined as a variable, equal to the number 5. At the end of the program, variable A is redefined as the number 6. The constant pi will remain 3.14 throughout the program. Variable B is equal to 6^2*3.14 = 113.

7.1.1.3 LOGIC STATEMENTS

The fundamental component of flow charts is the use of logic statements. Logic statements are used to determine which direction the flow chart should proceed. In the following examples, you should be able to follow the flow of commands in either the syntax form or the diagrammatic form.

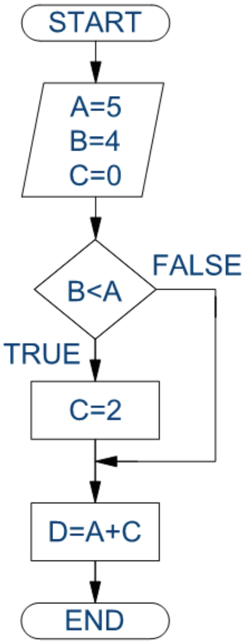
***"If" statements**: If-statements will start with a tested expression. If the expression is true, it will perform the process included within the if-statement, if it is false, then it will proceed around the process included in the if-statement.*

Example:

Syntax Format:

```
A=5;
B=4;
C=0;
IF B<A
        THEN C=2;
D=A+C;
END;
```

ANSI Flow Chart Format:



Solution: In the above command, the expression within the "if" statement: B<A is true, therefore C=2 and the program proceeds to compute D=7. Let's assume that B was equal to 6 instead of 4. Then the statement B<A would be false and the program would skip of over the C=2 command. C would remain being equal to 0 and the result would be D=5.

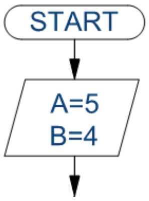
"If-Else" statements: If-else statements will start with a tested expression. If the expression is true, it will perform the process included within the if-statement, if it is false, then it will proceed to the process within the else statement. Then the flow will continue to the next commands outside of the if-else statement.

Example:

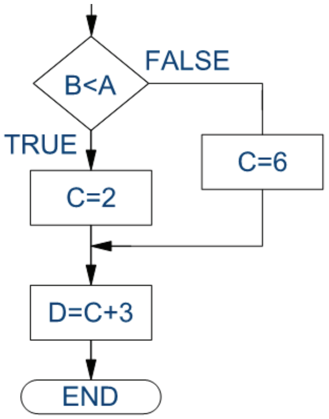
Syntax Format:

```
A=5;
B=4;
```

ANSI Flow Chart Format:



```
IF B<A
    THEN C=2;
    ELSE C=6;
D=C+3;
END;
```



Solution: In the flow chart above, B<A, therefore C=2 and the program skips over the "else" statement. The program ends with D=5. Again, for arguments sake, let's assume that B was 6 instead of 4. Then, B<A is a false statement, so the program will skip over the "then" statement and proceeds to the "end" statement, where C=6. The program ends with D=9.

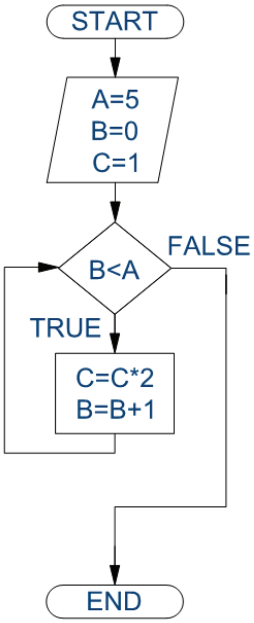
"Loop" statements: In programming, loop statements come in the form of "for", "while" and "do-while" loops. Differentiating the three will require additional syntax that will unlikely be tested. The test is more concerned with flow rather than syntax, so the syntax itself should be intuitive. The important concept to understand is that in a general looping scheme, a command is continually repeated within a loop until a test expression is satisfied. The test expression is located within the loop command and is also continuously updated.

Example:

Syntax Format:

ANSI Flow Chart Format:

```
A=5;
B=0;
C=1;
WHILE (B<A)
    C=C*2;
    B=B+1;
END;
```



Solution: In the example above, a loop continues under the while command until B<A is no longer true. The initial flow begins with A=5, B=0, and C=1. The test statement B<A is true, so the commands C=C*2 and B=B+1 are executed. In the first pass, C=2 and B=1. The loop continues

four more times (5 passes total) until B=5 and C=32. On the sixth pass, B=5 and the test statement B<A is no longer true, the command jumps over the while loop, and the program ends.

7.1.2 Critical Path Project Flow Chart

One application of flow charts is finding the critical path in a project and understanding the effects of delays on tasks within the critical path and outside of the critical path. On the exam each task of a project will most likely be given and the time it takes to complete each task. In addition, each task will have its dependencies that must be completed prior to the task being started. The best way to understand how to complete these types of problems is to go through a sample problem, step by step.

The following is a table of various tasks and their durations, along with what tasks must be completed before a task begins.

Task Name	Predecessors	Duration
A	N/A	1 day
B	A	3 days
C	A	2 days
D	B,C	4 days
E	D	3 days
F	B,D	2 days
G	E	1 day
H	F,G	5 days

The first step is to pick the task(s) that have no predecessors and draw these tasks on the left side of your paper. The understanding is that the progression of the project will proceed from left to right. The only task without a predecessor is task A. This task serves as your starting point. Next find the tasks that lead from A, which are B and C. Next, follow the same steps for tasks B, C and the rest of the tasks. Your final figure should have the following connections shown on the diagram below.

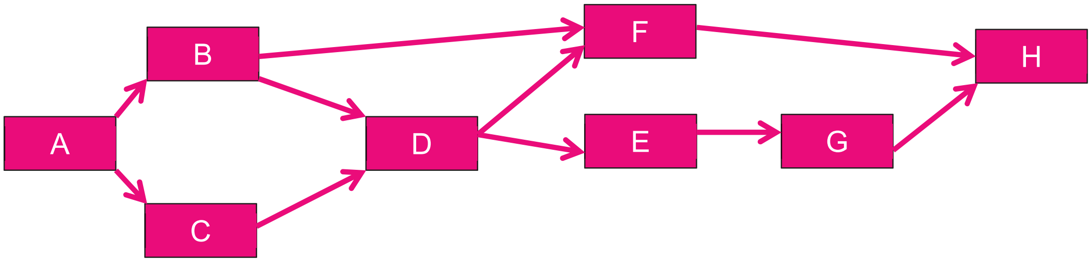


Figure 19: Project work flow diagram for example problem.

The next step is to create all the possible paths and determine the total number of days for each path.

Path	Total Duration	Path	Total Duration
A-B-F-H	1+3+2+5=11 days	A-C-D-E-G-H	1+2+4+3+1+5=16 days
A-B-D-F-H	1+3+4+2+5=15 days	A-C-D-F-H	1+2+4+2+5=14 days
A-B-D-E-G-H	**1+3+4+3+1+5=17 days**		

The critical path is “A-B-D-E-G-H”. This is also the total length of the project. Thus if any delay or improvement occurs to this path, then the total length of project will change. If a change occurs in a task not included in this path like C or F, then the total length of the project may or may not change, depending on the severity of the change.

7.2 PSEUDOCODE

Psuedocode uses simple, readable syntax and phrases to represent a computer programming code, without using any specific programming language. This allows anyone to be able to read and understand the intent of the code. Whereas the logic based flow charts in the previous section represented programming codes in a graphical format, pseudocode uses descriptive words to define the operations. Like flow charts, pseudocode can act as an intermediary step between developing and designing a program.

Computer programming is an important tool in mechanical engineering for solving repetitive and lengthy calculations, especially those that require iterations. For the exam, you should be familiar with common logic expressions used in computer languages, such as if-then, else-if, counters, etc, and be able to follow a pseudocode to solve the output of a simple program. Logic expressions were introduced in the Flow Chart section.

<u>Example:</u>

This program calculates if there is enough paint for one coat, given the dimensions of a surface.

```
Start
Set thickness to 0.00005

Input length in meters
Input width in meters
Input gallons of paint

required_gallons = length * width * thickness * 264

If gallons < required_gallons
    Then print “there is not enough paint”
Else
    Print “there is enough paint”
End
```

7.3 SPREADSHEETS

The spreadsheet topic was listed in the previous *NCEES Mechanical FE Exam Specifications*. Although it is not explicitly called for in the latest exam specifications, it is still reviewed here, since it is a commonly used algorithm and logic tool in the mechanical engineering field.

Spreadsheets refer to commonly used formulas that are used in Microsoft Excel® or other similar computational programs. The basic operations and commands are not included in the *NCEES FE Reference Handbook*, but rather are understood to be used in everyday practice. The intent of spreadsheets is to simplify the calculations, increase computational efficiency, be able to quickly change variables and outputs, and to output data in a presentable format. The other benefit of spreadsheets is being able to compute results for multiple input data either in list or array format.

7.3.1 Cells

A spreadsheet consists of an array of cells labeled by a letter for each column and a number for each row. A specific cell is identified by a letter-number combination. For example, the cells with an x is labeled A1, the cell with xx is labeled B2, and the cell with xxx is labeled D5.

	A	B	C	D	E	F
1	x					
2		xx				
3						
4						
5				xxx		
6						

Within each cell, a text, number, or formula can be inserted. A text is a string of letters, a number is a numerical value, and a formula is used for mathematical or logic operations.

7.3.2 Common Operations

The following are common mathematical formulas and operations used in spreadsheets. A formula use numbers or reference cells in an operation.

Common Operations

Operation	**Description**	**Example**	**Note**
=	Equals	=A1+B2	Required at the beginning of a formula in a cell
+	Addition	=5+5	
-	Subtraction	=10-5	
*	Multiplication	=3*5	
/	Division	=100/4	
%	Percent	80%	Equivalent to 0.8

sqrt(…)	Square root	=sqrt(100)	Square root calculated for the number inside the parenthesis
^	Exponent	=7^2	Square of 7 = 49.
E	Power of 10	=5E3	= 5*10^3 = 5,000
pi()	Pi	=pi()	Pi = 3.412159…
(…)	Order of Operation	=((5+4)*20/4)^2*sqrt(4)	Result equals 4050
=SUM(…)	Summation	=SUM(5,6,4)	Sum of all values in the parenthesis. Result is 15. Comma separates the values to be added.
=AVERAGE(…)	Average	=AVERAGE(5,2,9)	Finds the average of all values in the parenthesis. Result is 5.33
=COUNT(…)	Count	=COUNT(A1:A5)	Counts the number of cells within A1 thru A5 that has numbers.
=COUNTIF(…,…)	Count if	=COUNTIF(A1:A5,>2)	Counts the number of cells within A1 thru A5 that are greater than 2. =countif(range, criteria)
“…”	Text	“abc”	Identifies texts within formulas
&	And	=5 & 4 =“x”&“y”	First result is 54, second result is xy. Combines two statements.
>	Greater than	Used in Logic Formulas	See if, then example
>=	Greater than or equal	“	
<	Less than	“	
<=	Less than or equal	“	
=	Equal	“	
<>	Not Equal	“	
=IF(.., …, …)	If-Then, true or false statement	=if(5>4, 3, 1)	Result is 3 =if(logic_test, value_if_true, value_if_false)
AND	And	=IF(AND(5>4,4>3),1,0)	Both statements 5>4 and 4>3 must be true, so the output is 1
OR	And	=IF(OR(6>9,6>3),1,0)	At least one statement, 6>9 or 6>3 is true, so the output is 1
: (colon)	Range	=A1:C5	The range of cells contained within A1 and C5
, (comma)	Union	=sum(B4, C9)	Combines individual references within a formula

See below for examples of these operations used within a spreadsheet layout.

Example: Use the table below to solve for the results of the following formulas.

	A	B	C	D	E	F
1		January	February	March		
2	Apple	5	7	3		
3	Oranges	6	3	4		
4	Banana	2	5	8		
5	Grapes	9	6	4		
6						

Solution: The following formulas yield the following results.

Formula	Equivalent Formula	Result
=SUM(B3, D4)	=6+8	14
=SUM(B2:D5)	=5+6+2+9+7+3+5+6+3+4+8+4	62
=AVERAGE(B2:B5)	=(5+6+2+9)/4	5.5
=COUNTIF(B2:D5,>=5)	=1+1+0+1+1+0+1+1+0+0+1+0	7
=(C5^2+21)/4	=(6^2+21)/4	14.25

7.3.3 Relative, Absolute, and Mixed Reference

For ease of populating spreadsheets, a cell's position is relative to referenced cells. Refer to the spreadsheet used in the example above. Assume cell E2 was populated with the formula =B2+C2+D2. If cell E2 was copied and pasted into E3, then the formula would change to references are in a relative position to the new cell E3. The formula in E3 would result in =B3+C3+D3. If E2 was copied, then pasted into E4, then the formula in E4 would be shifted two rows down and become =B4+C4+D4. If E2 was copied, then pasted into F2, then the references would shift right and the formula in F2 would be =C2+D2+E2.

To lock the position of a reference, it can be converted to an absolute reference by adding dollar signs before the reference letter and number. In the example above, change the formula in E2 so that B2 becomes an absolute reference. The new formula becomes =$B$2+C2+D2. Then, when E2 is copied into F4, the reference B2 remains locked and the formula in F4 is =$B$2+D4+E4.

It is possible to freeze just the row or the columns of a reference. This is called a mixed reference. Adding a dollar sign before the letter locks the referenced column and a dollar sign before the number locks the referenced row. For example, assume the formula in E2 is B$2+C2+D2. If E2 is copied into F2, then the formula becomes =C$2+D2+E2. If E2 is copied into E3, then the formula becomes =B$2+C3+D3, since row 2 in the first reference remains locked. Similar rules apply when B2 becomes the mixed reference $B2 instead.

8.0 PRACTICE EXAM PROBLEMS

8.1 PRACTICE PROBLEM 1 – CALCULUS

What is the integral of the function below from 0 to 90 degrees?

$$\int_{0\ deg}^{90\ deg} 10 * \sin(\theta)\, d\theta$$

(a) 0

(b) 1

(c) π

(d) 10

8.2 PRACTICE PROBLEM 2 – CALCULUS

What is the integral of the function below from 1 to 2?

$$\int_{1}^{2} \ln(x) dx$$

(a) 0.39

(b) 1.01

(c) 1.77

(d) 2.00

8.3 PRACTICE PROBLEM 3 - CALCULUS

An object's speed is governed by the below equation. How much distance is covered between 0 and 3 seconds? Assume the location at time t = 0 sec is 0 ft.

$$v(t) = 10\left(\frac{ft}{s^2}\right) * t + 2\left(\frac{ft}{s}\right)$$

(a) 35 ft

(b) 50 ft

(c) 65 ft

(d) 90 ft

8.4 PRACTICE PROBLEM 4 – CALCULUS

What is the average value of a sine function from 90 to 180 degrees?

$$y = \sin(x)$$

(a) 0.04

(b) 0.31

(c) 0.64

(d) 1.12

8.5 PRACTICE PROBLEM 5 – CALCULUS

What is the derivative of the function below with respect to x?

$$y = \ln(x)$$

(a) 1/x

(b) 0.5 ln(x^2)

(c) 2*ln(x^2)

(d) e^x

8.6 PRACTICE PROBLEM 6 – CALCULUS

What is the integral of the function below?

$$y = \frac{\ln(x)}{x}$$

(a) $\frac{1}{2}(\ln(x))^2 + C$

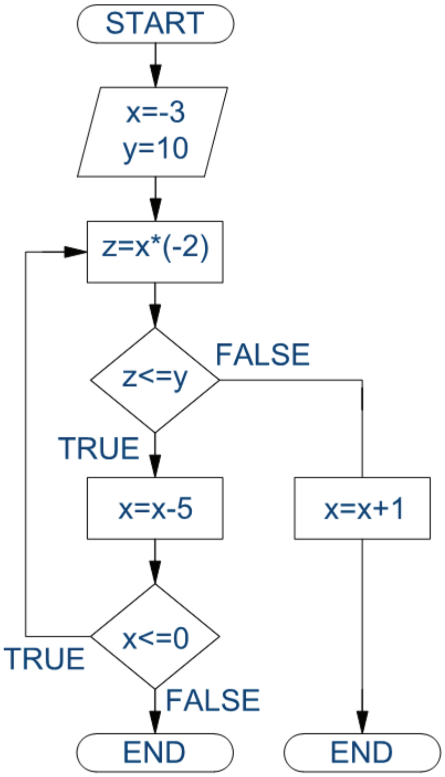
(b) $\ln(x) + C$

(c) $2(\ln(x))^2 + C$

(d) $e^x + C$

8.7 PRACTICE PROBLEM 7 – FLOW CHARTS

In the flow chart below, what is the final value of x?


START
x=-3
y=10
z=x*(-2)
z<=y
FALSE
TRUE
x=x-5
x=x+1
x<=0
TRUE
FALSE
END
END

(A) -2
(B) -9
(C) 7
(D) -7

8.8 PRACTICE PROBLEM 8 – FLOW CHARTS

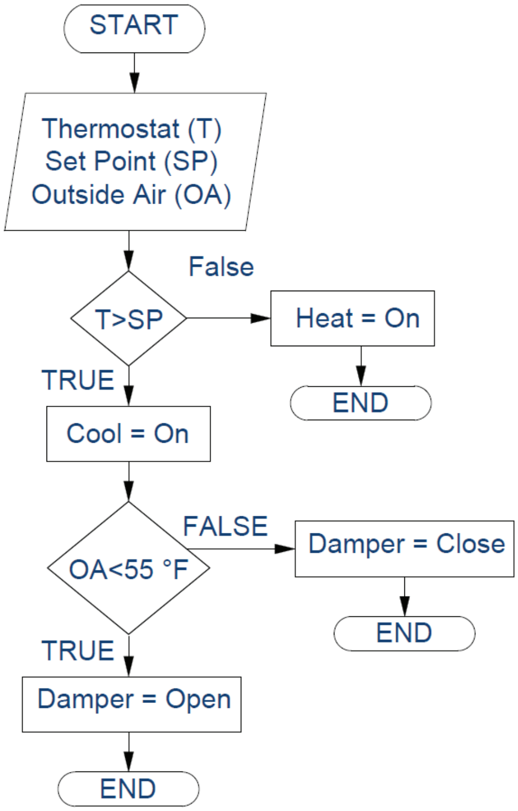
Which description below best describes an accurate set of outputs form the below flow chart?

(A) A temperature of 74 °F, set point of 78 °F and an outside air temperature of 50 °F will turn on the heating function of this air conditioner and open the outside air.

(B) A temperature of 80 °F, set point of 75 °F and an outside air temperature of 56 °F will turn on the cooling function of this air conditioner and open the outside air damper to let in cool outside air.

(C) A temperature of 68 °F, set point of 78 °F and an outside air temperature of 65 °F will turn on the heating function of this air conditioner and close the outside air damper.

(D) A temperature of 80 °F, set point of 75 °F and an outside air temperature of 52 °F will turn on the cooling function of this air conditioner and open the outside air damper to let in cool outside air.


START
Thermostat (T)
Set Point (SP)
Outside Air (OA)
False
T>SP
Heat = On
END
TRUE
Cool = On
FALSE
OA<55 °F
Damper = Close
END
TRUE
Damper = Open
END

8.9 Practice Problem 9 – Critical Path

Determine the length of the critical path for the following machining process.

Machining Process	Duration (minutes)	Dependency
A	5	Start
B	2	Start
C	3	A & B
D	5	Start
E	6	Start
F	4	D & E
G	2	C & F
H	5	G
I	7	G
J	3	H
Finish	0	I & J

(A) 17 minutes

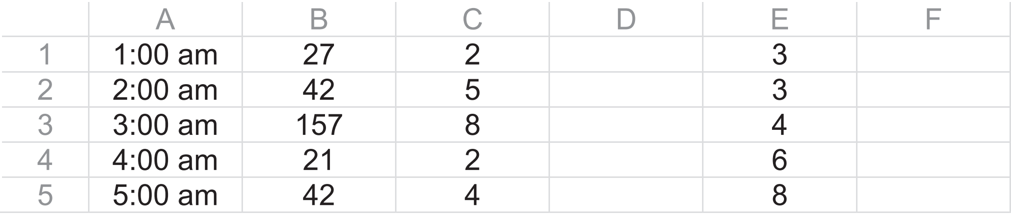
(B) 18 minutes

(C) 19 minutes

(D) 20 minutes

8.10 Practice Problem 10 - Spreadsheets

In the spreadsheet below, cell D4 contains the formula “=$B4/(pi()*C4^2/4)” is copied over to cell F2. What is the value in cell F2?



	A	B	C	D	E	F
1	1:00 am	27	2		3	
2	2:00 am	42	5		3	
3	3:00 am	157	8		4	
4	4:00 am	21	2		6	
5	5:00 am	42	4		8	

(A) 0.7
(B) 1.5
(C) 3.0
(D) 5.9

8.11 PRACTICE PROBLEM 11 – SPREADSHEETS

The formula in cell C4 is "=IF(AND(B1<>5,B2<0,B3>=0),3,1)" and the formula in C5 reads "=C4*1E3". What is the numerical result in cell C5?

	A	B	C	D	E
1	X	5		A	3
2	Y	-5		B	0
3	Z	0		C	-3
4					
5					

(A) -3
(B) -9
(C) 1000
(D) 3000

9.0 SOLUTIONS

9.1 SOLUTION 1 – CALCULUS

Use the integral equations from the *NCEES FE Reference Handbook*.

$$\int_{0\ deg}^{90\ deg} 10 * \sin(\theta)\, d\theta = -10\cos(\theta)|_{0\ deg}^{90\ deg}$$

$$= -10 * \cos(90\text{deg}) - (-10 * \cos(0\text{ deg}))$$

$$= 0 - (-10 * 1) = 10$$

The correct answer is most nearly, (d) 10.

9.2 SOLUTION 2 – CALCULUS

$$\int_{1}^{2} \ln(x)\, dx = x(\ln(x) - 1)|_1^2$$

$$= 2 * (\ln(2) - 1) - 1 * (\ln(1) - 1)$$

$$= -0.614 + 1 = 0.386$$

The correct answer is most nearly, (a) 0.39

9.3 SOLUTION 3 - CALCULUS

First, integrate the function.

$$\int v(t)\, dt = \frac{1}{2}t^2 * 10 + 2t$$

$$x(t) = \int v(t)\, dt = 5t^2 + 2t$$

Then evaluate from 0 to 3 seconds.

$$x(0) = 5(0)^2 + 2(0) = 0\ ft$$

$$x(3) = 5(3)^2 + 2(3) = 51\ ft$$

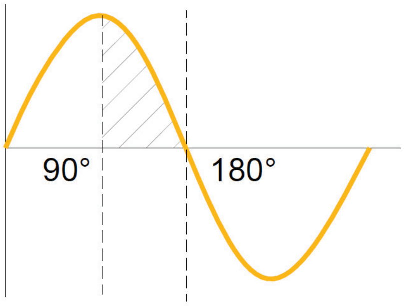
$$\Delta x = x(3) - x(0) = 51 ft$$

The correct answer is most nearly (b) 50 ft.

9.4 SOLUTION 4 - CALCULUS

For this problem, you need to integrate from 90 to 180 degrees and then divide by the total x-value between 180 and 90 degrees.

$$\mathrm{y_{avg}} = \frac{\int y(x)\,dx}{\Delta x}$$



$$\int_{90\ deg}^{180\ deg} \sin(x)\,dx = -\cos(180\mathrm{deg}) - (-\cos(90\ \mathrm{deg}))$$

$$1 - 0$$

Remember that the linear distance Δx must be taken in terms of radians and not degrees.

$$X - distance = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$Average\ Value = \frac{1}{\pi/2} = 0.64$$

The correct answer is most nearly (c) 0.64.

9.5 SOLUTION 5 - CALCULUS

For this one just use the *NCEES FE Reference Handbook*.

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x}$$

The correct answer is most nearly, (a) 1/x.

9.6 SOLUTION 6 - CALCULUS

This problem should be done with integration by parts.

$$\int u(x) * dv(x) = u(x) * v(x) - \int v * \, du(x)$$

For this problem, define u and dv/dx as the following.

$$u = \ln(x)\,; \quad dv(x) = \left(\frac{1}{x}\right)$$

Then the derivative of u and the integral of dv/dx are solved below.

$$du(x) = \frac{1}{x}; \quad v = \ln(\mathrm{x})$$

Plug in the u, du/x, v and dv/dx values.

$$\int \ln(\mathrm{x}) * \frac{1}{\mathrm{x}}\, dx = \ln(x) * \ln(x) - \int \ln(\mathrm{x}) * \frac{1}{\mathrm{x}}\, dx$$

$$2\int \ln(\mathrm{x}) * \frac{1}{\mathrm{x}}\, dx = (\ln(x))^2$$

$$\int \ln(\mathrm{x}) * \frac{1}{\mathrm{x}}\, dx = \frac{1}{2}(\ln(x))^2 + C$$

The correct answer is most nearly, (a) $\frac{1}{2}(\ln(x))^2 + C$

9.7 SOLUTION 7 – FLOW CHARTS

Start with the input variables x= –3 and y=10. Then z becomes (-3)*(-2)=6. Z is less than or equal to y=10, so the flow continues to the true path to x=x-5 = (-3)-5 = -8. X is less than or equal to 0, so the program continues to the true path and loops back up to z=x*(-2) = -8*(-2) = 16. Then in the next statement, z is no longer less than or equal to y, and the program follows the false path to x=x+1=-8+1=-7.

The correct answer is **(D) -7**

9.8 SOLUTION 8 – FLOW CHARTS

(A) A temperature of 74 °F, set point of 78 °F and an outside air temperature of 50 °F will turn on the heating function of this air conditioner and open the outside air.

[*The set point is above the current temperature, so the heating function will turn on. The damper will neither be closed nor opened because the flow chart is terminated before the damper operation is reached. The damper will remain in whatever position it was previously in. This answer will not work.*]

(B) A temperature of 80 °F, set point of 75 °F and an outside air temperature of 56 °F will turn on the cooling function of this air conditioner and open the outside air damper to let in cool outside air.

[*The set point is below the current temperature, so the cooling function will turn on. The damper will be closed because the outside air temperature is too high. This answer will not work.*]

(C) A temperature of 68 °F, set point of 78 °F and an outside air temperature of 65 °F will turn on the heating function of this air conditioner and close the outside air damper.

[*The set point is above the current temperature, so the heating function will turn on. The damper will be closed because the outside air temperature is too high. The damper will neither be closed nor opened because the flow chart is terminated before the damper operation is reached. The damper will remain in whatever position it was previously in. This answer will not work.*]

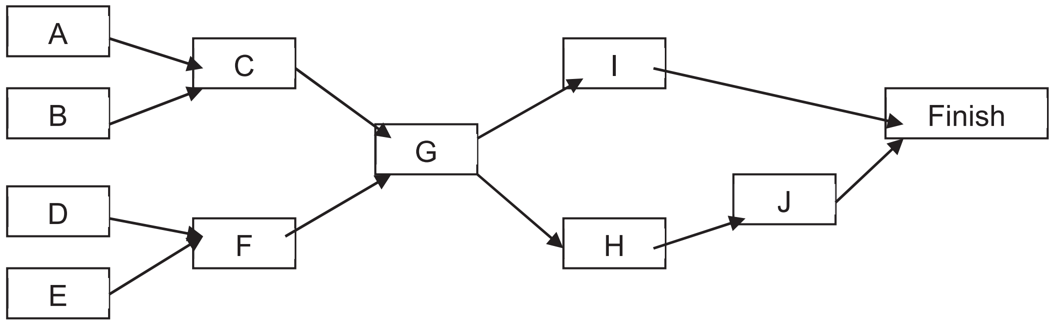
(D) A temperature of 80 °F, set point of 75 °F and an outside air temperature of 52 °F will turn on the cooling function of this air conditioner and open the outside air damper to let in cool outside air.

[*The set point is below the current temperature, so the cooling function will turn on. The damper will be opened because the outside air temperature is lower than 55 °F. This answer will work.*]

The correct answer is most nearly (D).

9.9 SOLUTION 9 – CRITICAL PATH

Determine the length of the critical path for the following machining process.



A->C->G->I-> 17 minutes

A->C->G->H->J-> 18 minutes

B->C->G->I-> 14 minutes

B->C->G->H->J-> 15 minutes

D->F->G->I-> 18 minutes

D->F->G->H-> J -> 19 minutes

E->F->G->I-> 19 minutes

E->F->G->H-> J -> 20 minutes

The critical path is most nearly, **(D) 20 minutes**.

9.10 SOLUTION 10 - SPREADSHEETS

The formula copied into cell F2 becomes “=$B2/(pi()*E2^2/4)”. Replacing the cell references with the cell values, the formula becomes

$$F2 = 42/(3.14 * 3\hat{}2/4) = 5.9$$

Correct Answer is **(D) 5.9**

9.11 SOLUTION 11 – SPREADSHEETS

In cell C4, all statements within the parenthesis after AND must be true. B1 is not equal to 5 is not true, B2 is less than 0 is true, and B3 is greater than or equal to 0 is true. Since the first statement is false, the output in cell C4 is equal to 1 (note: if the statement were true, the output in cell C4 would be 3). Then, the formula in C5 becomes:

$$C5 = C4 * 1E3 = 1 * 1000 = 1000$$

Remember that 1E3 is equal to 1*10^3.

The correct answer is **(C) 1000**

2 – Probability & Statistics

MECHANICAL FE TEXTBOOK

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Section 2.0 – Probability and Statistics

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1.0 INTRODUCTION

Probability and Statistics accounts for approximately 4 to 6 questions on the Mechanical FE exam. Statistics is primarily used in Machine Design for statistical quality control, which is covered under Section 14.0 Mechanical Design and Analysis, under the topics Quality & Reliability. This section focuses on the following NCEES Outline topics, Probability Distributions and Regression Curve Fitting.

Probability Distribution involves applying a mathematical formula to describe the probability of a measured variable occurring at a certain value. This is useful for characterizing the measured output of any mechanical system property when you are taking a sample of a larger number. For example, you measure the weight of 100 products, but this is only a sample of the 10,000 products that are produced. A probability distribution will help to characterize all 10,000 products.

Regression curve fitting involves measuring a variable as a function of another variable, then plotting the data points and assigning a mathematical formula to approximate the function. This is useful in predicting how a change in one variable will affect another.

Section 2.0 Probability and Statistics (4 to 6 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
2B	Section 2.0	Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation, confidence intervals)
2A	Section 3.0	Probability Distributions (e.g., normal, binomial, empirical, discrete, continuous)
2C	Section 4.0	Expected value (weighted average) in decision making
2D	Section 5.0	Regression (linear, multiple), curve fitting, and goodness of fit (e.g., correlation coefficient, least squares)
	Section 6.0	Practice Exam Problems

2.0 MEASURES OF CENTRAL TENDENCIES AND DISPERSIONS

Before you get to probability distributions, you need to understand some of the basic topics in probability like the difference between samples and population, mean, mode, standard deviation. As you go through these topics, you should remember that probability is used in mechanical engineering to measure the reliability of a set of data points.

2.1 MEAN OR AVERAGE

The mean of a set of data points is calculated by summing up all the values and dividing by the total number of data points. The mean is also known as the average.

$$\bar{x} = \frac{\sum_1^n x_i}{n}$$

$$n = number\ of\ data\ points;\, x_i = measured\ value;\ \bar{x} = mean\ or\ average$$

Sometimes this term can also be called arithmetic mean. An example calculation for finding the mean is shown below for the sample data set.

$$Data\ set \rightarrow \{1,2,2,2,3,4,10\}$$

$$\bar{x} = \frac{1+2+2+2+3+4+10}{7} = \mathbf{3.43}$$

2.2 MODE

The mode is the measured value that appears the most in a set of data points. In the sample data set from the previous topic, the mode will be "**2**".

2.3 MEDIAN

The median is the measured value that occurs in the middle of the data set. The median is found by first ordering all the data points in ascending or descending order, then finding the middle value. If there is no middle value (i.e. there are an even number of data points), then you must take the average between the two middle values.

$$Data\ set \rightarrow \{1,2,2,\quad \mathbf{2},\quad 3,4,10\}$$

The median in the previous example is "**2**".

2.4 GEOMETRIC MEAN

The geometric mean is used to give equal weight to a set of data points with high volatility. The geometric mean is found by multiplying the data points and taking the nth root of the product, where n is equal to the number of data points.

$$Geometric\ Mean = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

$$Geometric\ Mean = \sqrt[7]{1 * 2 * 2 * 2 * 3 * 4 * 10} = \mathbf{2.67}$$

On the FE exam, just be careful to read the problem correctly and to make sure you calculate the correct mean. If the question does not specifically state geometric mean, then you can assume that arithmetic mean is required.

2.5 STANDARD DEVIATION

The standard deviation is used to characterize the amount of variation between a set of values. A set of values with a large variation, may lead you to not trust the average of those values or the set of values. A set of values with a small variation, will more likely lead you to trust the average of those values and the set of values itself. Large variations imply volatility, while small variations imply consistency and reliability.

Standard deviation is calculated by first finding the mean, then calculating the difference for each value from the mean. Next, square these differences and sum up the square of the differences. Finally divide by the number of data points, then take the square root of the sum. The equation is shown below.

$$\sigma_{population} = \sqrt{\frac{\sum_1^n (x_i - \bar{x})^2}{n}}$$

$$\bar{x} = arithmetic\ mean;\ \sigma = standard\ deviation;\ n = number\ of\ data\ points;\ x_i = data\ point\ i;$$

It is important to note that you can only use this equation when you have a complete set of data points. In the *NCEES FE Reference Handbook*, this equation is called the standard deviation for a population. This just means that you have the complete set of data points and you are not taking a sample of the complete set of data points. If you are taking a sample of data points, then you must use this formula to calculate the standard deviation.

$$\sigma_{sample} = \sqrt{\frac{\sum_1^n (x_i - \bar{x})^2}{n-1}}$$

$$\bar{x} = arithmetic\ mean;\ \sigma = standard\ deviation;\ n = number\ of\ \boldsymbol{sample}\ points;$$

$$x_i = data\ point\ i;$$

3.0 PROBABILITY DISTRIBUTIONS

The *NCEES FE Reference Handbook* has a set of probability distribution tables. These tables take the probability formulas and put the results into tables. There are several probability distribution formulas, (1) Binomial, (2) Normal, (3) t-Distribution and (4) x^2 Distribution. These tables generally describe the probability of obtaining “x” successes, within “n” attempts or samples, with an individual probability of success equal to “P”. This is often used to predict and test the probability of success, given a number of samples.

3.1 BINOMIAL DISTRIBUTION

A binomial distribution is used when there are only 2 outcomes. Typically in mechanical engineering, these two outcomes are described as success or failure, pass or do not pass, and defective or satisfactory. The following table shows the *cumulative* binomial probability of success, given “n” independent trials or experiments, “x” successes and “p” individual probability. Say for example you are to flip a coin. The amount of times you flip a coin is “n”, the number of trials. These trials are independent because one does not affect the other. Let’s assume that heads is a success and tails is a failure. The number of heads flipped throughout the trials is “x” the number of successes. Finally, the likelihood of each flip being a heads is 50%, so “p” the individual probability is 0.5. For a set number of trials, n, the binomial distribution will graph out the probabilities of x successes from 0 to n trials. The cumulative binominal probability table includes all probabilities of success from 0 to x. The main objective is to understand how to read and use the Cumulative Binomial Distribution table.

Table 1: This table is read by first identifying the correct column to look at. The columns corresponds to the probability of success for each individual trial or sample. For example, you can assume the individual probability of success is 0.1, meaning that the probability that each product will come out successful is 10%. Next, determine the combination that you want. A combination is the number of successes “x” out of a total number of trials “n”. For example, if you want to know the probability of 0 successes out of 2 trials, then you would find the probability as 0.81. This means that there is an 81% chance that you will select two samples or run two trials and both will be failures, meaning you have zero successes. This table is shown as cumulative, so if you navigate to right below that point you will see a value of 0.99. This means that there is a 99% probability that out of two trials you will either have 0 successes or 1 success. If you calculate the difference between x=1 and x=0, then you will have the probability that there is exactly 1 success, which is 0.99-0.81=0.18 or 18%.

Cumulative Binomial Distribution

		0.1	0.2	0.3	0.4	0.5
n=2	x=0	0.81	0.64	0.49	0.36	0.25
	1	0.99	0.96	0.91	0.84	0.75
	2					

It is very important that you understand the probability of success is cumulative in these tables as you move from x= 0 down to x = 1.

3.2 NORMAL DISTRIBUTION

A normal distribution is shown as the figure below. In order to answer these problems, you have to use a different method from the binomial distribution table. The outline of the normal distribution curve may look similar to the binomial distribution at times, but its values are different.

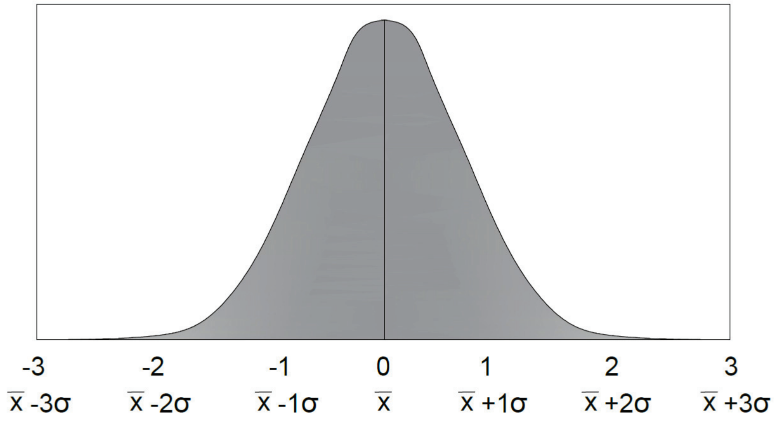


Figure 1: This figure shows the normal or Gaussian distribution. This can also be shown in table form. It shows the probability of a sample's value appearing a certain standard deviation away from the mean. There is a high probability of the value appearing at the mean, then 1 standard deviation away, then less probability at 2 deviations away and finally least probability at 3 standard deviations away.

This graph can be used as follows. The first way is through the R(x) values. These values tell you the probability that the sample will occur in the range to the right of the (x) value. The (x) value is the number of standard deviations away from the arithmetic mean. This tells you the probability that the sample will occur in the range right of the (x) value of standard deviations away from the arithmetic mean.

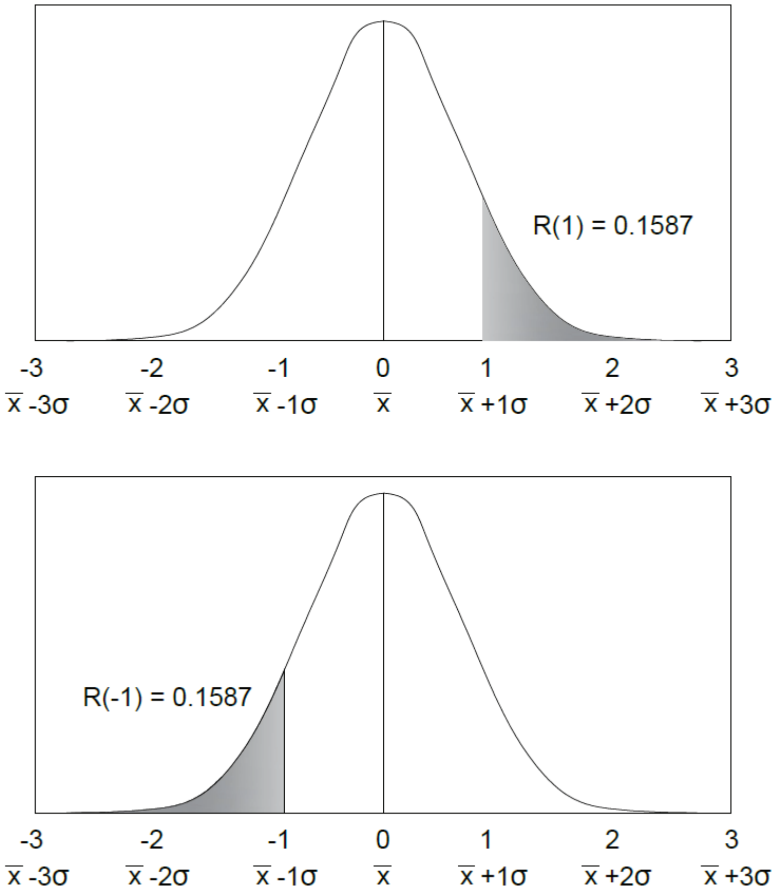


Figure 2: The first way you can look at the normal distribution tables is through the F(x) value. This value is normally used as shown in the top figure. But you can also use it in its inverse format. You can find out the probability that the sample will occur to the left of the negative standard deviations away from the arithmetic mean.

The next way you can use the normal distribution curve is through the F(x) values. These values tell you the probability that leads up to the (x) value when you move from left to right. As you can see this is the inverse of the R(x) values.

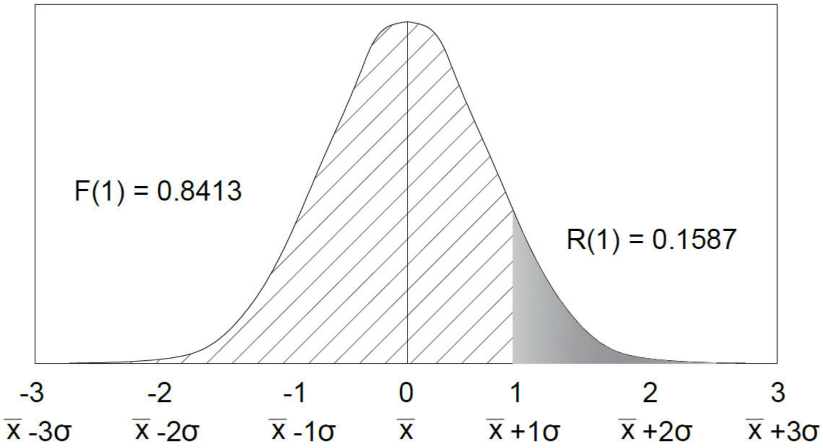


Figure 3: The second way you can look at the normal distribution tables is through the F(x) values. This is the inverse of the R(x) values, meaning that R(x) +F(x) = 1.0.

The third and fourth ways of analyzing the normal distribution is through the W(x) and 2R(x) values. The 2R(x) values is simply found by multiplying the R(x) values by 2. This value tells you the probability that the sample will occur at a value greater than the absolute value of (x) deviations away from the mean. The W(x) values show you the inverse of the 2R(x) values. This describes the probability that a sample will occur at a value less than or equal to the absolute value of (x) deviations within the mean.

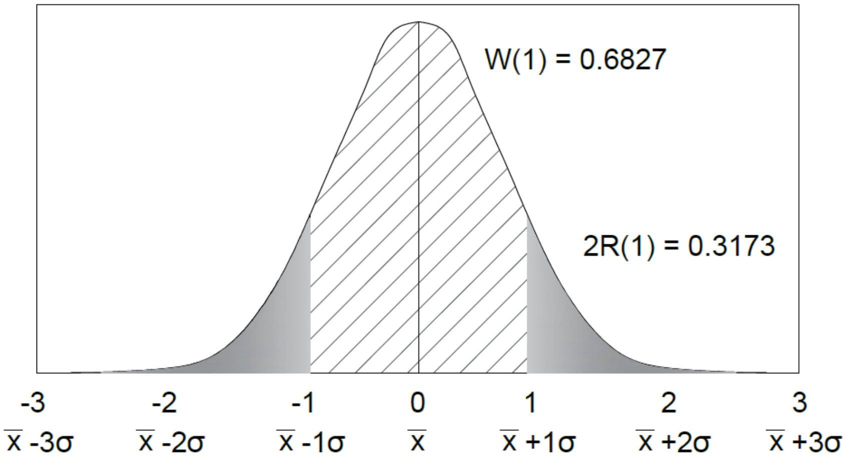


Figure 4: The third and fourth way you can look at the normal distribution tables is through the 2R(x) and W(x) values. The W(x) is the inverse of the 2R(x) values, meaning that 2R(x) +W(x) = 1.0.

3.3 T-DISTRIBUTION

A t distribution is used when there is a limited number of samples within a larger population. It is also used when the standard deviation of the population is not known. The t distribution can be used to estimate the probability distribution of the population given the sample size and the degrees of freedom.

The term degrees of freedom is used to describe the number of independent observations within the sample set. This is shown by the equation below.

$$DOF = n - 1; n = number\ of\ samples$$

In order to use the t-distribution tables, you must first find "t". The equation for t is shown below.

$$t = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$\bar{x} = sample\ mean;\ \mu = population\ mean;\ \sigma = sample\ standard\ deviation; n = \#\ of\ samples$$

As you can see from the previous equation you also need to know the population mean. An example of using the t-distribution is when you have a sample of 10 widgets, which have a sample mean of 10. The population mean was previously determined to be 11. The sample standard deviation is 1. The t distribution will give you an idea of the confidence you should have in the population mean.

$$t = \frac{10 - 11}{1/\sqrt{10}} = -3.17$$

The t distribution table shows that for 9 degrees of freedom (10 samples -1), alpha is equal to around 0.005.

$$\alpha = 0.005 = 0.5\%$$

The alpha value means that there was a 0.5% chance that the above sample situation would occur, that 10 widgets would be selected and their sample mean would be less than or equal to 10, given that the population mean is 11 and the sample standard deviation was 1.

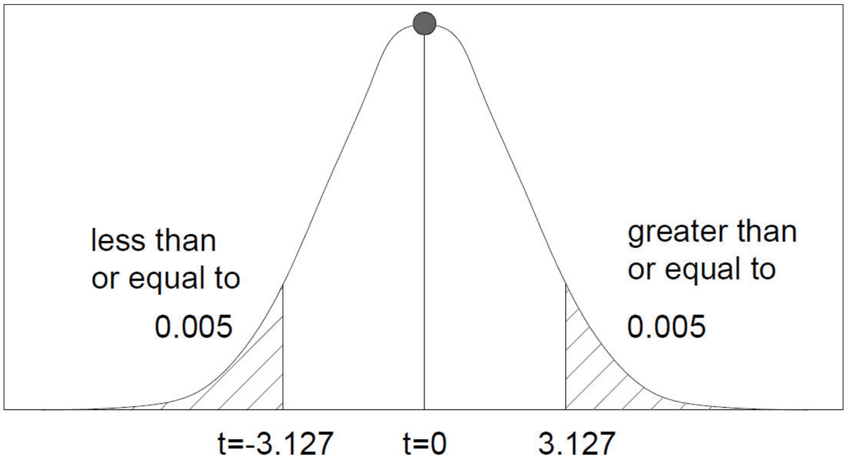


Figure 5: When you are using the t distribution tables, remember that the probabilities are cumulative and for a single side, but the curve is symmetric. So the table only shows positive

values for t, but you can use that value on the negative side as shown in the example. The table shows that for t=3.127, the probability that the t value will be greater than or equal to 3.127 is 0.005. This also means that the probability that the t-value will be less than or equal to -3.127 is 0.005.

3.4 x^2-DISTRIBUTION

An x^2 distribution is shown as the figure below. In order to answer these problems, use the same methods as the t distribution, but instead of solving for t, you will most likely be given the x^2 value. You can then use the number of samples to determine the degrees of freedom.

$$DOF = n - 1; n = number\ of\ samples$$

Then you should go to the table and find the appropriate row that corresponds to the DOF value. Then navigate to the column with the x^2 value. The alpha value at the top will correspond to the cumulative probability that the x^2value is greater than or equal to the value selected.

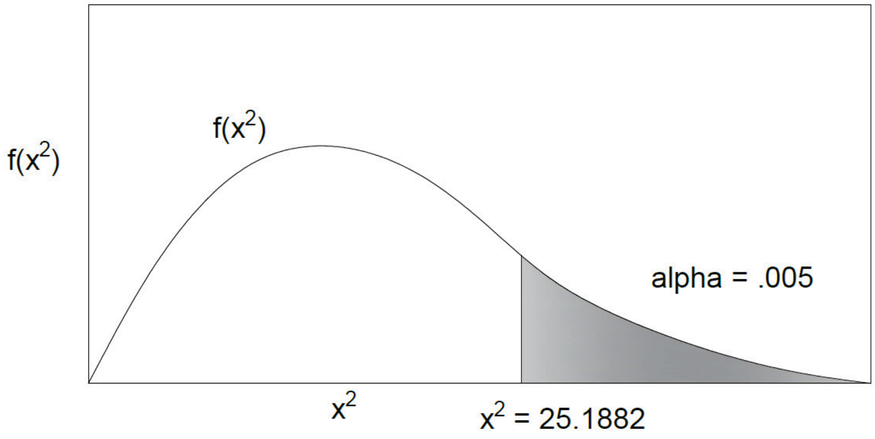


Figure 6: In this figure, 10 degrees of freedom was selected and an x^2 value of 25.1882 was found. The corresponding probability is .5%, which means that there is a 0.5% chance that the x^2 value could be greater than or equal to the one given.

4.0 EXPECTED VALUE

Expected value is calculated by taking the weighted averages of the possible outcomes. Weights are assigned to the possible outcomes based on their probability. The weighted factors allow you to see the relative importance of each independent variable. There are a few different ways that you can expect this problem on the exam, since you only have a few minutes to solve these problems.

4.1 EXPECTED VALUE BINOMIAL RANDOM VARIABLE

The first way expected value can appear on the exam is with a simple binomial random variable. As an example, an exam has 100 questions and 5 possible answers (A, B, C, D and E) for each question. If you select (C) for all 100 questions, what is your expected value? Assume the same probability for each possible answer and that each question is weighted as one point.

$$Expected\ Value = P(x) * x;$$

$$P(x) = probability\ of\ x; x = number\ of\ trials\ (questions\ in\ this\ example)$$

$$Expected\ Value = \left(\frac{1}{5}\right) * 100 = 20\ points$$

4.2 EXPECTED VALUE PROBABILITY TABLE

The second way is through the use of an expected value table. You could have five possible outcomes, but the weight of each outcome and the probability of each outcome could be different. Given the table below, you could use the following equation to solve for the expected value.

Outcome	P(x)	(x)
A	0.4	4
B	0.1	3
C	0.15	8
D	0.25	6
E	0.1	2

$$Expected\ Value = \sum P(x) * x$$

$$Expected\ Value = 0.4 * 4 + 0.1 * 3 + 0.15 * 8 + 0.25 * 6 + 0.1 * 2 = 4.8$$

A variation to the second way is to provide number of times an outcome occurs and you would have to convert that number to probability.

Outcome	#	(x)
A	40	4
B	10	3
C	15	8
D	25	6
E	10	2

$$Calculate\ P(x) \rightarrow A \rightarrow P(4) = \frac{40}{40 + 10 + 15 + 25 + 10} = 0.4$$

$$B \rightarrow P(3) = \frac{10}{40 + 10 + 15 + 25 + 10} = 0.1$$

$$C \rightarrow P(8) = \frac{15}{40 + 10 + 15 + 25 + 10} = 0.5$$

$$D \rightarrow P(6) = \frac{25}{40 + 10 + 15 + 25 + 10} = 0.25$$

$$E \rightarrow P(2) = \frac{10}{40 + 10 + 15 + 25 + 10} = 0.1$$

$$Expected\ Value = \sum P(x) * x$$

$$Expected\ Value = 0.4 * 4 + 0.1 * 3 + 0.15 * 8 + 0.25 * 6 + 0.1 * 2 = 4.8$$

4.3 EXPECTED VALUE PROBABILITY DENSITY FUNCTION

The third way is to find the expected value of a variable, given a probability density function. For example, say you are given the variable x, which can vary from greater than 0 to infinity. The probability of x varies in accordance with the following function.

$$P(x) = \frac{3}{8} * x^2; 0 \leq x \leq 2$$

The expected value equation changes from a discrete sum to an integration.

$$Expected\ Value = \int_0^2 x * \frac{3}{8} * x^2$$

$$Expected\ Value = \int_0^2 \frac{3}{8} * x^3$$

$$Expected\ Value = \frac{3}{32} * (2^4 - 0^4) = 1.5$$

The probability density function should integrate to 1, in order for it to be a probability density function. For example, if you were to integrate the probability density function from 0 to 2, you should find that it is equal to 1. This allows you to assign probabilities to the variable, such that the probability sum to 100%.

$$Integration = \int_0^2 \frac{3}{8} * x^2 = \frac{3}{24}(2^3 - 0^3) = 1$$

5.0 REGRESSION ANALYSIS

Regression analysis is the technique of creating a mathematical formula that describes the relationship between two or more measured variables. Curve fitting for the purposes of the FE exam is the same as regression analysis. In practice, curve fitting and regression analysis are done through the use of a computer software program. Therefore it is highly unlikely that you will be asked to curve fit a dataset by hand. However, you should know how to interpret that data that results from a curve fit.

In the software program, you will have the opportunity to choose the type of curve fit. You can choose a linear fit, square fit or any series of polynomials to fit the data points.

$$Linear\ Fit \rightarrow y = Ax + B$$

$$Square\ Fit \rightarrow y = Ax^2 + Bx + C$$

$$Cubed\ Fit \rightarrow y = Ax^3 + Bx^2 + Cx + D$$

$$A, B, C\ \&\ D\ are\ constants\ that\ are\ automatically\ determined\ by\ the\ software$$

The software program then will output an R^2 value. This value is called the coefficient of determination and it describes the fit of the curve as compared to the data points. Please see more in the goodness of fit topic below.

5.1 GOODNESS OF FIT

After you input data into your software program and apply a curve fit to approximate the data with a formula, you will be given an equation and an R^2 value. This value is called the coefficient of determination.

$$Coefficient\ of\ Determination \rightarrow R^2$$

The coefficient of determination will range from 0 to 1. The closer the value is to 1, the better the fit. This means that the equation used to characterize the points is a very good representation of the data points. The equations for this value will be explained later in this section.

You may also have to work with the R value, which is called the correlation coefficient.

$$Correlation\ Coefficient \rightarrow R$$

The correlation coefficient describes how the x & y values are correlated. The range of this variable is from -1 to 1. If the correlation coefficient is -1, then the variables are perfectly correlated but in the opposite manner. If the correlation coefficient is +1, then the variables are perfectly correlated in a direct manner as opposed to indirectly. If the correlation coefficient is equal to 0, then the variables are not correlated at all.

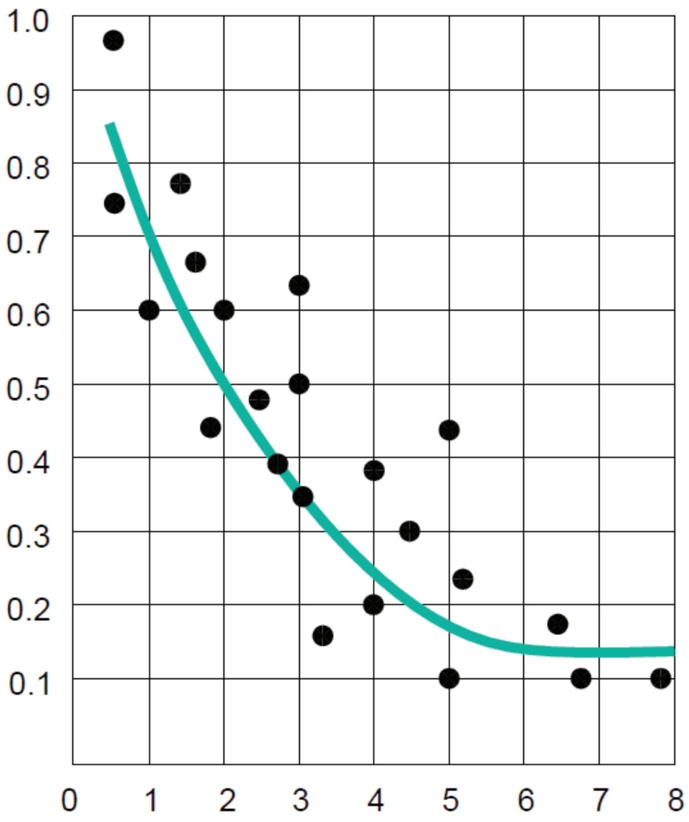


Figure 7: This figure shows data points in black and a mathematical curve fitted to that data.

For a linear fit, the coefficient of determination (R^2) value can be found through the below equation. In this equation you need the sum of squares. The sum of squares describe how far apart individual data points are from the mean. The mean is calculated on the right hand side of the equation. There are two sum of squares required for the coefficient of determination, first the sum of squares for the x-values and the sum of squares for the product of the y-values.

$$S_{xx} = \sum_{i=1}^{n} (x_i)^2 - \frac{(\sum_1^n x_i)^2}{n}$$

$$S_{yy} = \sum_{i=1}^{n} (y_i)^2 - \frac{(\sum_1^n y_i)^2}{n}$$

$$S_{xy} = \sum_{i=1}^{n} (x_i * y_i)^2 - \frac{(\sum_1^n x_i) * (\sum_1^n y_i)}{n}$$

You also should understand that for a linear fit, the equation can be found by using the above sum of squares too.

$$slope = \frac{S_{xy}}{S_{xx}}; y - intercept = y_{mean} - slope * x_{mean}$$

Finally, the coefficient of determination can be determined with the following equation.

$$R^2 = \frac{SS_{xy}}{SS_{xx} * SS_{yy}}$$

For all other fits, the FE exam will have to give you the following values to calculate the R^2 value. However, this equation is not shown in the NCEES FE Reference Handbook, so it is unlikely that the exam will test you on this topic.

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$$

$$SS_{res} = sum\ of\ residual\ squares = \sum_1^n (y_i - f_i)^2$$

$$SS_{tot} = sum\ of\ squares = \sum_1^n (y_i - \bar{y})^2$$

$$f_i = y - value\ from\ curve\ fit\ function; y_i = value\ from\ data\ point;$$

$$n = total\ number\ of\ data\ points$$

5.2 EXAMPLE LINEAR FIT – LEAST SQUARES

Since, you only have a simple calculator on the FE exam, the only fit that seems likely will be a linear fit. This example will take you through the possible calculations that you could be asked to be completed. The equations are included in the FE Handbook under the Probability & Statistics section.

Given the following points, find the linear fit equation.

x	1	2	3	4	5
y	1.5	2.5	4	7	8.2

First, calculate the average-x and average-y.

$$x_{mean} = \frac{1+2+3+4+5}{5} = 3\ ;\ \ y_{mean} = \frac{1.5+2.5+4+7+8.2}{5} = 4.64$$

Next, calculate the x and y squared values and the product of x and y for all points.

x	y	x^2	y^2	xy
1	1.5	1	2.25	1.5
2	2.5	4	6.25	5
3	4	9	16	12
4	7	16	49	28
5	8.2	25	67.24	41

Next, calculate the sum of squares x, sum of squares y and sum of x-y products. The equations are shown below.

	x	y	x^2	y^2	xy
	1	1.5	1	2.25	1.5
	2	2.5	4	6.25	5
	3	4	9	16	12
	4	7	16	49	28
	5	8.2	25	67.24	41
Sum	15	23.2	55	140.74	87.5

Calculate the sum of x squared.

$$S_{xx} = \sum_{i=1}^{n} (x_i)^2 - \frac{(\sum_1^n x_i)^2}{n}$$

This equation is much easier to use.

$$S_{xx} = (Sum\ of\ x^2) - \frac{(Sum\ of\ x)^2}{n}$$

$$S_{xx} = (55) - \frac{(15)^2}{5} = 10$$

Next, calculate the sum of y squared.

$$S_{yy} = (Sum\ of\ y^2) - \frac{(Sum\ of\ y)^2}{n}$$

$$S_{yy} = (140.74) - \frac{(23.2)^2}{5} = 33.092$$

Next, calculate the sum of the x-y products.

$$S_{xy} = (Sum\ of xy) - \frac{(Sum\ of\ x)(Sum\ of\ y)}{n}$$

$$S_{xy} = (87.5) - \frac{15 * 23.2}{5} = 17.9$$

Now, you can find the slope and y-intercept.

$$b\ (slope) = \frac{S_{xy}}{S_{xx}} = \frac{17.9}{10} = 1.79\ ;\ a\ (y - interecept) = \ y_{mean} - bx_{mean} = 4.64 - 1.79 * (3) = -.73$$

The linear fit equation is shown below.

$$y = -0.73 + 1.79x$$

Once you have the linear fit equation, then you can calculate your coefficient of determination and correlation coefficient.

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{17.9}{\sqrt{10(33.092)}} = 0.984$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}} = \frac{17.9^2}{10(33.092)} = 0.968$$

6.0 PRACTICE EXAM PROBLEMS

6.1 PRACTICE PROBLEM 1 – BINOMIAL DISTRIBUTION

A product has shown to have a satisfactory probability of 0.5. If there are 10 samples, then what is the probability that only 3 of the samples will pass? The table shown below is a cumulative binomial probability table.

		P=0.5
n=10	x=0	0.001
	1	0.0107
	2	0.0547
	3	0.1719
	4	0.377
	5	0.623
	6	0.8281
	7	0.9453
	8	0.9893
	9	0.999

(A) 0.1%

(B) 1.07%

(C) 5.47%

(D) 11.7%

6.2 PRACTICE PROBLEM 2 – STANDARD DEVIATION

A manufacturing plant produces 500 parts per day. Five parts are sampled and measured with the following tolerances: 1mm, 0.5mm, 0.4mm, 0.4mm, 0.3mm. What is the sample standard deviation of the measured tolerances?

(A) 0.14 mm

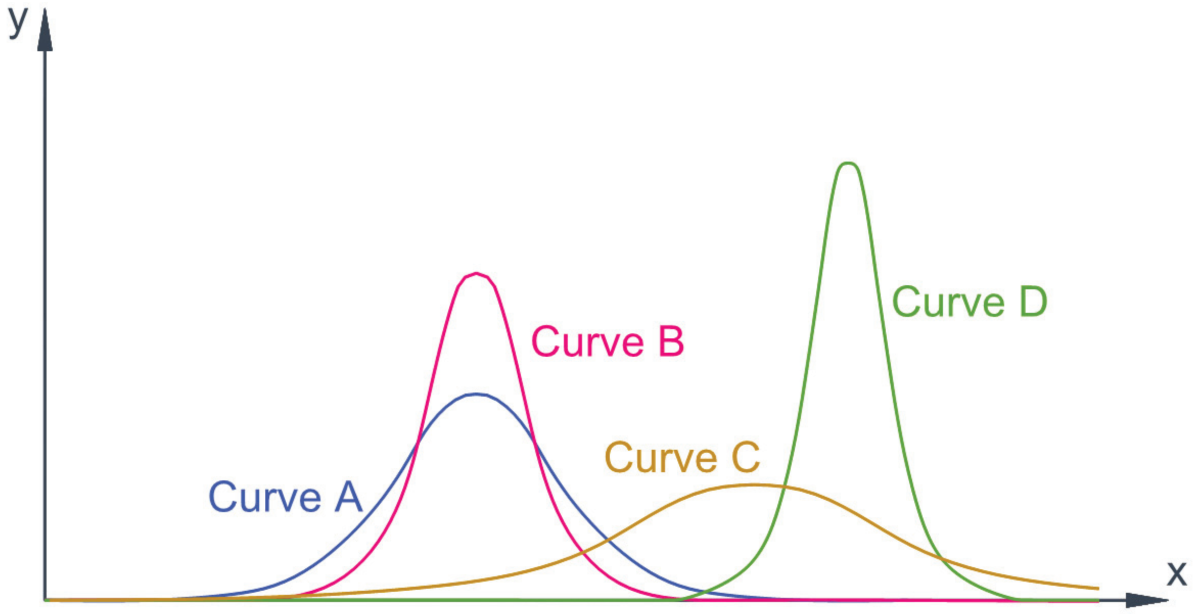
(B) 0.25 mm

(C) 0.28 mm

(D) 0.31 mm

6.3 PRACTICE PROBLEM 3 – PROBABILITY DISTRIBUTION

Answer the questions with the following options: *Curve A, Curve B, Curve C, Curve D, Unknown*



Which curve has the smallest variance? ________

Which curve has the largest mean? _______

Which curve has the largest standard deviation? ________

Which curve has the smallest mode? ________

6.4 PRACTICE PROBLEM 4 – GEOMETRIC MEAN

Temperature differentials measured across a coil over the past 6 months were recorded as: 22F, 30F, 32F, 36F, 37F, and 46F. Find the geometric mean of these measurements.

(A) 32F

(B) 33F

(C) 34F

(D) 35F

6.5 PRACTICE PROBLEM 5 – STATISTICS

The number of defective parts for the last 10 batches of a manufacturing process that produces 1000 parts per batch are:

3, 0, 0, 4, 5, 3, 8, 12, 7, 3

What is the mean, median, mode, and range, of the number of defective parts in the last 10 batches?

(A) Mean: 4, Median: 5, Mode: 3, Range: 12
(B) Mean: 4.5, Median: 4, Mode: 3.5, Range: 12
(C) Mean: 3.5, Median: 4, Mode: 3, Range: 10
(D) Mean: 4.5, Median: 3.5, Mode: 3, Range: 12

6.6 PRACTICE PROBLEM 6 – NORMAL DISTRIBUTION

Assume a normal distribution curve for this problem. The mean is 10 and the standard deviation is 2. What is the probability that the measured value will be greater than 13?

(A) 6.7%

(B) 11.1%

(C) 14.9%

(D) 22.2%

7.0 SOLUTIONS

7.1 SOLUTION 1 – BINOMIAL DISTRIBUTION

In order to complete this problem, you need to understand that the probability of 0 out of 10 successes is shown as 0.001. Then the probability of 0 and 1 out of 10 successes is shown as 0.0107. Then the probability of 0 and 1 and 2 out of 10 successes is shown as 0.0547. Then the probability of 0 and 1 and 2 and 3 out of 10 successes is shown as 0.1719. This means that in order to just get 3 successes you need to subtract 0.0547 from 0.1719.

$$only\ 3\ sucesses \rightarrow 0.1719 - 0.0547 = 0.1172$$

The correct answer is most nearly, (D) 11.7%.

7.2 SOLUTION 2 – STANDARD DEVIATION

Use the sample standard deviation equation. Note: be sure to use the equation for sample standard deviation, s, and not population standard deviation, σ.

$$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$$

First find the mean of the sample set, where the sample size, n = 5.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{(1 + 0.5 + 0.4 + 0.4 + 0.3)mm}{5} = 0.52mm$$

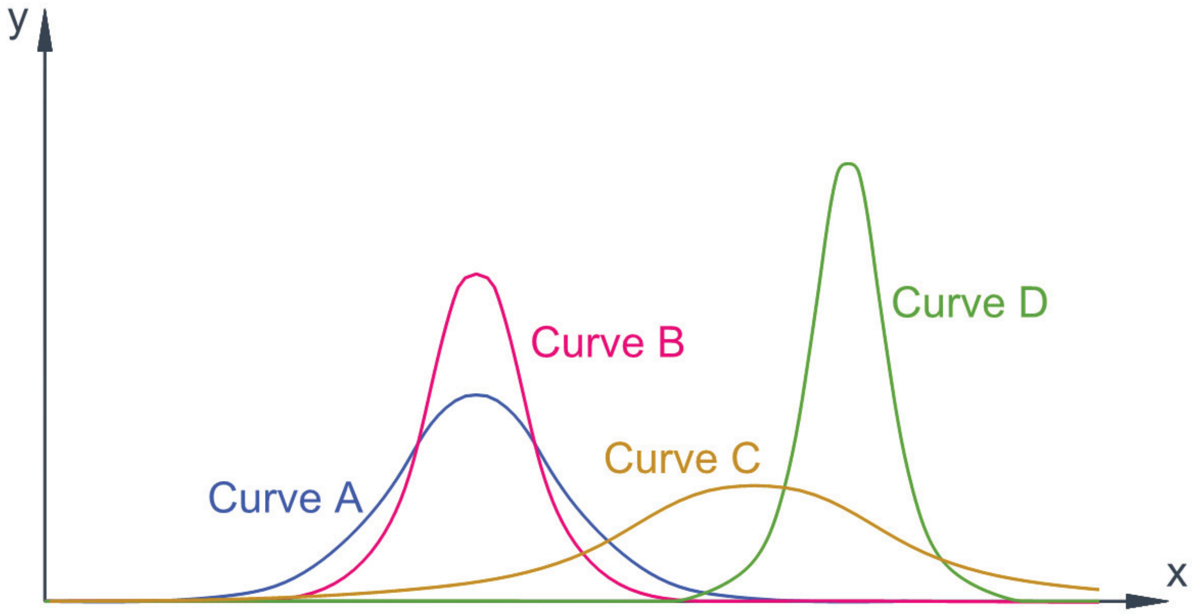
Then, solve for the sample standard deviation.

$$s = \sqrt{\frac{(1-0.52)^2 + (0.5-0.52)^2 + (0.4-0.52)^2 + (0.4-0.52)^2 + (0.3-0.52)^2}{5-1}} = 0.277mm$$

The answer is (C) 0.28 mm.

7.3 SOLUTION 3 – PROBABILITY DISTRIBUTION

Answer the questions with the following options: *Curve A, Curve B, Curve C, Curve D, Unknown*



Which curve has the smallest variance? _Curve D_

Which curve has the largest mean? _Curve D_

Which curve has the largest standard deviation? _Curve C_

Which curve has the smallest mode? _Unknown_

7.4 SOLUTION 4 – GEOMETRIC MEAN

Solve for the geometric mean with the following equation:

$$Geometric\ Mean = \sqrt[n]{X_1 X_2 \dots X_n} = \sqrt[6]{22F * 30F * 32F * 36F * 37F * 46F} = 33.0\ F$$

The answer is (B) 33F.

7.5 SOLUTION 5 - STATISTICS

The mean is calculated as:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{(3+0+0+4+5+3+8+12+7+3)}{10} = 4.5$$

Since the number of batches is even, the median is the average of the two middle numbers.

First rate the set in ascending order:

0, 0, 3, 3, 3, 4, 5, 7, 8, 12

The two middle numbers are the 5th and 6th number, which are 3 and 4. Therefore the median is:

$$Median = \frac{3+4}{2} = 3.5$$

The mode is the number that occurs the most, in this sent, the mode is 3.

The range is the difference between the highest and lowest number.

$$Range = 12 - 0 = 12$$

The answer is (D) Mean: 4.5, Median: 3.5, Mode: 3, Range: 12

7.6 SOLUTION 6 – NORMAL DISTRIBUTION

Assume a normal distribution curve for this problem. The mean is 10 and the standard deviation is 2. What is the probability that the measured value will be greater than 13?

First, convert the tested measured value to standard deviations.

$$\sigma = \frac{13-10}{2} = 1.5\,standard\ deviations$$

Next, you need to find the probability that the value will be greater than 13 so on the positive side or right side of 1.5 standard deviations.

$$P(x > 13) = 0.0668 = 6.68\%$$

The correct answer is most nearly, (A) 6.7%.

3 – Ethics & Professional Practice

MECHANICAL FE TEXTBOOK

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Section 3.0 – Ethics and Professional Practice

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1.0 INTRODUCTION

Ethics and Professional Practice accounts for approximately 4 to 6 questions on the Mechanical FE exam. The NCEES outline provides the following topics on its outline, Codes of Ethics, Agreements and Contracts, Ethical and Legal Considerations, Professional Liability, Public Health, Safety and Welfare. There is no readily available or commonly used content that covers these topics, except for what is shown on the NCEES website and the *NCEES FE Reference Handbook*. In addition, the topics cover concepts that are open for interpretation. These two facts make it very difficult to fairly test this topic.

Section 3.0 Ethics and Professional Practice (4 to 6 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
3A	Section 1.1	Codes of ethics (e.g., NCEES Model Law, professional and technical societies, ethical and legal considerations)
3B	Section 1.2	Public health, safety, and welfare
3C	Section 1.3	Intellectual property (e.g., copyright, trade secrets, patents, trademarks)
3D	Section 1.4	Societal considerations (e.g., economic, sustainability, life-cycle analysis, environmental)
	Section 2.0	Practice Exam Problems

1.1 CODES OF ETHICS

There are a couple of things you can do to prepare for questions on ethics.

(1) The first can be found in the *NCEES FE Reference Handbook*. There are about 8 pages on Ethics. This section covers the Codes of Ethics, Intellectual Property and the NCEES Model Law and Model Rules.

(2) The second is to read through the Model Law. The model law is published by NCEES and it is on the NCEES website. The link is shown below. The model law has very general regulations to govern engineering for the purposes of keeping the public safe. It is important to note that these laws are only ideas and it is up to each authority having jurisdiction like your State board to come up with their own laws.

NCEES Publications Website: https://ncees.org/about/publications/

Model Law Website: https://ncees.org/wp-content/uploads/Model_Law_2018.pdf

(3) You should also read through the Model rules. The model rules are published by NCEES and it is on the NCEES website. The link is shown below. The model rules complement the model laws. One key section in the model rules is the Rules of Professional Conduct.

Model Rules Website: https://ncees.org/wp-content/uploads/Model_Law_2018.pdf

Once you have read through these two writings, the following practice problems will help you to become familiar with the problems that may appear on the exam.

(4) This final task is optional, because the first three tasks should give you enough preparation for these 4-6 problems. The National Society of Professional Engineers or NSPE has information on a lot of legal cases and can provide insight on the topic of ethics and professional

practice. The following link has a list of Ethics Resources. You should read the NSPE Ethics Reference Guide.

NSPE Ethics Resources:

https://www.nspe.org/resources/ethics/ethics-resources/other-ethics-resources

NSPE Ethics Reference Guide:

https://www.nspe.org/sites/default/files/resources/pdfs/Ethics/CodeofEthics/NSPECodeofEthicsforEngineers.pdf

1.2 PUBLIC HEALTH, SAFETY & WELFARE

An engineer has the knowledge and duty to protect the public. The best resource for learning about an engineer's duty to public health, safety and welfare are shown on the National Society of Professional Engineer's website and specifically the codes of ethics. If an engineer is an expert in a certain topic and has facts about a situation that goes against a building code, plumbing code or any other engineering design code that puts the public's health, safety or welfare at risk, then the engineer has an ethical responsibility to report the situation to the appropriate parties.

https://www.nspe.org/resources/ethics/ethics-resources/board-ethical-review-cases/public-health-and-safety-engineer-s

An engineer must always design to the appropriate engineering standards applicable to the situation. If a client insists on designing below the minimum engineering standards, then the engineer must withdraw from the design and inform the proper authorities.

1.3 INTELLECTUAL PROPERTY

Intellectual property in engineering includes copyrights, patents, trade secrets and trademarks. For the purposes of the exam, you should understand the difference between each type of intellectual property and be able to come up with examples for each type of intellectual property. An excellent source to learn about intellectual property is on the United States Patent and Trademark Office website. https://www.uspto.gov/ip-policy

1.3.1 Copyrights

Copyrights are used to grant exclusive rights to artistic or intellectual work. Books, music, lyrics, literary works, dramatic works, movies, software and even architectural designs. Websites, literary writing, artwork and photographs can also be copyrighted. Recipes, names, titles, slogans, logos, ideas cannot be copyrighted. You can access copyright.gov for more information on what can and cannot be copyrighted. https://www.copyright.gov/

A copyright allows the owner of the copyright to maintain exclusive rights to the intellectual property and can rightfully sue others for using the intellectual property without written permission. Others may use the intellectual property, if it falls under fair use. Fair use is limited to a small portion of the intellectual property and the use of that portion must be in the context of criticism, review or parody. There is a large amount of cases that have been collected to provide better insight on what is considered fair use. The location of these cases is on the following website. https://www.copyright.gov/fair-use/fair-index.html

1.3.2 Patents

Patents are also used to grant exclusive rights to intellectual property, but the type of intellectual property is focused on designs and inventions. Patents are given to designs or inventions, including machines, tools, parts, compositions (drugs or other mixtures), plants, processes, methods of construction, etc. The patent must be for something that isn't obvious, must be brand new and must solve a problem, meaning that it has to be for something useful.

There are three main types of patents, (1) Utility, (2) Design and (3) Plant. The plant type is a new variety of plant that is successfully reproduced. A utility patent is for a discovery of a new process, machine, part or composition. A design patent is for a discovery of a new, original and ornamental design of a part.

A patent is issued by the United States Patent and Trademark Office (USPTO). A patent gives the inventor the right to exclude anyone from reproducing their invention. This allows the inventor to enforce this exclusion. For more information on patents please read the patent basics on the USPTO website. https://www.uspto.gov/patents-getting-started/general-information-concerning-patents

1.3.3 Trade Secrets

Trade secrets are the third type of intellectual property. Trade secrets are the opposite of patents. In a patent, you must explicitly disclose your invention to the USPTO and thus the entire public. This allows you to protect your intellectual property for typically 20 years, but after that length of time, the patent becomes no longer exclusive. Trade secrets do not disclose any information and keeps the invention a secret. This type of intellectual property is typically protected through the use of non-disclosure agreements or non-compete clauses. Any individual that is given a look at the trade secret must sign those documents, which allows the owner of the trade secret to sue that individual for the release of the trade secret. More information on this topic can be found on the USPTO website. https://www.uspto.gov/ip-policy/trade-secret-policy

1.3.4 Trademarks

The last type of intellectual property is trademarks. Trademarks are used to protect names, slogans, phrases, etc. Trademarks are protected similar to patents. Trademarks are registered and approved by the USPTO. Patents have an expiration date, but trademarks do not. The trademark does not expire until the owner of the trademark relinquishes it or no longer uses it in ordinary commerce. More information on this topic can be found on the USPTO website. https://www.uspto.gov/ip-policy/trademark-policy

1.4 SOCIETAL CONSIDERATIONS

Engineers must also consider the societal impacts of their design. An engineering design may have economic and/or environmental impacts. Although an engineering design may meet the appropriate engineering design standard, the engineering design may have a longer term impact that isn't covered by the standard. The engineer should always consider the way the design will perform over the lifetime of the design. This includes ongoing maintenance, utility usage cost and the sustainability of the design. A list of the societal considerations shown in the handbook are shown below.

- Safety, public health and welfare
- Quality of life
- Resource allocation
- Non-renewable resources
- Landscape aesthetics
- Protection of ecosystems
- Resource conservation
- Air and water pollution
- Atmospheric emissions
- Collection and processing of waste

2.0 PRACTICE EXAM PROBLEMS

2.1 PRACTICE PROBLEM 1 – CODES OF ETHICS

An engineer has submitted a large donation to a public official to secure a state funded condominium project. Which section of the *Model Rules* code of ethics does this violate?

(A) Licensee's Obligation to the Public
(B) Licensee's Obligation to Other Licensees
(C) Licensee's Obligation to Employers and Clients
(D) Licensee's Obligation to Engineering Etiquette

2.2 PRACTICE PROBLEM 2 – AGREEMENTS AND CONTRACTS

You are seeking to secure a contract to design a new fuel facility and distribution system for the local gas company. Based on the Model Rules, which of the following would not allow you to proceed with the project?

(A) You sit on the board of directors for the local gas company, but you have disclosed all affiliations with the gas company.
(B) You are unfamiliar with the particular intricacies required for this fuel design, but you have a licensed engineer within your firm that will oversee, stamp, and sign for the plans and documents.
(C) With the agreement of the local gas company, your firm will use the information gathered from this design to upgrade facilities in other states.
(D) In addition to the payment you will receive from the gas company, you will also receive payment from the county the for the new distribution mains that will support the city. The gas company is unaware of your contract with the county.

2.3 PRACTICE PROBLEM 3 – ETHICAL AND LEGAL CONSIDERATIONS

True or False? As an engineer you can perform engineering services for a client outside of your technical expertise as long as you truthfully inform your client.

(A) True

(B) False

2.4 PRACTICE PROBLEM 4 – PROFESSIONAL LIABILITY

Which of the following party is not liable for negligently performed engineering drawings that resulted in a claim? Assume the claim and drawing were conducted within the same year.

(A) Licensed engineer still employed by the company, who stamped and signed the drawing.

(B) Engineering firm that employs or employed the licensed engineer who stamped and signed the drawing.

(C) Licensed engineer who left the company, but who also stamped and signed the drawing.

(D) None of the above

2.5 PRACTICE PROBLEM 5 – PUBLIC, HEALTH, SAFETY & WELFARE

As a licensed engineer, you make a professional engineering recommendation on a project that is subsequently overruled. The overruling of your recommendation will cause the health, safety or welfare of the public to be endangered. Which of the following actions is best?

(A) You should notify your employer or client and the appropriate authority.

(B) You should not injure the professional reputation of another professional engineer.

(C) You should express your opinion publicly only when it is founded upon facts and proper analysis.

(D) You should not provide false information about a potential professional engineering applicant.

3.0 SOLUTIONS

3.1 SOLUTION 1 – CODES OF ETHICS

Engineers are expected to follow a code of ethics for the expertise and work they perform as a professional. One of these codes is the Model Rules. Refer to the Ethics section of the *NCEES FE Reference Manual.* Answers A, B, and C are major sections of the Model Rules, answer D is irrelevant. In reviewing the sections, paragraph 2 under the Licensee's Obligation to Other

Licensees states that a licensed engineer shall not give gifts to secure work, nor shall they give political gifts with the expectation of receiving work for a public official.

The answer is (B) Licensee's Obligation to Other Licensees.

3.2 SOLUTION 2 – AGREEMENTS AND CONTRACTS

The answer can be found under the "Licensee's Obligation to Employer and Clients" section of the Model Code. The correct answer is (D). The contracted firm is receiving payment for the same work from two separate parties, without disclosing these payments to both parties. Under paragraph 7, this is not acceptable.

(A) is incorrect. Paragraph 6 allows the continuation of the work, as long as all conflicts of interests are disclosed. Unless the board of directors have their own rules preventing this work, the contract is still allowed under the Model Code.

(B) is incorrect because the full design will be overseen by a licensed engineer.

(C) is incorrect. Under paragraph 4 and 9, information collected from a project may be used in other capacities, as long as the consent of the client is received and other rules are not broken.

3.3 SOLUTION 3 – ETHICAL AND LEGAL CONSIDERATIONS

You should not perform work that you are not qualified to conduct. The correct answer is (B) False.

3.4 SOLUTION 4 – PROFESSIONAL LIABILITY

The correct answer is (D). The engineering firm will be liable for the work of the engineers that are employed or who have been employed. The licensed engineer will still be liable for work that he or she stamped even after leaving a company, up to the statute of limitations which is typically much longer than a year. The following link has more information on this topic.

https://www.nspe.org/resources/professional-liability/liability-employed-engineers

3.5 SOLUTION 5 – PUBLIC, HEALTH, SAFETY & WELFARE

The correct answer is most nearly, (A) you should notify your employer or client and the appropriate authority. Yes, you should not injure the professional reputation of another professional engineer and you should only express you opinion publicly when you have the facts and have done a sound analysis. However, in this problem it is stated that you have already completed the analysis and have come up with a professional engineering recommendation and you are not injuring another professional engineer's reputation. You simply are making the appropriate entities aware that by overruling your professional engineering recommendation, the public will be endangered.

4 – Engineering Economics

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Section 4.0 – Engineering Economics

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1.0 INTRODUCTION

Engineering Economics accounts for approximately 4 to 6 questions on the Mechanical FE exam. As an engineer, you will be tasked with determining the course of action for a design. Often times this will entail choosing one alternative instead of several other design alternatives. You need to be able to present engineering economic analysis to their clients in order to justify why a certain alternative is more financially sound than other alternatives. The following topics will present only the engineering economic concepts that you need for the FE exam and does not present a comprehensive look into the study of engineering economics. For the FE exam you are required to know the following concepts shown in the table below. Applicable equations for these topics can be found in the Engineering Economics section of the *NCEES FE Reference Handbook*.

Section 4.0 Engineering Economics (4 to 6 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
4A	Section 2.0	Time value of money (e.g., equivalence, present worth, equivalent annual worth, future worth, rate of return, annuities)

4B	Section 3.0	Cost types and breakdowns (e.g., fixed, variable, incremental, average, sunk)
4C	Section 4.0	Economic analyses (e.g., cost-benefit, break-even, minimum cost, overhead, life cycle)
	Section 5.0	Depreciation (not on FE Mechanical outline, but included just for your information)
	Section 6.0	Practice Exam Problems

2.0 TIME VALUE OF MONEY

The concept of time value of money is that money today is worth more than money in the future, because of the interest that could be earned by having the money today.

2.1 FUTURE AND PRESENT VALUE

Having money today is worth more than having that same amount of money in the future, due to the interest that could be earned. This is the time value of money concept. For example, if you were given the option to have \$1,000 today or to have \$1,000 ten years from now, most people will choose \$1,000 today, without understand why this option is worth more. The reason \$1,000 today is worth more is because of what could have done with that money; in the financial world, this means the amount of interest that could have been earned with that money. If you took \$1,000 today and invested it at 4% per year, you would have \$1,040 dollars at the end of the first year.

$$\$1{,}000\ x\ (1+.04)=\$1{,}040$$

- If you kept the \$1,040 in the investment for another year, then you would have \$1,081.60.

$$\$1{,}040\ x\ (1+.04)=\$1{,}081.60$$

- At the end of the 10 years the investment would have earned, \$1,480.24.

$$\$1{,}000\ x\ (1+.04)x(1.04)x(1.04)\ldots\ldots=\$1{,}000\ x\ (1.04)^{10}=\$1{,}480.24$$

- An important formula to remember is the Future Value (FV) is equal to the Present Value (PV) multiplied by (1+interest rate), raised to the number of years.

$$PV\ x\ (1+i)^{10}=FV$$

- As an example, what is the present value of receiving \$1,000 ten years from now, if the interest rate is 4%?

$$PV\ x\ (1+.04)^{10}=\$1{,}000$$

$$PV\ =\$675.46$$

- Thus in the previous example, receiving $1,000, 10 years from now, is only worth $675.46 today.

It is important to understand present value because when analyzing alternatives, cash values will vary with time and the best way to make a uniform analysis is to *first convert all values to consistent terms, like present value.*

For example, if you were asked whether you would like $1,000 today or $1,500 in ten years (interest rate at 4%), then it would be a much more difficult question than the previous question. But with an understanding of present value, the "correct" answer would be to accept $1,500 ten years from now, because the $1000 today at 4% interest is only worth $1,480 ten years from now. In this example, the $1,000 today was converted to its future value 10 years from now. Once this value was converted, it was then compared to the $1,500, which was presented as future value in 10 years. Notice how all values were converted to *future value* for comparison.

2.2 ANNUAL VALUE OR ANNUITIES

The previous section described the difference between present value and future value. It also showed how a lump sum given at certain times are worth different amounts in present terms. In engineering, there are often times when annual sums are given in lieu of one time lump sums. An example would be annual energy savings due to the implementation of a more efficient system. Thus, it is important for the engineer to be able to determine the present/future value of future **annual** gains or losses.

For example, let's assume that a solar hot water project, provides an annual savings of $200. Using the equations from the previous section, each annual savings can be converted to either present or future value. Then these values can be summed up to determine the future and present value of annual savings of $200 for four years at an interest rate of 4%.

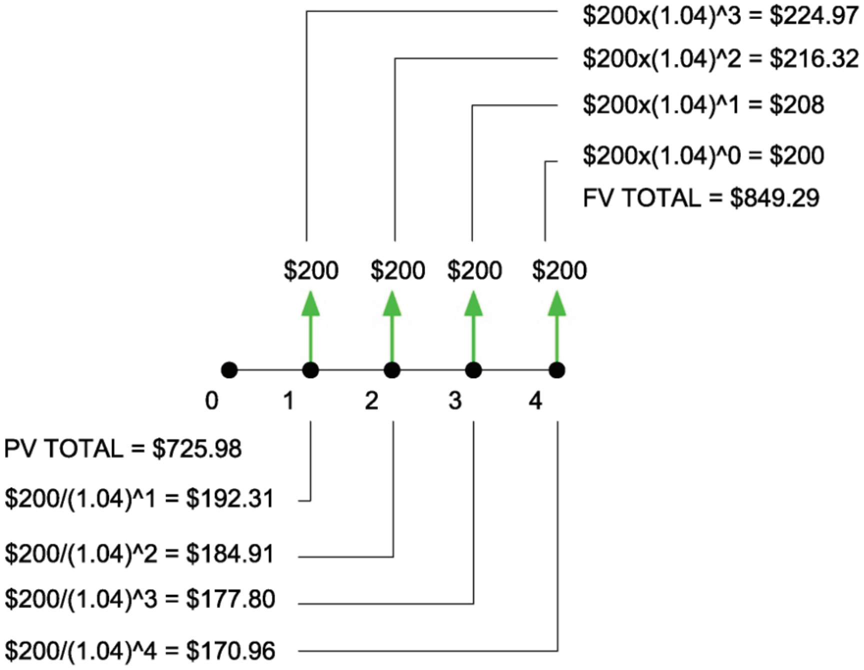
$200x(1.04)^3 = $224.97
$200x(1.04)^2 = $216.32
$200x(1.04)^1 = $208
$200x(1.04)^0 = $200
FV TOTAL = $849.29
$200
$200
$200
$200
0
1
2
3
4
PV TOTAL = $725.98
$200/(1.04)^1 = $192.31
$200/(1.04)^2 = $184.91
$200/(1.04)^3 = $177.80
$200/(1.04)^4 = $170.96

Figure 1: Annual value shown on a cash flow diagram

For longer terms, this method could become tedious. Luckily there is a formula that can be used to speed up the process in converting annuities (A) to present value and future value.

$$\mathrm{FV} = \mathrm{A} * \left(\frac{(1+\mathrm{i})^{\mathrm{n}} - 1}{\mathrm{i}}\right)$$

$$FV = 200 * \left(\frac{(1+.04)^4 - 1}{.04}\right) = \$849.29$$

$$\mathrm{PV} = \mathrm{A} * \left(\frac{1 - (1+\mathrm{i})^{-\mathrm{n}}}{\mathrm{i}}\right)$$

$$PV = 200 * \left(\frac{1 - (1+.04)^{-4}}{.04}\right) = \$725.98$$

- Reverse Equations, where annual value is solved:

$$\mathrm{A} = \mathrm{PV} * \left(\frac{\mathrm{i} * (1+\mathrm{i})^{\mathrm{n}}}{(1+\mathrm{i})^{\mathrm{n}} - 1}\right)$$

$$\mathrm{A} = \mathrm{FV}\left(\frac{\mathrm{i}}{(1+\mathrm{i})^{\mathrm{n}} - 1}\right)$$

2.3 EQUIVALENCE CALCULATIONS

Often times the engineer must convert money at various times, like annual, future or present. This section will take you through these important calculations. But first, another way of presenting future value is through the term salvage value.

Salvage value is the amount a piece of equipment will be worth at the end of its lifetime. Lifetime is typically given by a manufacturer as the average lifespan (years) of a piece of equipment. Looking at the figure below, initial cost is shown as a downward arrow at year 0. Annual gains are shown as the upward arrow and maintenance costs and other costs to run the piece of equipment are shown as downward arrows starting at year 1 and proceeding to the end of the lifetime. Finally, at the end of the lifetime there is an upward arrow indicating the salvage value.

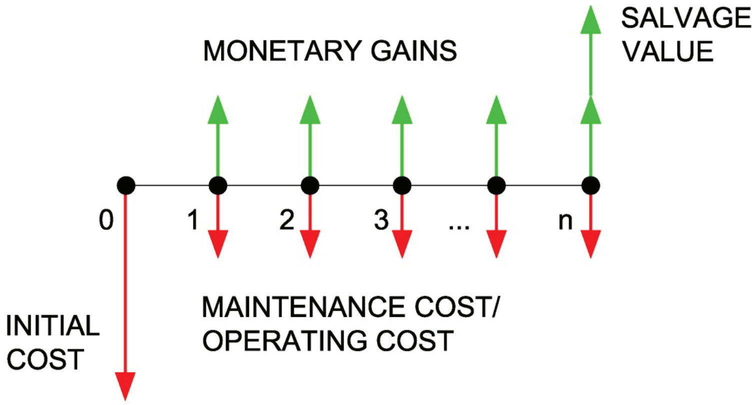

Figure 2: Adding salvage value to cash flow diagram.

As previously stated, the most important thing in engineering economic analysis is to *convert all monetary gains and costs to like terms*, whether it is present value, future value, annual value or rate of return. Each specific conversion will be discussed in the following sections.

Each of the following sections will use the same example, in order to illustrate the difference in converting between each of the different terms.

Example: A new chiller has an initial cost of $50,000 and a yearly maintenance cost of $1,000. At the end of its 15 year lifetime, the chiller will have a salvage value of $5,000. It is estimated that by installing this new chiller, there will be an energy savings of $5,000 per year. The interest rate is 4%.

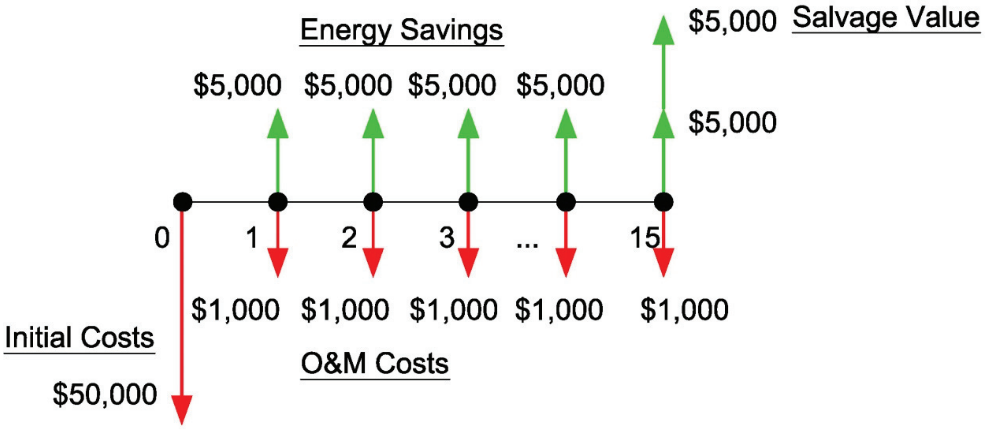
Energy Savings
$5,000
$5,000
$5,000
$5,000
$5,000
Salvage Value
$5,000
0
1
2
3
...
15
$1,000
$1,000
$1,000
$1,000
$1,000
Initial Costs
$50,000
O&M Costs

Figure 3: Sample cash flow figure for example problem

2.3.1 Finding Equivalent Present Value

What is the equivalent Present Value (Present Worth) of this chiller?

The first term, initial cost is already in present value.

$$PV_{initial\ cost} = -\$50{,}000$$

The second term, maintenance cost must be converted from an annual cost to present value. However, we can add the annual energy savings to this amount to save time.

$$A_{O\&M\ cost} + A_{energy\ savings} = -\$1{,}000 + \$5{,}000 = \$4{,}000$$

$$PV_{O\&M+energy\ savings} = \$4{,}000 * \left(\frac{1-(1+.04)^{-15}}{.04}\right) = \$44{,}473.55$$

The third term, salvage value must be converted from future value to present value.

$$PV_{salvage\ value} = \frac{\$5{,}000}{(1+.04)^{15}} = \ \$2{,}776.32$$

Finally, you should sum up all like terms.

$$PV_{total} = \ PV_{initial\ cost} + PV_{O\&M+energy\ savings} + PV_{salvage\ value}$$

$$PV_{total} = -\$50{,}000 + \$44{,}473.55 + \$2{,}776.32 = \$-2750.13$$

A negative Present Value indicates that the investment does not recoup the initial investment.

2.3.2 Finding Equivalent Future Value

What is the Future Value (Future Worth) of this chiller at the end of its lifetime?

The first term, initial cost is in present value and must be converted to future value.

$$FV_{initial\ cost} = -\$50{,}000 * (1+.04)^{15} = \$-90{,}047.18$$

The second term, maintenance cost must be converted from an annual cost to future value. However, we can add the annual energy savings to this amount to save time.

$$A_{O\&M\ cost} + A_{energy\ savings} = -\$1{,}000 + \$5{,}000 = \$4{,}000$$

$$FV_{O\&M+energy\ savings} = \$4{,}000 * \left(\frac{(1+.04)^{15}-1}{.04}\right) = \$80{,}094.35$$

The third term, salvage value is already in future value.

$$FV_{salvage\ value} = \$5{,}000$$

Finally, you should sum up all the like terms.

$$FV_{total} = FV_{salvage\ value} + FV_{initial\ cost} + FV_{O\&M+energy\ savings}$$

$$FV_{total} = \$80{,}094.35 - \$90{,}047.18 + \$5{,}000 = \$-4{,}952.83$$

2.3.3 Finding Equivalent Annualized Value

What is the Annual Value of this chiller?

The first term, initial cost is in present value and must be converted to annual value.

$$AV_{initial\ cost} = -\$50{,}000 * \left(\frac{.04 * (1+.04)^{15}}{(1+.04)^{15}-1}\right) = \$-4{,}497.06$$

The second term, maintenance cost is already annualized. However, we can add the annual energy savings to this amount to save time.

$$AV_{O\&M\ cost} + AV_{energy\ savings} = -\$1{,}000 + \$5{,}000 = \$4{,}000$$

The third term, salvage value is in future value and must be annualized.

$$AV_{salvage\ value} = \$5{,}000\left(\frac{.04}{(1+.04)^{15}-1}\right) = \$249.71$$

Finally, you should sum up all the like terms.

$$AV_{total} = AV_{salvage\ value} + AV_{initial\ cost} + AV_{O\&M+energy\ savings}$$

$$AV_{total} = \$249.71 - \$4{,}497.06 + \$4{,}000 = \$ - 247.35$$

2.3.4 Finding Equivalent Rate of Return

What is the rate of return on the investment of $50,000 for the new chiller?

The rate of return is a tool used by engineers to describe how profitable or un-profitable an investment is over the equipment's lifetime. The calculation involves determining the equivalent interest rate for a monetary investment and a monetary gain or loss.

In the previous example, $50,000 is invested in a new chiller and the returns on this chiller are $4,000 a year ($5,000 energy savings minus $1,000 O&M) and a salvage value of $5,000 at the end of the 15 years. For the calculation of rate of return (ROR) or return on investment (ROI), **the salvage value is assumed to be $0 only to simplify the problem**.

The ROR is calculated as the "i" value that is required in the below equation to make both sides equal. This approach takes trial and error, unless you have a computer or financial calculator.

$$\$4{,}000 * \left(\frac{1-(1+i)^{-15}}{i}\right) = \$50{,}000$$

First try, i= .04 (4%).

$$\$44{,}473.55 < \$50{,}000$$

Second try, i= .03 (3%).

$$\$47{,}751.74 < \$50{,}000$$

Third try, i= .025 (2.5%).

$$\$49{,}525 < \$50{,}000$$

Fourth try, i= .023 (2.3%).

$$\$50{,}262 > \$50{,}000$$

Correct answer is approximately, 2.4% ROR. Since, the ROR is less than the interest rate of 4%, this investment is not wise.

2.3.5 Factor Tables

When conducting engineering economic analyses, factor values are used in lieu of formulas. Factor values are pre-calculated values that correspond to

(1) A specific equation (convert present value to annual, convert present value to future, etc.)

(2) An interest rate.

(3) Number of years.

Looking up these values in a table is sometimes quicker than using the equations and lessens the possibility of calculator error. It is recommended that the *engineer use the tables in the NCEES FE Reference Handbook*. A summary of the factory values are shown below.

Conversion	Formula	Factor Value
Present Value to Future Value	$FV = PV\ x\ (1+i)^n$	Multiply PV by (F/P, i, n)
Future Value to Present Value	$PV = \frac{FV}{(1+i)^n}$	Multiply FV by (P/F, i, n)
Present Value to Annual Value	$A = PV * (\frac{i*(1+i)^n}{(1+i)^n - 1})$	Multiply PV by (A/P, i, n)
Annual Value to Present Value	$PV = A * (\frac{1-(1+i)^{-n}}{i})$	Multiply A by (P/A, i, n)
Future Value to Annual Value	$A = FV(\frac{i}{(1+i)^n - 1})$	Multiply FV by (A/F, i, n)
Annual Value to Future Value	$FV = A * (\frac{(1+i)^n - 1}{i})$	Multiply A by (F/A, i, n)

3.0 COST TYPES & BREAKDOWNS

The cost types used in engineering economics include fixed, variable, incremental, average and sunk costs. These cost types are best described by looking at the manufacturing process. In a manufacturing process, there will be fixed costs that do not change whether or not you manufacture 0 units or product or 1,000 units. These costs include rent, overhead, insurance, etc. Variable costs and incremental costs are similar. Variable costs depend on the production of units, so for example if you manufacture 500 units, there will be an associated cost for that unit like labor, utilities, materials, etc. The incremental cost is the cost of producing one additional unit. The incremental cost can change based on which additional unit is produced. For example, to produce 2 units instead of 1, will typically require a large incremental cost. But producing 1,001 units instead of 1,000 units would most likely require a much smaller incremental cost. Another cost that can be used for analysis is the average cost, which is found by taking the total cost and divided it by the number of units produced. Lastly, sunk costs are a type of fixed cost. These costs cannot be recouped. A sunk cost does not change with volume of units produced, so it is a fixed cost. But not all fixed costs are sunk, because some fixed costs can be salvaged and some fixed costs can be canceled, like insurance or rent.

3.1 FIXED COSTS

Fixed costs do not change with changes in production. This does not mean that fixed costs do not change, it just means that they do not change in a set period of time. For example, rent is a fixed cost, but that cost may have to be re-evaluated at the end of the rental period. Other examples of fixed costs include insurance, depreciation of purchased equipment, interest costs on borrowed money, property taxes, fixed salaries (not hourly) and any utility costs that are fixed.

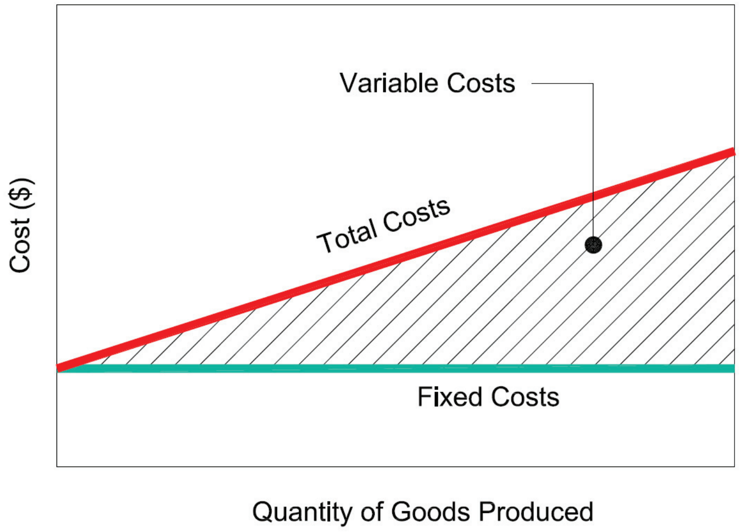

Figure 4: Fixed costs do not change with the quantity of goods produced. Variable costs will change as the quantity of goods changes.

3.2 VARIABLE COSTS

Variable costs change with changes in production. These costs are directly related to the manufacturing of the product. This includes material costs, labor costs, utility costs to run equipment, etc.

3.3 AVERAGE COSTS

Average costs is the total cost to produce one unit. It is the total cost, which is equal to fixed plus variable costs, divided by the total quantity of units produced.

$$Average\ Cost\ \left(\frac{\$}{Unit}\right) = \frac{Fixed\ Cost + Variable\ Cost}{Total\ Quantity\ of\ Units}$$

3.4 INCREMENTAL COSTS

Incremental costs are the changes in expenses due to a variation in productions or services. It is associated with costs that are variable, and are used to make decisions on whether the variation is worthwhile. For example, if it costs $10,000 to produce 200 units and it takes $12,000 to produce 300 units, then the incremental cost is $2,000. The $2000 could include costs such as additional material costs, electricity expenses to operate machines, the pay for manufacturing and sales personnel to work additional hours, and advertising costs. It does not include costs that are fixed or do not change with the decision, such as building rent, the initial investment of a production machine, or the cost of personnel healthcare. The business must decide if it is worth the incremental cost to produce additional units. Marginal costs are similar, except it is the variable cost to produce one additional unit.

$$Incremental\ Cost\ (\$) = New\ Cost\ (\$) - Original\ Cost\ (\$)$$

$$Marginal\ Cost\ (\$\ per\ unit) = \frac{[New\ Cost\ (\$) - Original\ Cost\ (\$)]}{Additional\ \#\ of\ units}$$

3.5 SUNK COSTS

Sunk costs are fixed costs that have already been generated and cannot be recovered, such as the purchase of a machine or the payment of a lease. These costs are not included in business decision making processes, since it is already lost and has no effect on the future.

$$Sunk\ Costs\ (\$) = Unrecoverable\ Expenses$$

Sunk costs can be compared with incremental costs for a business decision making process with the following example.

Example: A facility is determining whether to replace a pump with a more efficient one. The cost for the replacement is the incremental cost, while the cost already incurred for the old pump is a sunk cost and is not used to determine whether the pump should be replaced.

3.6 ESTIMATING COSTS

Cost estimations in mechanical engineering are used to determine the material and installation costs of equipment or systems. Usually this will require the sizing of the equipment based on loads and are used to budget the project. The estimate can then be used in economic analysis to determine the optimal route or to decide whether an investment is worthwhile.

Refer to the "Cost Estimation" section of the *NCEES FE Reference Handbook* for equations and tables related to cost indexes, capital cost estimation, and scaling of equipment costs. Note that this information is provided under the Chemical Engineering section, but some items are applicable.

3.6.1 Cost Indexes

Cost indexes can be used to convert outdated costs given for a specific year to present day values. A cost index is established and reported by various sources, so the index values should be given to you in the problem. The index is a value representing the average cost of a set combination of goods or services, taken over a set period of time, and calculated for a specific category, such as construction, housing, or food. A cost index can also vary based on an area. Typically a base year is chosen with an index value of 100, then each subsequent or previous year varies with the base. Using this yearly cost index, a ratio can be taken to convert a past cost to a present year value, essentially simplifying the calculations for inflation and deflation for a given category at a given location.

$$Cost_{Present\ Year}\ (\$) = Cost_{Past\ Year,N} * \frac{Cost\ Index_{Present\ Year}}{Cost\ Index_{Past\ Year,N}}$$

Example: A building in the US is constructed for $1.2 Million in 1978, what is the cost of the building in 2018? The cost index in 1978 and 2018 is 78 and 240, respectively.

Solution: Use the cost index to solve for 2018 values.

$$Cost_{2018}\ (\$) = \$1.2M * \frac{240}{78} = \$3.7M$$

The construction cost for the same building is $3.7 million in 2018.

3.6.2 Capital Cost Estimation

Capital costs for mechanical engineering are estimated by the sum of the materials, installation labor, installation equipment costs, and adding a percentage of overhead, taxes, and other indirect fees.

$$Captial\ Cost\ (\$) = \left(100\% + Overhead(\%) + Taxes(\%) + Other\ Indrect\ Cost\ (\%)\right) * \sum_{i}^{n} Quantity_i * [Material_{Unit\ Cost,i} + Labor_{Unit\ Cost,i} + Equipment_{Unit\ Cost,i}]$$

The Capital Cost Estimation tables in the *NCEES FE Reference Handbook* are meant for chemical process facilities, such as coal or oil, and will likely not be tested. The tables use an estimation factor method, specifically the Lang factors, which are based on sampled data. In these types of problems, the major equipment costs for the facility is given. It is then multiplied by the appropriate factor, between 4 and 6, to find the facility's total installed cost. The second table lists specific components of the facility and also uses factors. The values listed under the range are given as percentages. Use the total major equipment cost, then multiply by the percentage factor to get the estimated cost for each component.

3.6.3 Equipment Scaling Cost

Finally, the table listed under the "Scaling of Equipment Cost" in the *NCEES FE Reference Handbook* can be used to estimate equipment of different sizes using an exponential factor "n" that is based on an equipment type.

$$Equipment\ Cost_{Size\ A} = Equipment\ Cost_{Size\ B} * \left(\frac{Capacity\ of\ A}{Capacity\ of\ B}\right)^n$$

Example: A 1,000 cfm centrifugal exhaust fan costs \$700. What is the estimated cost of a similar fan with 8,000 cfm capacity.

Solution: The table indicates that the exponential factor for a centrifugal fan within the desired capacity range is n=0.44. Solve for the cost of the larger capacity fan with the following equation.

$$Fan\ Cost_{8,000\ CFM} = \$700 * \left(\frac{8,000\ cfm}{1,000\ cfm}\right)^{0.44} = \$1,748$$

The material cost of the 8,000 CFM fan is approximately \$1,748.

4.0 ECONOMIC ANALYSES

Engineers use economic analysis to compare various alternatives or to analyze an alternative on its own. The analysis may or may not use the time value of money. On the FE exam, if you see an interest rate in the problem, then you most likely will have to conduct the analysis with the time value of money and the factor tables presented in the Engineering Economics section of the *NCEES FE Reference Handbook*.

4.1 COST/BENEFIT ANALYSIS

A cost-benefit analysis is used to determine whether or not to complete an engineering project. The costs are calculated and the benefits are calculated. If the benefits outweigh the costs, then the project is viable. Other projects can also be included in the analysis to see which one has the most favorable cost-benefit analysis.

$$Benefits - Costs > 0 \rightarrow Project\ viable$$

4.2 BREAK-EVEN ANALYSIS

Break even analysis is the point at which revenues equals cost. This is most commonly used in manufacturing. The most common application is finding how many products must be produced until the facility reaches the breakeven point, or until the facility will finally start seeing profits. A break even analysis in manufacturing does not use the time value of money.

As an example, assume the fixed costs of a manufacturing facility is $100,000. The average cost per unit of product is equal to $20 per unit. Each product is sold at $100. What is the break-even point?

$$Total\ revenue = Total\ costs$$

$$\$100\ x\ (\#\ of\ units) = \$100{,}000 + \$20\ x\ (\#\ of\ units)$$

$$\$80\ (\#\ of\ units) = \$100{,}000$$

$$\#\ of\ units\ to\ break\ even = 1{,}250$$

This means that in order for the facility to break even, it must produce and sell 1,250 units. It can also start to see a profit if it produces and sells more than 1,250 units and a loss if it produces and sells less than that amount.

4.3 MINIMUM COST ANALYSIS

The minimum cost analysis is a form of life cycle cost analysis used to compare various alternatives. You will calculate the total owning and operating cost of each alternative and then choose the alternative that has the lowest cost. This type of analysis will use the time value of money.

For example, consider alternative 1, which has an initial cost of $100,000 and an annual owning and operating cost of $15,000 per year over a 10 year life. Alternative 2 has an initial cost of $50,000 and an annual owning and operating cost of $30,000 per year over a 5 year life. The interest rate is 6% and the analysis period is 10 years. Which alternative has the minimum cost?

The first step is to convert the annual value to present value.

$$Alternative\ 1 \rightarrow \$15{,}000\ x\ \left(\frac{P}{A}, n = 10, i = 6\%\right) \rightarrow \$15{,}000 * 7.3601 = \$110{,}401.50$$

Next, add the previous value with the initial cost. This results in the total cost of alternative 1, in present value.

$$Alternative\ 1\ Cost\ (Present\ Value) \rightarrow \$100{,}000 + \$110{,}401.50 = \$210{,}401.50$$

Next, repeat the steps for Alternative 2.

$$Alternative\ 2 \rightarrow \$30{,}000\ x\ \left(\frac{P}{A}, n = 5, i = 6\%\right) \rightarrow \$30{,}000 * 4.2124 = \$127{,}237$$

Next, add the previous value with the initial cost. This results in the total cost of alternative 2, in present value for the first 5 years.

$$Alternative\ 2\ Cost_{1-5\ yrs}\ (Present\ Value - Year\ 0) \rightarrow \$50{,}000 + \$127{,}237 = \$177{,}237$$

Alternative 2 has a life of 5 years, so alternative 2 must be completed again for years 6-10. This results in a present value at year 5 that is the same as Year 0. This value must be converted back to year 0.

$$Alternative\ 2\ Cost_{6-10\ yrs}\ (Present\ Value - Year\ 5) \rightarrow \$50{,}000 + \$127{,}237 = \$177{,}237$$

$$Alternative\ 2\ Cost_{6-10\ yrs}\ (Present\ Value - Year\ 0) \rightarrow \$183{,}554 * \left(\frac{P}{F}, n = 5; i = 6\%\right)$$

$$Alternative\ 2\ Cost_{6-10\ yrs} \rightarrow \$177{,}237 * 0.7473 = \$132{,}449$$

The total cost for alternative 2 is the sum of the costs for years 1-5 and years 6-10.

$$Alternative\ 2\ Cost = \$177{,}237 + \$132{,}449 = \$309{,}687$$

You can also complete Alternative 2 in this manner.

$$\$50{,}000 + \$50{,}000\left(\frac{P}{F}, n = 5; i = 6\%\right) + \$30{,}000\ x\ \left(\frac{P}{A}, n = 10, i = 6\%\right)$$

$$\$50{,}000 + \$50{,}000 * (0.7473) + \$30{,}000\ x\ (7.3601) = \$308{,}168$$

The alternative with the minimum cost is Alternative 1. On the FE exam, the costs will be much simpler and the analysis periods will be the same, because it would take too long to complete this entire analysis in the exam. The factors are located in the *NCEES FE Reference Handbook* in the Engineering Economics section.

4.4 OVERHEAD ANALYSIS

Overhead analysis sums up the costs due to overhead. The counterpart to overhead costs are operating expenses. Overhead costs are not directly related to production, while operating expenses are directly related to production. Production or operating costs are those costs that are directly accounting for sales.

For example, in a manufacturing type business, the cost of materials and the labor used to produce a product would fall under the operating costs. The electricity and other utilities that are used to produce the product would also be operating costs. The depreciation of the equipment, the insurance, the rent, the accounting costs, administrative costs, marketing costs would be classified as overhead.

In a service type business, like an engineering consulting business, the hours spent by engineers to produce plans and documents would be direct operating costs. The printing costs used to produce the plans and documents would also be direct operating costs. The marketing, accounting, administrative, rent, insurance, etc. would be overhead costs.

4.5 LIFE CYCLE ANALYSIS

Life cycle analysis uses the time value of money to calculate the total owning and operating costs of an engineering project. It takes into account the initial cost, annual costs like maintenance and utility costs and the salvage value of the project. The life of the project is also taken into account, along with the current or projected interest rate for the life of the project. There are various versions of life cycle analyses. The list is shown below.

The first two methods were mentioned in the minimum cost analysis section. A project's initial costs, annual costs and salvage costs are all converted to either (1) Annual value or (2) Present value. The project alternative with the lowest costs should be selected.

1. Comparing Alternatives with Annual Owning and Operating Costs
2. Comparing Alternatives with Total Owning and Operating Costs

The next methods are used to compare a project to a baseline. The project is an improvement on the baseline that will incur savings each year, but there will be an initial cost. The annual savings are converted to initial cost with either (1) a given n = years and the rate of return (interest rate is calculated or (2) a given interest rate and a number of years value is calculated.

3. Comparing Alternatives with Rate of Return
4. Comparing Alternatives with Pay Back Period

The calculation of the rate of return was shown in the Finding Equivalent Rate of Return section. The calculation of the payback period is shown in the example below.

As an example, assume an alternative has a savings of \$200 per year and the interest rate is 12%. If the initial cost is \$1,000, then what is the payback period?

$$\$1{,}000 = \$200 * \left(\frac{P}{A}, n = ?, i = 12\%\right)$$

$$5 = \left(\frac{P}{A}, n = ?, i = 12\%\right)$$

Next, go to the 12% table in the Engineering Economics section in the *NCEES FE Reference Handbook*. Look at the column for P/A and find the number of years that most closely matches a factor of 5. That value is approximately, n = 8 years.

Interest Rate Tables – Factor Table
***i* = 12.00%**

n	**P/F**	**P/A**	**P/G**	…
…	…	…	…	…
7	0.4523	4.5638	11.6443	…
8	0.4039	4.9676	14.4714	…
9	0.3606	5.3282	17.3563	…
10	0.3220	5.6502	20.2541	…
…	…	…	…	…

4.6 SIMPLE PAYBACK

Calculating simple payback is similar to a life cycle analysis. Simple payback is calculated by taking the initial cost and divided it by the yearly savings that will be achieved through the project or the installation of a new piece of equipment. Simple payback is the number of year that will be required to recoup the initial cost. It is called simple because it does not take into account the time value of money.

Example: A new chiller has an initial cost of \$50,000 and a yearly maintenance cost of \$1,000. At the end of its 15 year lifetime, the chiller will have a salvage value of \$5,000. It is estimated that by installing this new chiller, there will be an energy savings of \$5,000 per year. The interest rate is 6%. What is the simple payback of the new chiller?

$$Simple\ Payback = \frac{Initial\ Cost}{Profit\ (Income - Expenses)} = \frac{\$50{,}000}{\$5{,}000 - \$1{,}000} = \frac{\$50{,}000}{\$4{,}000} = 12.5\ years$$

5.0 DEPRECIATION

Depreciation is the value that an asset decreases over time. For example, as a building or an equipment gets older, it starts to gradually deteriorate and reduce in useful life over time. Depreciation values can be represented as either a straight line or accelerated form.

5.1 STRAIGHT LINE

Straight line depreciation distributes the depreciation values evenly over the life of the asset. This is the simplest method for calculating depreciation and is represented by the following equation.

$$Straight\ Line\ Depreciation\ (\$/Year) = \frac{Capital\ Cost\ (\$) - Salvage\ Value\ (\$)}{Useful\ Life\ (Years)}$$

For example, a machine is purchased at \$100,000 and has a salvage value of \$10,000. If the machine has a useful life of 10 years, then the straight line depreciation value is:

$$Straight\ Line\ Depreciation = \frac{\$100{,}000 - \$10{,}000}{10\ Years} = \$9{,}000/Year$$

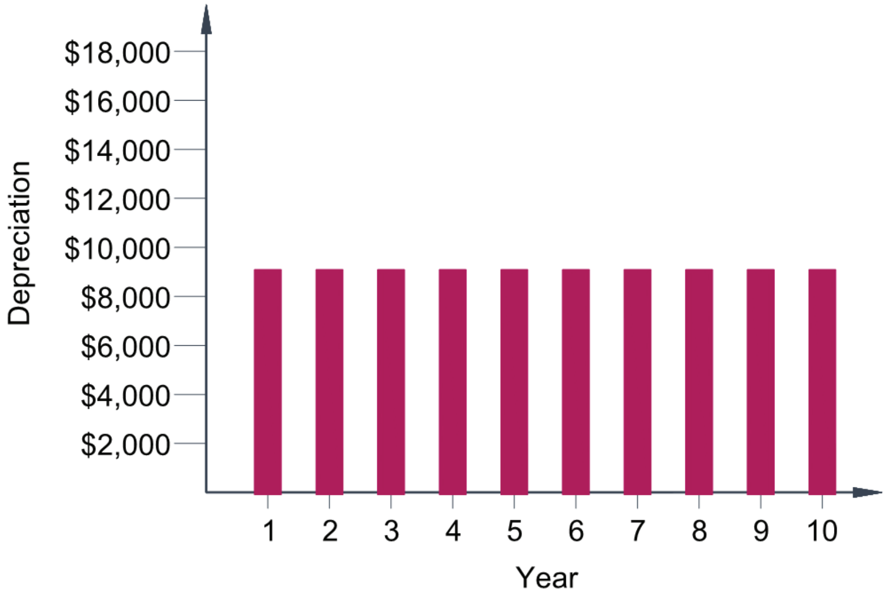

Figure 5: Example of Straight Line Depreciation for an asset with ten years of usable life

5.2 MODIFIED ACCELERATED COST RECOVERY SYSTEM (MACRS)

The modified accelerated cost recovery depreciation system distributes the depreciation to be heavily weighted in the earlier years of the asset's usable life and less weighted in the later years. In other words, it accelerates the depreciation to earlier in the lifetime of the asset. This system is used for taxes in the United States. It allows for the company to take larger depreciation credits in the earlier years, thereby deferring taxes to later in the asset's lifetime.

There are two main differences between this depreciation method and the straight line method. First, the depreciation occurs over n+1 years, where "n" is the lifetime of the asset. In addition, there is no salvage value for MACRS depreciation. At the end of the "n+1" years, the asset will have a salvage value of \$0.

$$MACRS\ Depreciation_{year,j}\ (\$\ at\ Year\ j) = Recovery\ Rate\ (\%)_{year,j} * Capital\ Cost\ (\$)$$

In practice, the MACRS tables are published by the IRS. For the exam, use the table in the *NCEES Mechanical PE Reference Handbook* for the recovery rate based on the recovery period. The same equipment used in the straight line example above (\$100,000 initial cost, 10 year lifespan) will have a recovery rate at year 2 of 18%, and a recovery rate at year 8 at 6.55%.

$$MACRS\ Depreciation_{year\ 2} = 18\% * \$100{,}000 = \$18{,}000$$

$$MACRS\ Depreciation_{year\ 8} = 6.55\% * \$100{,}000 = \$6{,}550$$

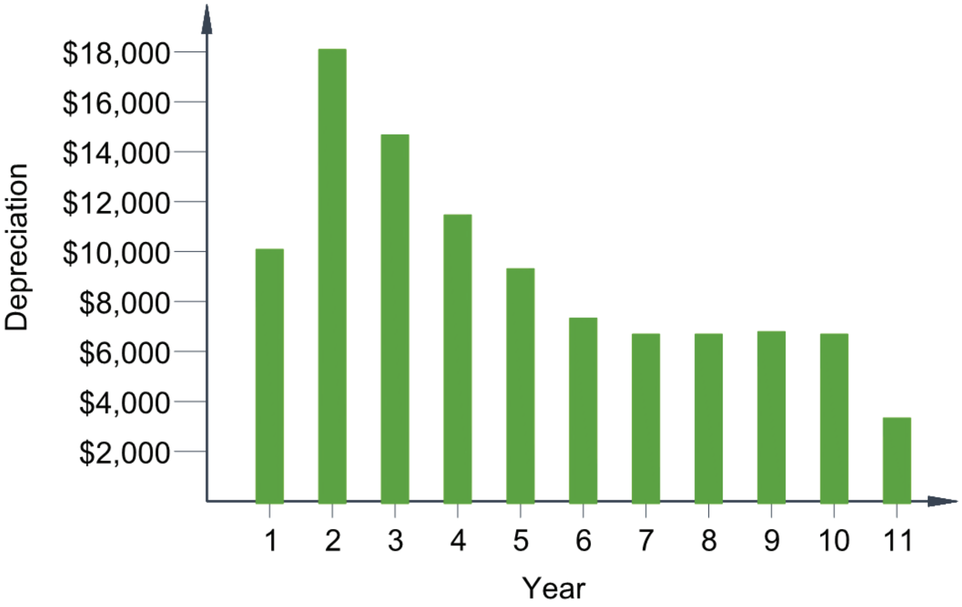

Figure 6: Example of MACRS Depreciation for an asset with ten years of usable life

5.3 SUM OF YEARS DIGITS (SYD)

The sum of the years digits method also uses accelerated depreciation, applying more depreciation in the earlier part of an equipment's life. The difference between MACRS is the amount of depreciation that is distributed throughout the years. SYD is also different to MACRS, but similar to straight line depreciation, in that it incorporates a salvage value and depreciates the equipment over its lifetime, instead of the n+1 years that MACRS uses. SYD is an older method that was used in taxes before MACRS. In present day, it is used for accounting, and is not allowed for taxes.

$$SYD\ Depreciation_{year\ j}(\$\ at\ Year\ j) = \frac{2 * [Capital\ Cost\ (\$) - Salvage\ Value\ (\$)] * (n - j + 1)}{n * (n + 1)}$$

$$n = Useful\ Life\ (years);\ \ j = year\ of\ depreciation$$

Using the same example, a machine is purchased at $100,000 and has a salvage value of $10,000. If the machine has a useful life of 10 years, then the sum of the years depreciation value at year 2 and year 8 is found below.

$$SYD\ Depreciation_{year\ 2} = \frac{2 * (\$100{,}000 - \$10{,}000) * (10 - 2 + 1)}{10 * (10 + 1)} = \$14{,}727$$

$$SYD\ Depreciation_{year\ 8} = \frac{2 * (\$100{,}000 - \$10{,}000) * (10 - 8 + 1)}{10 * (10 + 1)} = \$4{,}909$$

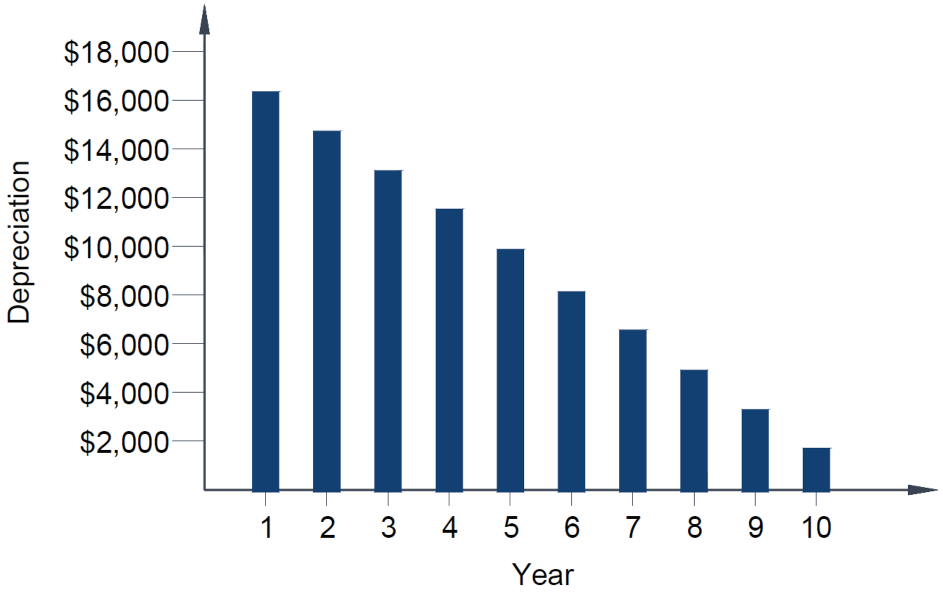

Figure 7: Example of SYD Depreciation for an asset with ten years of usable life.

5.4 DEPRECIATION COMPARISON

The following table compares the straight line, MACRS, and SYD depreciation methods for an equipment with initial cost of $100,000, lifespan of 10 years, and $0 salvage value.

Year (n)	Straight Line		MACRS		SYD	
1	10%	$10,000	10%	$10,000	18%	$18,182
2	10%	$10,000	18%	$18,000	16%	$16,364
3	10%	$10,000	14%	$14,400	15%	$14,545
4	10%	$10,000	12%	$11,520	13%	$12,727
5	10%	$10,000	9%	$9,220	11%	$10,909
6	10%	$10,000	7%	$7,370	9%	$9,091
7	10%	$10,000	7%	$6,550	7%	$7,273
8	10%	$10,000	7%	$6,550	5%	$5,455
9	10%	$10,000	7%	$6,560	4%	$3,636
10	10%	$10,000	7%	$6,550	2%	$1,818
11			3%	$3,280		

8: Example of MACRS Depreciation for an asset with ten years of usable life

6.0 PRACTICE EXAM PROBLEMS

6.1 PRACTICE PROBLEM 1 – ECONOMIC ANALYSIS

Background: A client is contemplating on purchasing a new high efficiency pump and motor, with an initial cost of $10,000. The pump has a lifetime of 15 years and is estimated to save approximately $1,000 per year. There is an additional maintenance cost of $300 per year associated with this new pump. The pump will have a salvage value of $0 at the end of its lifetime. Assume the interest rate is 4%.

Problem: What is the annual value of the pump?

(a) -$499

(b) -$199

(c) $199

(d) $499

6.2 PRACTICE PROBLEM 2 – ECONOMIC ANALYSIS

Background: A client is contemplating between two separate chillers. Chiller 1 has a life of 25 years, an initial cost of $50,000, an ongoing maintenance/electricity cost totaling $1,000 per year. Chiller 2 has a life of 25 years, an initial cost of $35,000 and an ongoing maintenance/electricity cost totaling $1,500 per year. Assume interest rate is equal to 4%.

Problem: What is the present value of the two turbines?

(a) Chiller 1 = -$65,622 ; Chiller 2 = -$58,433

(b) Chiller 1 = -$91,646 ; Chiller 2 = -$103,455

(c) Chiller 1 = $23,646 ; Chiller 2 = $97,469

(d) Chiller 1 = $12,646 ; Chiller 2 = $103,455

6.3 PRACTICE PROBLEM 3 – ECONOMIC ANALYSIS

Background: An existing A/C control system is inefficient and you are researching whether or not to replace system. You develop a new system that will cost $30,000 and require an ongoing maintenance of $1,000 per year, but it will save $4,000 per year in energy savings. The new A/C control system will have a lifetime of 30 years.

Problem: If the minimum rate of return is 8%, what will be the annual cost of the new system? Economically, should the new system be installed?

(a) -665, Yes, it provides a negative annual cost at the minimum rate of return.

(b) -335, No it provides a negative annual cost at the minimum rate of return

(c) 335, Yes, it provides a positive annual cost at the minimum rate of return.

(d) 665, No, it provides a positive annual cost at the minimum rate of return.

6.4 PRACTICE PROBLEM 4 – ECONOMIC ANALYSIS

Background: A new high efficiency chiller with a lifetime is planned on being purchased. It has an initial cost of $200,000 and an ongoing maintenance cost of $2,000. However, this chiller will provide an energy savings of $10,000 per year. The chiller has a lifetime of 25 years and the minimum attractive rate of return is 4%. At the end of its lifetime, the chiller will have a salvage value of $25,000.

Problem: What is the annual value of the chiller at the minimum attractive rate of return? What is the simple payback?

(a) -$8,210, 25 years

(b) -$4,202, 25 years

(c) -$4,202, 20 years

(d) -$2,820, 20 years

6.5 PRACTICE PROBLEM 5 – ECONOMIC ANALYSIS

Background: A new boiler is planned on being purchased. It has an initial cost of $10,000 and an ongoing maintenance cost of $500 per year. This new boiler is much more efficient than the existing boiler and will have a yearly energy savings of a$1,500 per year. The boiler has a lifetime of 15 years and the minimum attractive rate of return is 5%. At the end of its lifetime, the boiler will have a salvage value of $1,000.

Problem: What is the present value of the boiler at the minimum attractive rate of return?

(a) -$860

(b) -$380

(c) $380

(d) $860

6.6 PRACTICE PROBLEM 6 – ECONOMIC ANALYSIS

A new chiller was purchased for $50,000 in 2016. Using MACRS depreciation, what is the depreciation amount of the chiller at year 9? Assume the equipment has a life of 10 years.

(a) $1,640

(b) $3,280

(c) $4,920

(d) $46,720

7.0 SOLUTIONS

7.1 SOLUTION 1 – ECONOMIC ANALYSIS

Background: A client is contemplating on purchasing a new high efficiency pump and motor, with an initial cost of $10,000. The pump has a lifetime of 15 years and is estimated to save approximately $1,000 per year. There is an additional maintenance cost of $300 per year associated with this new pump. The pump will have a salvage value of $0 at the end of its lifetime. Assume the interest rate is 4%.

Problem: What is the annual value of the pump?

First convert all terms to annual values.

Maintenance cost and cost savings are already annual values.

$$A_{maint} = -\$300$$

$$A_{savings} = \$1{,}000$$

Convert initial cost (present value) to annual value.

$$A_{initial\ cost} = -\$10{,}000 * (\frac{A}{P}, 4\%, 15)$$

Refer to economics tables for value.

$$A_{initial\ cost} = -\$10{,}000 * (.0899)$$

$$A_{total} = A_{maint} + A_{savings} + A_{initial\ cost}$$

$$A_{total} = -\$300 + \$1{,}000 - \$899$$

$$A_{total} = -\$199$$

Correct Answer: B) -$199

7.2 SOLUTION 2 – ECONOMIC ANALYSIS

Background: A client is contemplating between two separate chillers. Chiller 1 has a life of 25 years, an initial cost of $50,000, an ongoing maintenance/electricity cost totaling $1,000 per year. Chiller 2 has a life of 25 years, an initial cost of $35,000 and an ongoing maintenance/electricity cost totaling $1,500 per year. Assume interest rate is equal to 4%.

Problem: What is the present value of the two chillers?

First convert all terms to present worth, Chiller 1.

$$P_{chiller\ 1} = P_{initial\ cost} + A_{maint/elec} * (\frac{P}{A}, 4\%, 25)$$

$$P_{chiller\ 1} = -\$50{,}000 - \$1{,}000 * 15.6221$$

$$P_{chiller\ 1} = -\$65{,}622$$

Second convert all terms to present worth, Chiller 2.

$$P_{chiller\ 2} = P_{initial\ cost} + A_{maint/elec} * (\frac{P}{A}, 4\%, 25)$$

$$P_{chiller\ 2} = -\$35{,}000 - \$1{,}500 * 15.6221$$

$$P_{chiller\ 2} = -\$58{,}433$$

The correct answer is most nearly, (a) Chiller 1 = -\$65,622 ; Chiller 2 = -\$58,433.

7.3 SOLUTION 3 – ECONOMIC ANALYSIS

Background: An existing A/C control system is inefficient and you are researching whether or not to replace system. You develop a new system that will cost \$30,000 and require an ongoing maintenance of \$1,000 per year, but it will save \$4,000 per year in energy savings. The new A/C control system will have a lifetime of 30 years.

Problem: If the minimum rate of return is 8%, what will be the annual value of the new system? Economically, should the new system be installed?

First convert all terms to annual values.

Maintenance cost and energy savings are already annual values.

$$A_{maint} = -\$1{,}000$$

$$A_{savings} = \$4{,}000$$

Convert initial cost (present value) to annual value.

$$A_{initial\ cost} = -\$30{,}000 * (\frac{A}{P}, 8\%, 30)$$

Refer to economics tables for value.

$$A_{initial\ cost} = -\$30{,}000 * (.08883)$$

$$A_{total} = A_{maint} + A_{savings} + A_{initial\ cost}$$

$$A_{total} = \$4{,}000 - \$1{,}000 - \$2{,}664.90$$

$$A_{total} = \$335$$

Correct Answer: (c) $335; Yes, it provides a positive annual cost at the minimum rate of return.

7.4 SOLUTION 4 – ECONOMIC ANALYSIS

Background: A new high efficiency chiller with a lifetime is planned on being purchased. It has an initial cost of $200,000 and an ongoing maintenance cost of $2,000. However, this chiller will provide an energy savings of $10,000 per year. The chiller has a lifetime of 25 years and the minimum attractive rate of return is 4%. At the end of its lifetime, the chiller will have a salvage value of $25,000.

Problem: What is the annual value of the chiller at the minimum attractive rate of return? What is the simple payback?

First convert all terms to annual values.

Maintenance cost and energy savings are already annual values.

$$A_{maint} = -\$2{,}000$$

$$A_{savings} = \$10{,}000$$

Convert initial cost/ (present value) and salvage value (future) to annual value.

$$A_{initial\ cost} = -\$200{,}000 * (\frac{A}{P}, 4\%, 25)$$

Refer to economics tables for value.

$$A_{initial\ cost} = -\$200{,}000 * (.06401)$$

$$A_{salvage\ value} = \$25{,}000 * (\frac{A}{F}, 4\%, 25)$$

$$A_{salvage\ value} = \$25{,}000 * (.02401)$$

$$A_{total} = A_{maint} + A_{savings} + A_{salvage\ value} + A_{initial\ cost}$$

$$A_{total} = -\$2{,}000 + \$10{,}0000 + 600.25 - \$12{,}802$$

$$A_{total} = \$ - 4{,}202$$

$$Simple\ Pay\ Back = \frac{initial\ cost}{yearly\ gain} = \frac{\$200{,}000}{\$10{,}000 - \$2{,}000} = 25\ years$$

Correct Answer: B) $-4,202; 25 years

7.5 SOLUTION 5 – ECONOMIC ANALYSIS

Background: A new boiler is planned on being purchased. It has an initial cost of $10,000 and an ongoing maintenance cost of $500 per year. This new boiler is much more efficient than the existing boiler and will have a yearly energy savings of a $1,500 per year. The boiler has a lifetime of 15 years and the minimum attractive rate of return is 5%. At the end of its lifetime, the boiler will have a salvage value of $1,000.

Problem: What is the present value of the boiler at the minimum attractive rate of return?

First convert all terms to present value.

Maintenance cost and energy savings can be combined and converted to present value.

$$A_{maint} = -\$500$$

$$A_{savings} = \$1{,}500$$

$$P_{maint/savings} = \$1{,}000 * (\frac{P}{A}, 5\%, 15)$$

$$P_{maint/savings} = \$1{,}000 * (10.3797) = \$10{,}380$$

Convert salvage value (future value) to present value.

$$P_{salvage\ value} = \$1{,}000 * (\frac{P}{F}, 5\%, 15)$$

$$P_{salvage\ value} = \$1{,}000 * (0.4810) = \$481$$

$$P_{initial\ cost} = -\$10{,}000$$

$$P_{total} = P_{maint/savings} + P_{salvage\ value} + P_{initial\ cost}$$

$$P_{total} = \$10{,}380 + 481 - \$10{,}000$$

$$P_{total} = \$861$$

Correct Answer: (D) $860

7.6 SOLUTION 6 – ECONOMIC ANALYSIS

A new chiller was purchased for $50,000 in 2016. Using MACRS depreciation, what is the depreciation amount of the chiller at the end of its 10 year life?

In order to solve this problem you need the MACRS depreciation schedule for 10 years, which is found in the *NCEES FE Reference Handbook*. This table shows that the depreciation at year 9 is as shown below.

$$D_{10} = 6.56\%$$

$$Depreciation\ Value\ @\ Year\ 10 = \$50{,}000 * (6.55\%) = \$3{,}280$$

5 – Electricity & Magnetism

MECHANICAL FE TEXTBOOK

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Section 5.0 – Electricity and Magnetism

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1.0 INTRODUCTION

Electricity and Magnetism accounts for approximately 5 to 8 questions on the Mechanical FE exam. The mechanical discipline intersects with the electrical when mechanical energy is converted to electrical and vice versa. Hence, the mechanical engineer should have a basic understanding of electricity and magnetism. The most common application is the production of power via a generator and the reception of power through equipment via motors. Other applications include control circuits, variable frequency drives, and the effects of mechanical equipment on power quality. Remember that only the basics of the "Electricity and Computer Engineering" section of the *NCEES FE Reference Handbook* will be tested during the exam. The questions must also relate to the topics below.

Section 5.0 Electricity & Magnetism (5 to 8 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
5A	Section 2.0	Electrical Fundamentals (e.g., charge, current, voltage, resistance, power, energy, magnetic flux)
5B	Section 3.0	DC circuit analysis (e.g., Kirchhoff's laws, Ohm's law, series, parallel)
5C	Section 4.0	AC circuit analysis (e.g., resistors, capacitors, inductors)
5D	Section 5.0	Motors and generators
	Section 6.0	Practice Exam Problems

2.0 ELECTRICAL FUNDAMENTALS

2.1 CHARGE

An electric charge, Q, describes the number of electrons or protons there are. It can be positive (protons) or negative (electrons) and is measured in Coulombs (C). For example, one electron has -1.6×10^{-19} C of charge and one proton has $+1.6 \times 10^{-19}$ C. The movement of electrons is the foundation of how electricity works. It is unlikely that charge itself will be tested. It is more important to understand how charge is used to describe other concepts like current, power, voltage, and energy. However, here is more information on charges and the equations presented in the FE Handbook.

2.1.1 Point Charge

The electric field due to a point charge will be a function of the radial distance away from the point charge.

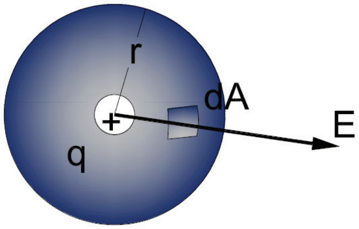


Figure 1: A point charge will cause an electric field in proportional to its charge value. The electric field will emanate in a spherical shape. This is why the charge is divided by the equation for the surface area of a sphere.

$$Point\ Charge \rightarrow E = \frac{Q}{4\pi\varepsilon r^2} a_{r,1-2}$$

$$Q = charge\ (C); r = distance\ from\ point\ charge\ (m);\ \varepsilon = permittivity\ of\ medium\ (\frac{C^2}{N-m^2})$$

$$if\ in\ free\ space, then\ use\ permittivity\ of\ air\ \varepsilon_0 = 8.854\ x\ 10^{-12}\ (\frac{C^2}{N-m^2})$$

$$a_{r,1-2} = directional\ vector$$

The units for electric field work out to Newton per Coulomb.

$$E = \frac{Q\ (C)}{4\pi\varepsilon(\frac{C^2}{N-m^2})r^2\ (m^2)} a_{r,1-2} \rightarrow Units = \frac{Newton}{Coulomb}$$

In order to find the force on a point charge due to an electric field, you simply multiply the electric field by the charge.

$$F = E\left(\frac{N}{C}\right) * Q\ (C) \rightarrow Units = Newton$$

2.1.2 Line Charge

This same process can be used for other geometries besides points, like line charges and sheet charges. You can check to make sure the units still work out below.

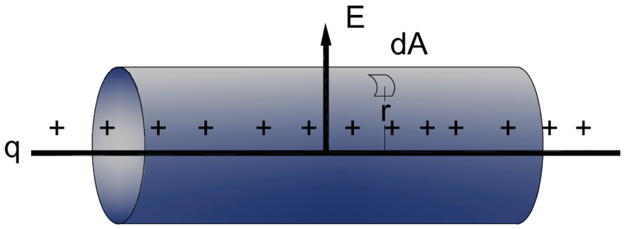


Figure 2: A line charge will cause an electric field that is proportional to its line charge density which will have units of charge per meter. The electric field will emanate from the line in

multiple circles (rings) along its length. This is why the charge density is divided by the surface area of a circle.

$$Electric\ Field\ due\ to\ Line\ Charge\ Density \rightarrow E = \frac{\rho_L}{2\pi\varepsilon r}a_r$$

$$\rho_L = line\ charge\ density\ \left(\frac{Coulomb}{meter}\right); r = perpendicular\ distance\ away\ from\ line\ charge\ (m)$$

$$E_{line\ charge} = \frac{\rho_L(\frac{C}{m})}{2\pi\varepsilon\left(\frac{C^2}{N-m}\right)r(m)}a_r \rightarrow Units = \frac{N}{C}$$

The force on a point charge due to a line charge can be found by multiplying the electric field by the point charge value.

$$Force\ due\ to\ Line\ Charge\ on\ a\ Point\ Charge \rightarrow F_{line\ charge} = Q * \frac{\rho_L}{2\pi\varepsilon r}a_r$$

2.1.3 Sheet Charge

The equation for a sheet charge is shown below. For this equation, there is no distance from the sheet charge.

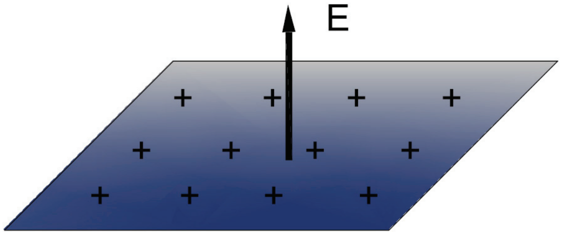


Figure 3: A sheet charge will cause an electric field in accordance with its sheet charge density, which will have units of charge per square meter. The electric field will emanate in straight lines from the sheet.

$$Electric\ Field\ due\ to\ Sheet\ Charge\ Density \rightarrow E = \frac{\rho_s}{2\varepsilon}a_r$$

$$\rho_s = sheet\ charge\ density\ \left(\frac{Coulomb}{meter^2}\right);$$

The force due to the sheet charge on a point charge is found similarly to the line charge.

$$Force\ due\ to\ Sheet\ Charge\ on\ a\ Point\ Charge \rightarrow F = QE = Q * \frac{\rho_s}{2\varepsilon}a_r$$

The work due to the electric field can be found by multiplying the force by the distance the charge moves. Another way to calculate the work due to the electric field is to integrate the electric field along the distance that the point charge moves.

$$Work\ due\ to\ Electric\ Field \rightarrow W = -Q \int E \cdot dl$$

$$Energy\ due\ to\ Electric\ Field \rightarrow W_E = \left(\frac{1}{2}\right) \int \varepsilon |E|^2 \cdot dV$$

2.2 CURRENT

Current, I, is the movement of charge and is more specifically defined as the rate at which charge flows. It is represented in terms of Amps, where one amp is equal to the movement of one Coulomb of charge per second.

$$Current, I(Amps) = \frac{Charge\ (C)}{Time\ (\mathrm{sec})}$$

$$I = \frac{dq}{dt} \quad \rightarrow \quad Charge, Q = \int_{t1}^{t2} i(t)\, dt$$

For steady flow, current can be calculated as:

$$I\ (Amps) = \frac{\Delta Q}{\Delta t}$$

One characteristic to distinguish is that current flows in the opposite direction of electrons. Current flows from positive to negative, see the green arrow in the figure below, start at the positive end of the battery, loop around the circuit and end at the negative end. Electrons on the other hand are attracted to positive charge, so it will flow from negative to positive, as shown in red below.

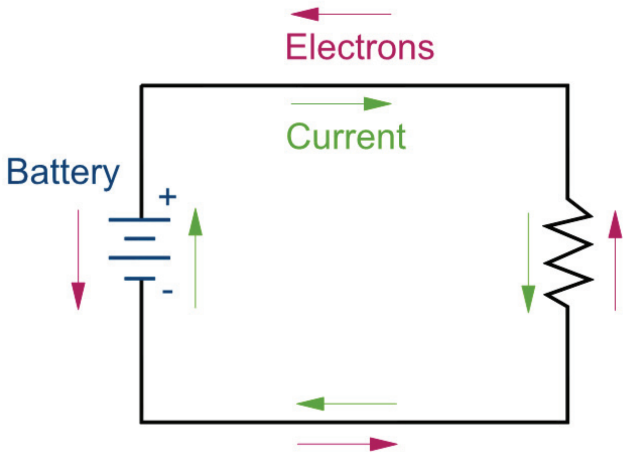


Figure 4: Current flows in a circuit from the positive end of the battery to the negative, as shown in green, while electrons flow from negative to positive.

Direct current (DC) is the supply of current in one direction. As mentioned previously, current flows from the positive voltage terminal to the negative terminal in a circuit. Current is deemed positive when it flows in this direction. Current is considered negative when it flows from a negative terminal to a positive terminal. DC current is a constant source and does not switch between negative and positive. The simplest example of a DC source is a battery.

Alternating current (AC) is able to supply current in both directions, positive to negative and negative to positive. This is shown in the graph below, where the current can be positive (above the 0-axis) or negative (below the 0-axis). Alternating current is what is supplied by the electric company to buildings. Alternating current is further discussed in the Alternating Circuits topic.

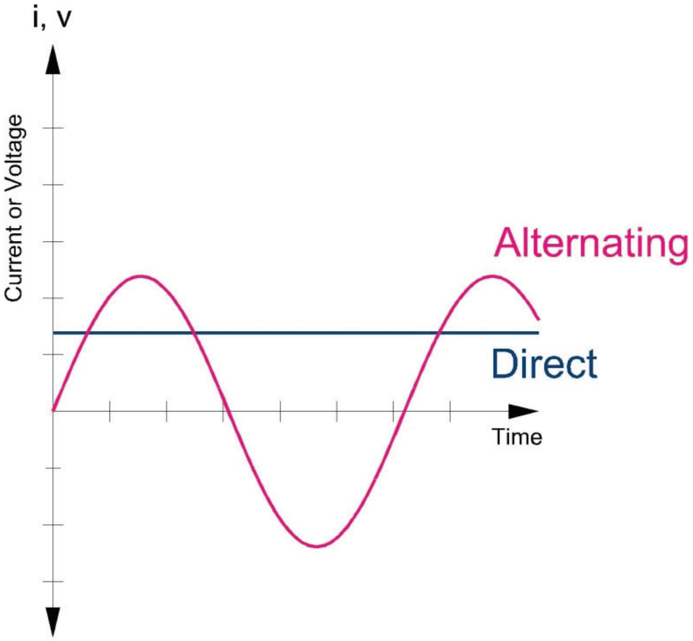


Figure 5: In an AC circuit, current can alternate its flow from positive to negative. In a DC circuit, current is constant.

2.3 VOLTAGE

Voltage, V is the potential energy in electricity; it is the amount of energy held in one charge and is given in units of volts.

$$Voltage\ (V) = \frac{Potential\ Energy\ (J)}{Charge\ (C)}$$

Voltage is measured between two points because potential energy is the difference in energy. This potential energy in a circuit is what drives the flow of electrons, and therefore the current, from one point to the next. A battery is typically a source voltage, supplying energy into a circuit. A voltage can also be measured across a load or a resistor, this is known as voltage drop since the energy is being absorbed by the load.

2.4 RESISTORS

To complete a circuit, wires are connected from a voltage source to a load, which is represented by the resistance. Resistance is measured in units of ohms (Ω). It is the opposition to the flow of current. One ohm is described as the level of resistance that will allow 1 ampere to flow when 1 volt is applied to a circuit. The following figure depicts a basic circuit with current flowing from a voltage source, then through a resistor.

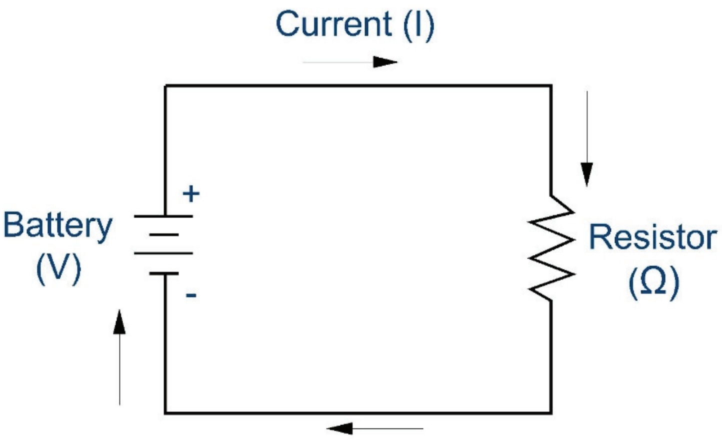


Figure 6: A basic DC circuit, current flows from positive to negative.

Ohm’s law describes the relationship between voltage, resistance and current. The voltage in a circuit is equal to the product of the current and the resistance in the circuit.

$$Ohm's\ Law \rightarrow\ V = IR$$

$$V = voltage\ (volts);\ \ I = current\ (amperes); R = ohms\ (\Omega)$$

As voltage increases, in a constant resistance circuit, current will rise. As current increases, in a constant resistance circuit, voltage will rise.

2.4.1 Resistivity

Resistivity is used to calculate the resistance of an object like a copper wire. Resistance is given in units of ohms and is a function of the material’s resistivity, the cross sectional area of the material and the length of the material.

$$R\ (\Omega) = \frac{\rho L}{A}$$

$$A = area\ (m^2); L = length\ (m)$$

$$\rho = resistivity\ (\Omega - \mathrm{m}); C = conductivity\ (\frac{1}{\Omega - \mathrm{m}})$$

2.4.2 Conductivity

The important electrical properties of materials are conductivity and capacitance. The conductivity describes the material’s ability to conduct electricity and capacitance describes the material’s ability to hold a charge. Conductivity is often expressed as its inverse, which is called resistivity. This is the material’s ability to resist an electrical current.

$$Conductivity(C) = \frac{1}{Resistivity\ (\rho)}$$

2.5 POWER

The next concept that you must understand is electrical power. Power is the rate at which energy flows and is given in units of Watts (Joules per second). The power equation in basic circuits is given below and states that real power is equal to the product of the voltage and the current. Real power is given in terms of watts.

$$Real\ Power \rightarrow\ P = IV$$

$$P = real\ power\ (watts)$$

If you combine Ohm's law and the real power equation, then you will find two equations that show power as a function of resistance. These equations, along with Ohm's Law can be found in the *NCEES FE Reference Handbook*.

$$P = I^2R\ ; \quad P = \frac{V^2}{R}$$

The term *power factor* will commonly be introduced when connecting to the main power grid. Power factor is essentially the quality of the power that enters a building. Power factor is discussed in detail under the AC Circuits topic.

2.6 ENERGY

Electric energy can be represented in two forms, kinetic and potential. Energy is potential when it is stored in a battery, capacitor, or inductor. As the energy is distributed to a circuit, it is converted to kinetic energy. Finally, when it flows through a load, the energy is consumed in forms such as heat, light, or mechanical energy. The following equations represent energy across various elements. They may be found in the *NCEES FE Reference Handbook*. The first is the most important to know. In mechanical engineering, electrical energy is normally represented as Wh, kWh, or MWh.

$$Energy\ (Joules) = Power\ (Watts) * Time\ (\mathrm{sec})$$

$$Energy\ (Wh) = Power\ (Watts) * Time\ (\mathrm{hours})$$

Example: A 20 hp motor operates for 8 hours a day on a daily basis. How much energy (kWh) is consumed in 5 years. Assume the motor is operating at full load and is 100% efficient.

Solution: Since efficiency is not a factor, the motor will consume 20hp of electric power. Then, the energy consumption is calculated as:

$$Power = 20hp * \frac{745.7W}{hp} * \frac{1kW}{1000W} = 14.9kW$$

$$Energy = 14.9kW * \frac{8hours}{day} * \frac{365days}{year} * 5years = 217{,}744kWh$$

After 5 years, the motor will consume 217,744 kWh of electricity.

The following three will likely not be used, but is shown for its relationship between capacitance, inductance, and charge to energy. The AC Circuits section discusses capacitors and inductors in further detail.

$$Energy\ in\ Battery\ (Joules) = Voltage\ (V) * Charge\ (Coulomb)$$

$$Energy\ in\ Capacitor\ (Joules) = \frac{Capacitance\ (Farads) * Voltage^2}{2} = \frac{Voltage * Charge}{2}$$

$$Energy\ in\ Inductor\ (Joules) = \frac{Inductance\ (Henry) * Current\ (Amps)^2}{2}$$

2.7 MAGNETIC FLUX DENSITY (B)

Magnetic flux, which is described in units of Webers is the amount of magnetic field lines. Magnetic field lines are the imaginary lines that leave magnets. These lines are used to represent magnetic fields.

$$\Phi = magnetic\ flux\ (Wb = \ Weber = 1\ x\ 10^8\ magnetic\ field\ lines)$$

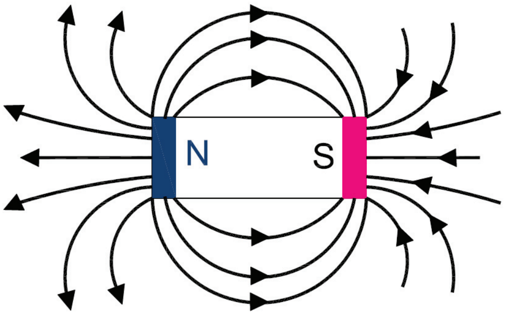


Figure 7: Magnetic flux lines travel from north to south poles.

Magnetic flux density is the amount of magnetic flux per a unit area. The area must be perpendicular to the direction of flux. The equation for magnetic flux density is shown below. Magnetic flux is the amount of lines that cross a certain area.

$$B = \frac{\Phi}{Area};$$

$$B = magnetic\ flux\ density\ (Tesla);\ \Phi = magnetic\ flux\ (Weber); A = area\ (m^2)$$

Another way to look at the previous formula is that the number of magnetic field lines or flux is equal to the magnetic flux density times the area.

$$\Phi = \vec{B}.\vec{A}$$

However, the area plane must be perpendicular to the magnetic flux density vectors. This is shown by the dot product of the two vectors. The dot product can also be rewritten as the multiplication of the magnitudes of the flux density vector and the vector of the area plane and the cosine of the angle between the two vectors.

$$\Phi = BAcos(\theta)$$

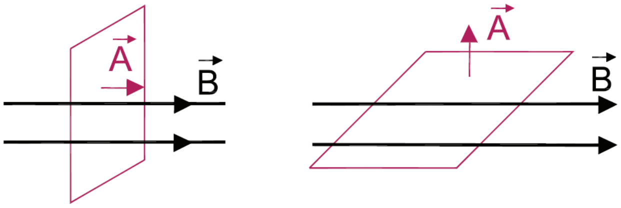


Figure 8: On the figure on the left the angle between the two vectors is equal to 0 degrees. On the figure to the right, the angle between the two vectors is equal to 90 degrees. The area vector is shown as perpendicular to the area plane.

The next term is the magnetic field strength or force. The strength of a magnetic field in a coil of wire depends on how much current flows in the turns of the coil. The more current, the stronger the magnetic field. Also, the more turns, the more concentrated are the lines of force. The product of the current times the number of turns of the coil, which is expressed in units called ampere-turns (At), is known as the magneto-motive force. As a formula,

$$F = N * I$$

$$F = ampere\ turns, N = turns; I = current\ (amps)$$

2.7.1 Magnetic Field Strength/Intensity (H)

Magnetic flux density and magnetic field strength are related based on Gauss's Law for Magnetism. The version of the law shown in the FE Reference Handbook is shown in the equation below.

$$Gauss'Law\ for\ Magnetic\ Fields\ (Integral\ Form) \rightarrow \oiint_{S_V} B \cdot dS = 0$$

The left hand side of the equation describes taking the integral of a magnetic flux over a length or surface. The special integral symbol reminds you that this integral is over a closed surface. The dot product between the electric field and the incremental change in length or surface is to show that the magnetic flux lines must be perpendicular to the surface or length. The right hand side of the equation is equal to zero, because there is no magnetic charge density.

$$B = magnetic\ flux\ (Tesla\ or \frac{Weber}{meter^2})$$

The next equation is the differential form of the same equation. Again, there is no such thing as a magnetic charge density, so unlike the first Gauss's law which had the electric flux equal to the electric charge density, there is no magnetic charge density, so the right side is equal to 0. Another way to think about the reason for the zero is that magnets always have a north and south poles or a positive and negative side, so the positive and negative sides are equal in magnitude but opposite in sign, thus the sum of the magnetic flux must be equal to zero.

$$Gauss'Law\ for\ Magnetic\ Fields\ (Differential\ Form) \rightarrow \nabla \cdot B = 0$$

Magnetic flux is related to magnetic field strength (intensity) through the following equation. This equation introduces the permeability of free space or simply permeability. This is a constant that defines the ability of a medium to promote magnetic fields. The permeability shown in the NCEES FE Reference Handbook is for air or free space. It has units of Henries per meter, which can also be shown as Newton per ampere squared.

$$B = \mu H$$

$$H = magnetic\ field\ strength\ \left(\frac{A}{m}\right); \mu = permeability\ constant\ 4\pi\ x\ 10^{-7}\ (H/m\ )$$

$$Deriviation\ of\ Units\ of\ B \rightarrow \mu H = \left(\frac{H}{m}\right)\left(\frac{V}{m}\right) = \left(\frac{N}{A^2}\right)\left(\frac{A}{m}\right) = \frac{N}{A - m}$$

Magnetic field strength has units of ampere per meter. This is analogous to the electric field strength which has units of volt per meter.

The useful equations for magnetism will be shown in the next sections, where the magnetic field is related to current flowing through a coil.

2.7.2 Induced Voltage (V)

The third Maxwell Equation combines the previous two concepts. It states that a changing magnetic field will cause an electric field. So basically a change in magnetic field will cause a current to be induced.

$$Farad's\ Law\ (Differential\ Form) \rightarrow \nabla\ x\ E = -\frac{\partial B}{\partial t}$$

In the equation above, the right hand side is the derivative of a magnetic field strength over time. A change in magnetic field strength will cause an opposite reaction in the electric field intensity. So if the magnetic field strength is reducing, then the electric field intensity will be positive and if the magnetic field strength is increasing, then the electric field intensity will be negative. The left hand side of the equation describes the divergence of the electric field or the measure of the electric field intensity through the same surface that is affected by the changing magnetic field.

In more practical terms, Faraday's Law of Induction states that for a coil of wire wrapped around a metal core, exposed to a changing magnetic field, will have an induced voltage. This is the key concept to how transformers, motors and generators operated. The coils are exposed to a changing magnetic field, which induces voltage, which will cause current to flow. The next figure is an attempt at showing Faraday's law in graphical format.

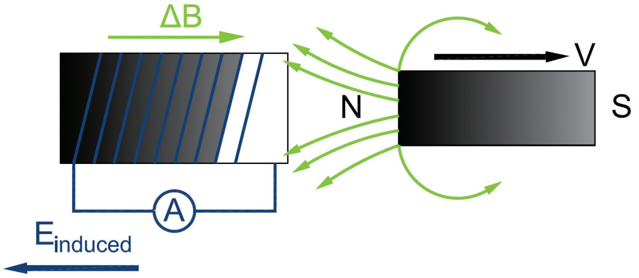


Figure 9: This figure shows a magnetic moving away from the coil wrapped around a metal core. The moving magnetic causes a changing magnetic field on the coil. This is shown as less magnetic flux lines hitting the coil. This causes a positive voltage, which causes a positive current to flow, which is symbolized by flow in the clockwise direction.

This next equation shows the same law, but in integral form. The left hand side of the equation takes the integral of the electric field around a path, like a circuit. The right hand side of the equation takes the integral of the rate of change of the magnetic field across a surface.

$$Farad's\ Law\ (Integral\ Form) \rightarrow \oint E \cdot dl = -\iint_S \left(\frac{\partial B}{\partial t}\right) \cdot dS$$

As the magnetic field changes, a voltage is induced in the coil. The size of the voltage will depend on the number of coil turns and the rate of change of the magnetic flux. This introduces another term as shown in the equation below. The previous variable, "B" was a magnetic flux per area (Weber per square meter). This flux term is simply in units of Weber.

$$v_{induced} = N * \frac{\Delta\Phi}{\Delta t}$$

$$v_{induced} = induced\ voltage\ (V); N = number\ of\ coil\ turns$$

$$\frac{\Delta\Phi}{\Delta t} = change\ in\ flux\ over\ time\ \left(\frac{Wb}{s}\right)$$

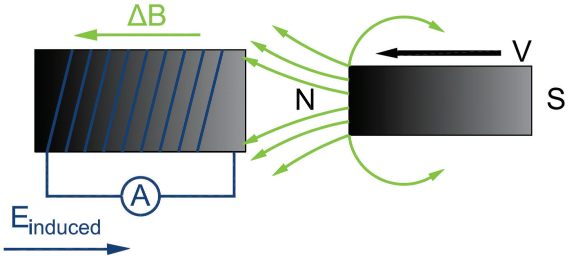


Figure 10: This figure shows a magnetic moving towards the coil wrapped around a metal core. The moving magnetic causes a changing magnetic field on the coil. This is shown as more magnetic flux lines hitting the coil. This causes a negative voltage, which causes a negative current to flow, which is symbolized by flow in the counterclockwise direction.

2.8 ELECTRIC FLUX DENSITY (D)

Electric Field Density is governed by Gauss' Law for Electricity. The version of the law shown in the FE Reference Handbook is shown in the equation below.

$$Gauss'Law\ for\ Electric\ Fields\ (Integral\ Form) \rightarrow \oiint_{S_V} D \cdot dS = \iiint_V \rho dv$$

The left hand side of this equation calculates the sum of all the "D" electric flux lines that are entering and exiting the surface of some volume. The right hand side of the equation calculates the total electric charge in a volume. The left hand side of the equation will result in the units Coulomb and so will the right hand side of the equation. You should also see that on the left hand side of the equation the "D" or electric flux is multiplied by an area (m^2). So you can deduce that the units of electric flux are C/m^2.

The next equation is the differential form of the same integral equation.

$$Gauss'Law\ for\ Electric\ Fields\ (Differential\ Form) \rightarrow \nabla \cdot D = \rho$$

$$D = Electric\ flux\ (C/m^2);\ \nabla = \text{divergence operator};\ \rho = electric\ charge\ density;$$

The divergence vector says a similar thing as the previous left side of the equation. It calculates the "D" electric flux lines that are leaving a point. The right hand side of the equation is the total electric charge.

Electric flux is the rate at which the electric field flows through a given area. The electric field is the area around a charge or the area between a voltage potential. The greater the charge or the greater the voltage potential, then the greater the electric field. Electric flux will also depend on the way the area is drawn and the math can get a little complicated, but for the purposes of this FE exam, you should not need anything more than the equations shown in this section and in the NCEES FE Reference Handbook.

2.8.1 Electric Field Strength/Intensity (E)

Electric flux is related to electric field intensity through the following equation. This equation introduces the permittivity of free space or simply permittivity. This is a constant that defines the ability of a medium to promote electric fields. The permittivity shown in the NCEES FE Reference Handbook is for air or free space. It has units of Farad per meter, which can also be shown as Coulomb per volt-meter.

$$D = \varepsilon_0 E$$

$$E = electric\ field\ intensity\ \left(\frac{V}{m}\right);\ \varepsilon_0 = permittivity\ constant\ 8.85\ x\ 10^{-12}\ (F/m\ )$$

$$Deriviation\ of\ Units\ of\ D\ (Electric\ Flux) \rightarrow \varepsilon_0 E = \left(\frac{F}{m}\right)\left(\frac{V}{m}\right) = \left(\frac{C}{v-m}\right)\left(\frac{V}{m}\right) = \frac{C}{m^2}$$

The useful equation that you should use on the exam is shown below. This equation can be used for various conditions like an enclosed point charge, an enclosed line charge and an enclosed sheet charge. This is discussed more in Coulomb's law in this section.

$$\oint_S E \cdot \hat{n}\, dA = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$E = electric\ field\ vector\ \left(\frac{V}{m}\right); Q_{enclosed} = enclosed\ charge\ (C);$$

This equation says that a charge will produce an electric field and the sum of the electric field intensity across an area will be equal to the enclosed charge divided by the permittivity. The units for permittivity can also be rewritten as Coulomb per Volt-Meter. This will result in the units of volt per meter for the division of the enclosed charge by the permittivity.

$$Deriviation\ of\ Units\ of\ E \rightarrow \frac{Q_{enclosed}}{(Area)\varepsilon_0} = \left(\frac{C}{F/m}\right) = \left(\frac{C}{\left(\frac{C}{v-m}\right)m^2}\right) = \frac{V}{m}$$

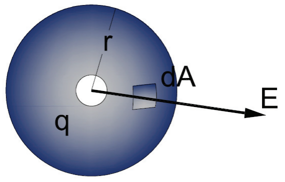


Figure 11: This figure shows a charge enclosed by an area. The resulting electric field can be found in units, volt per meter.

Basically, Gauss's law for electricity states that an electric charge will either attract or repel electric field lines. The intensity of electric field lines will be in direct proportion to the quantity of the charge. The intensity of electric field lines will be inversely proportional to the surface area in question.

2.8.2 Induced Current (I)

Ampere's law is used to calculate the magnetic field created by an electrical current. It is the counterpart to Faraday's Law of Induction. The Ampere's Law formulas presented in the FE Reference Handbook are shown below.

$$\oint H \cdot dl = I_{enc} + \iiint_S \left(\frac{\delta D}{\delta t}\right) \cdot dS$$

$$dl = incremental\ length$$

This first equation states that a magnetic field is created that circles around a wire carrying current. The left hand side of the equation sums up the strength of the magnetic field surrounding the wire. The right hand side is the current value plus the rate of change of any electric flux over an area. Remember that electric flux (D) is charge per area, so by multiplying by area and dividing by time, the result will be a current (charge per time).

$$\nabla \text{ x } H = J + \frac{\delta D}{\delta t}$$

The next equation is the differential form of Ampere's Law. The left hand side of the equation finds the strength of the magnetic field. The new introduction is the "J" variable which is just the current per area. Also on the right hand side is the rate of change of the electric flux. The derivative of electric flux results in current per area.

$$Derivation\ of\ Units \rightarrow \frac{\delta D}{\delta t} = \left(\frac{\frac{C}{m^2}}{s}\right) = \left(\frac{C/s}{m^2}\right) = \left(\frac{Ampere}{m^2}\right)$$

Ampere's law can also be viewed in another way. The magnetic field is proportional to the current flowing in the conductor and the permeability of the medium (typically air).

$$Ampere's\ Law \rightarrow B\ proportional\ to\ \mu_0 * I$$

The simple equation of Ampere's Law is found by integrating a surface around the conductor and finding all the electric field lines that cross that surface. The sum of these electric field lines will equal the permeability multiplied by the current encompassed by that surface.

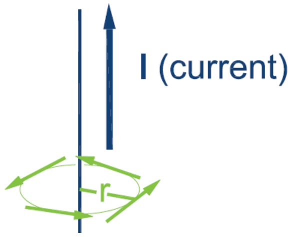


Figure 12: If you integrate all the electric field lines along a surface around a current carrying conductor, then the sum will equal the current multiplied by the permeability.

The permeability will change based on the medium, which will most likely be air (μ_0 aka vacuum or free space) or soil.

$$\mu_0 = 4\pi \times 10 - 7\frac{N}{A^2}$$

Ampere's law in its integral form can now be written into another format by converting magnetic field strength to magnetic flux. In addition, electric flux can be converted to electric field.

$$H = magnetic\ field\ strength\ \left(\frac{amperes}{meter}\right) = \frac{B\ (magnetic\ flux\ [\frac{Newton}{ampere - meter}])}{\mu_0\ (magnetic\ permeability\ \left[\frac{Newton}{ampere^2}\right])}$$

$$E = electric\ field\left(\frac{amperes}{meter}\right) = \frac{D\ (electric\ flux\ [\frac{Newton}{ampere - meter}])}{\mu_0\ (magnetic\ permeability\ \left[\frac{Newton}{ampere^2}\right])}$$

This is the equation that will most likely be used on the exam.

$$\oint B \cdot dS = \mu_0 * I_{enc}\ ;\ dS = incremental\ surface;$$

For example, if you were to integrate the below figure, you would result in the following equation.

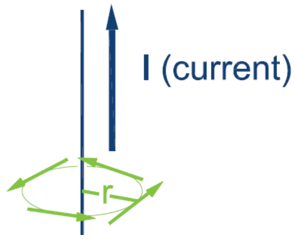


Figure 13: Ampere's law for this figure, can be found by calculating the circumference of the circle. Just make sure you use the correct units for permeability and radius.

$$B = \frac{\mu_0 * I}{2\pi r};\ r = radius\ of\ circle\ (m)$$

There are no other completed integrations for other geometries like for a cylinder or a solenoid. Integrating these other complex geometries would be unlikely to occur on the exam, so it is not covered in this section.

3.0 DC CIRCUIT ANALYSIS

3.1 OHM'S LAW

Ohm's law is explained in the resistors section above.

$$Ohm's\ Law \rightarrow\ V = IR$$

3.2 KIRCHHOFF'S LAWS

There are two Kirchhoff laws that are necessary for the FE exam, (1) Voltage Law and (2) Current Law. The voltage law or KVL (Kirchhoff's Voltage Law) states that the sum of voltages around a circuit loop is equal to zero. The current law or KCL (Kirchhoff's Current Law) states that the sum of the current into and out of a node is equal to zero.

3.2.1 Kirchhoff's Voltage Law (KVL)

KVL describes the principle that in a circuit loop all voltage produced must be used by the devices in the circuit. There cannot be a change in voltage in a circuit loop.

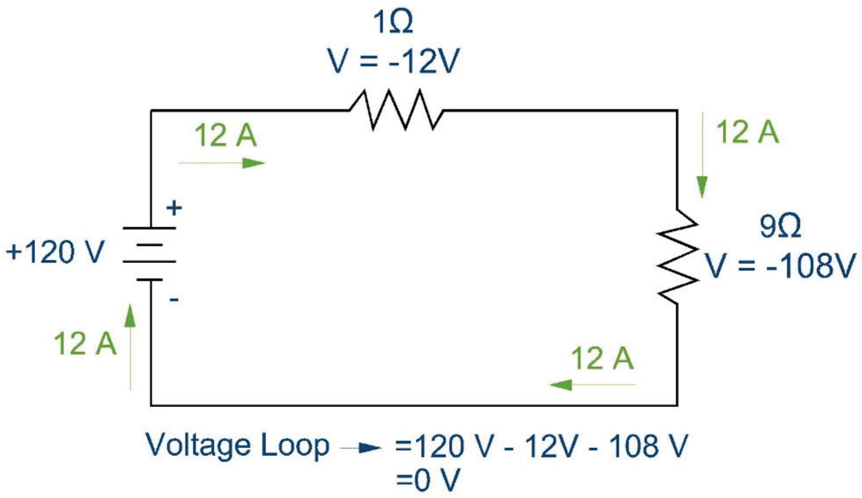


Figure 14: The sum of the voltages around a loop must equal zero.

KVL is also applicable when there is more than one voltage source in a circuit. Notice that the positive and negative terminals of the voltage sources are oriented in the same way, such that both voltage sources contribute positive voltage to the circuit.

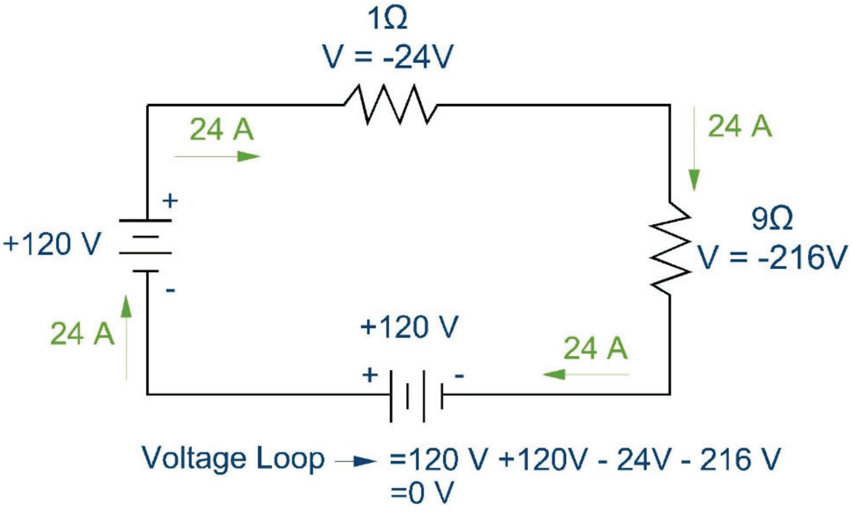

1Ω
V = -24V
24 A
24 A
+
+120 V
-
9Ω
V = -216V
24 A
+120 V
24 A
+
-
Voltage Loop → =120 V +120V - 24V - 216 V
=0 V

Figure 15: The sum of the voltage around a loop with multiple voltage sources will equal zero.

If the terminals of the two voltage sources were situated opposite from one another, then one voltage source would contribute positive voltage and the other would contribute negative voltage. The voltage source that contributed negative voltage would be similar to charging a battery and the positive voltage contribution would be a discharging battery.

KVL also applies to circuits with multiple loops. Each loop must have its voltage sum equal to zero.

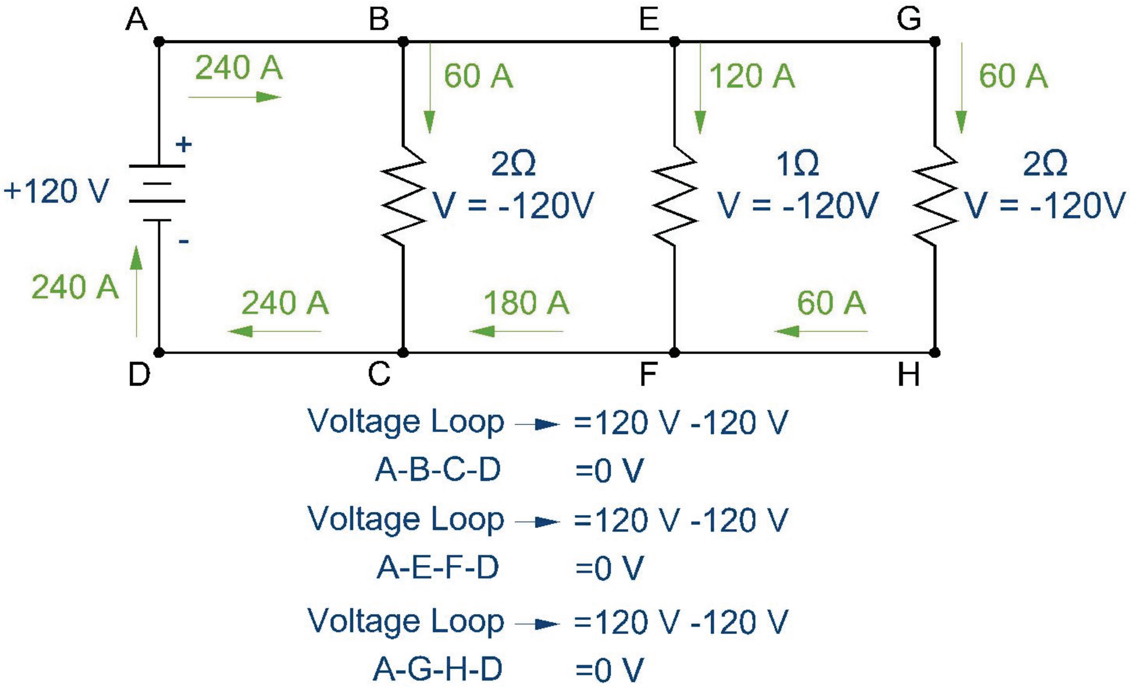

A
B
E
G
240 A
60 A
120 A
60 A
+
+120 V
-
2Ω
V = -120V
1Ω
V = -120V
2Ω
V = -120V
240 A
240 A
180 A
60 A
D
C
F
H
Voltage Loop → =120 V -120 V
A-B-C-D =0 V
Voltage Loop → =120 V -120 V
A-E-F-D =0 V
Voltage Loop → =120 V -120 V
A-G-H-D =0 V

Figure 16: KVL is applicable in any loop.

3.2.2 Kirchhoff's Current Law (KCL)

KCL describes the principle that the current entering any node in a circuit must equal the current leaving the node. If you look at the previous circuit, specifically node "B", you will notice that there is 240 A entering the node and 240 A leaving the node, with 60 A leaving on path B-C and 180 A leaving on path B-E. The next equations show more examples of KCL for various nodes in the previous figure.

$$Node\ B\!:\ 240\ A\ (A \rightarrow B) = 60\ A(B \rightarrow C) + 180\ A\ (B \rightarrow E)$$

$$Node\ E\!:\ 180\ A\ (B \rightarrow E) = 120\ A(E \rightarrow F) + 60\ A\ (E \rightarrow G)$$

$$Node\ F\!:\ 120\ A(E \rightarrow F) + 60\ A\ (H \rightarrow F) = 180\ A\ (F \rightarrow C)$$

$$Node\ C\!:\ 60\ A(B \rightarrow C) + 180\ A\ (F \rightarrow C) = 240\ A\ (C \rightarrow D)$$

3.3 EQUIVALENT CIRCUITS (SERIES, PARALLEL)

A complex circuit is made up of multiple devices and wires arranged in either series or parallel. These arrangements define the flow of current and the change in voltage with the following principles.

3.3.1 Series Circuits

In a series circuit, the resistances are added to determine the equivalent resistance.

$$R_{equiv,series} = R_1 + R_2 + R_3 + \cdots R_N$$

Capacitors in series are found by taking the inverse of the total sum of one over each capacitor.

$$C_{equiv,series} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3 + \cdots 1/C_N}$$

Inductors in series are added to determine the equivalent inductance.

$$L_{equiv,series} = L_1 + L_2 + L_3 + \cdots L_N$$

Further explanations of capacitors and inductors are described in the AC circuit topic.

3.3.2 Parallel Circuits

In a parallel circuit, the voltage drop across each path is the same and the circuit splits between the different paths.

In a parallel circuit, resistors are found by taking the inverse of the total sum of one over each resistor.

$$R_{equiv,parallel} = \frac{1}{1/R_1 + 1/R_2 + 1/R_3 + \cdots 1/R_N}$$

Capacitors in parallel are found by taking sum of all capacitors.

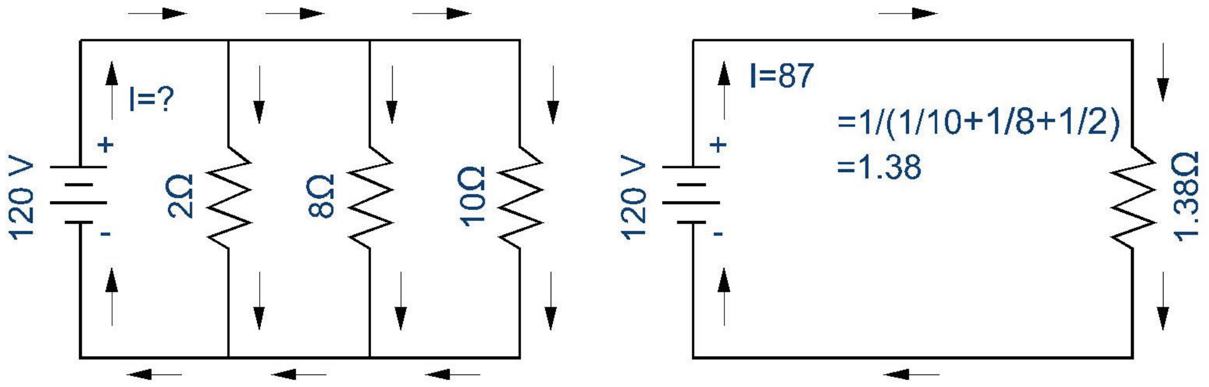
$$C_{equiv,parallel} = C_1 + C_2 + C_3 + \cdots C_N$$

Inductors in parallel are found by taking the inverse of the total sum of one over each inductor.

$$L_{equiv,parallel} = \frac{1}{1/L_1 + 1/L_2 + 1/L_3 + \cdots 1/L_N}$$

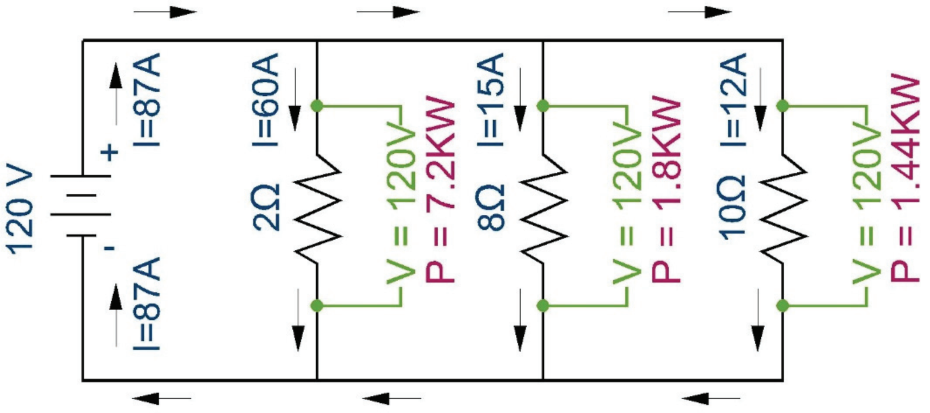
Further explanations of capacitors and inductors are described in the AC circuit topic.

3.3.3 Example: Solving for Resistors in Parallel



I=?
+
-
120 V
2Ω
8Ω
10Ω
I=87
=1/(1/10+1/8+1/2)
=1.38
120 V
+
-
1.38Ω

Figure 17: Equivalent resistance of resistances arranged in parallel.



120 V
+
-
I=87A
I=87A
I=60A
2Ω
V = 120V
P = 7.2KW
I=15A
8Ω
V = 120V
P = 1.8KW
I=12A
10Ω
V = 120V
P = 1.44KW

Figure 18: Parallel circuits

- Voltage Drop

In the previous figure, the voltage drop is found through each resistor by using KVL and realizing that in every loop the total voltage drop must equal to zero. Thus the voltage drop through resistors, (2 Ω), (8 Ω) and (10 Ω) are all equal to 120 V. The voltage drop through each resistor and the resistance value will also help to determine the amount of current through each resistor.

- Power Consumption

In the previous figure, the power consumption at each resistor is found by taking the current flow through each resistor and multiplying it by the voltage drop.

3.3.4 Example: Solving for Equivalent Capacitor

In the example below, find the equivalent capacitors:

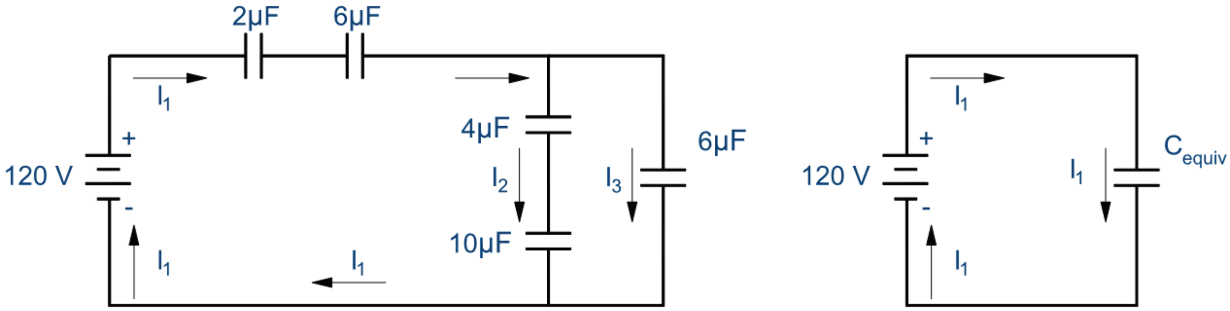


Figure 19: Example for Finding Equivalent Capacitance

$$C_{equiv} = \frac{1}{\frac{1}{2\mu F} + \frac{1}{6\mu F} + \left(\frac{1}{1/4\mu F + 1/10\mu F} + 6\mu F\right)} = 1.28\ \mu F$$

3.3.5 Example: Solving for Equivalent Inductors

In the example below, find the equivalent inductor:

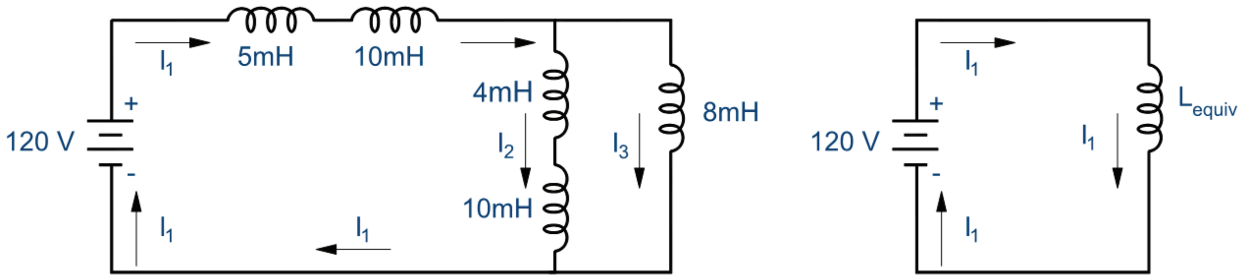

I1
5mH
10mH
4mH
120 V
+
-
I2
I3
8mH
10mH
I1
I1
I1
120 V
+
-
Lequiv
I1
I1

Figure 20: Example of Finding Equivalent Inductance

$$L_{equiv} = 5mH + 10mH + \frac{1}{\frac{1}{4mH + 10mH} + \frac{1}{8mH}} = 20.1mH$$

3.3.6 Example: Solving for Resistors in Series

Once you have a single equivalent resistance value, then you can determine the current flow through the circuit. The main concepts to understand are as follows:

(1) Current is constant through a series circuit and (2) resistances are summed to determine equivalent resistance.

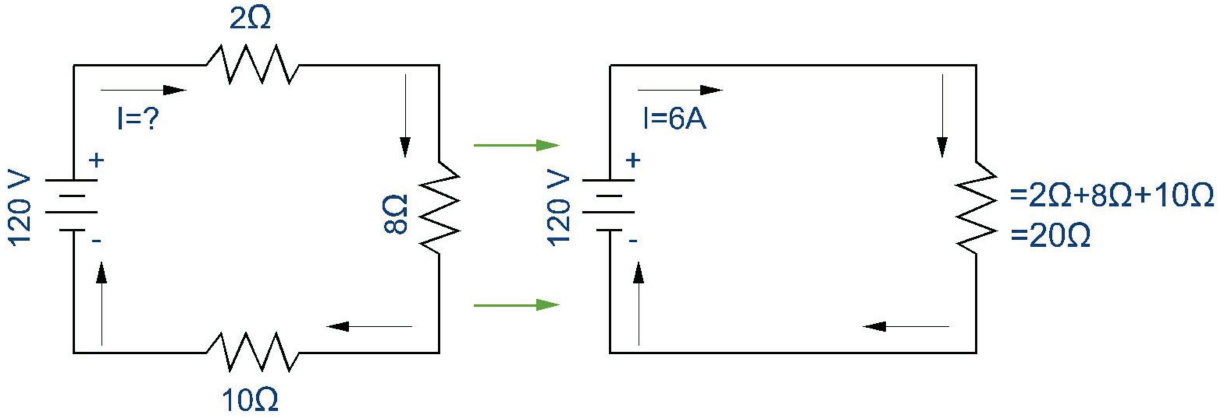

2Ω
I=?
+
120 V
-
8Ω
10Ω
I=6A
+
120 V
-
=2Ω+8Ω+10Ω
=20Ω

Figure 21: The equivalent resistance for resistors in series is found by adding the resistances.

$$R_{equivalent\ series} = R_1 + R_2 + \ldots R_n$$

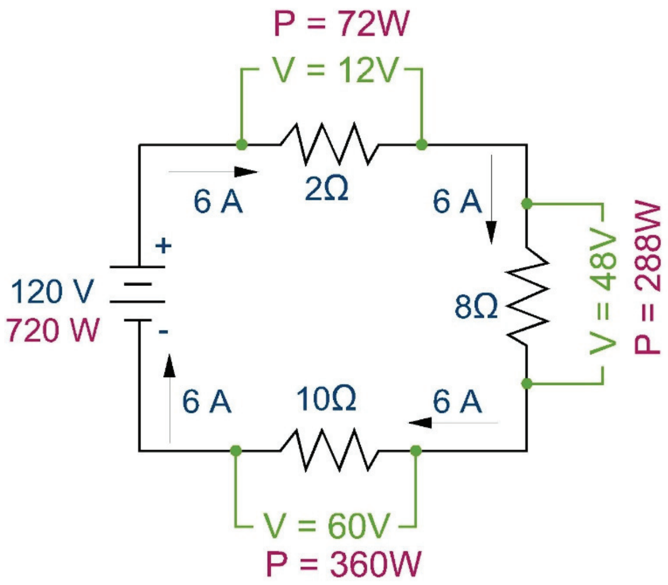


Figure 22: Voltage drop and power use in a series circuit

- Voltage Drop

In the previous figure, the voltage drop is found through each resistor by taking the current flow and multiplying it by the resistance. As you can see, the voltage drops add up to 120V in accordance with KVL and the voltage drop magnitude varies directly with the resistance level in accordance with Ohm's law.

- Power Consumption

In the previous figure, the power consumption is found through each resistor by taking the current flow and multiplying it by the voltage drop.

4.0 AC CIRCUIT ANALYSIS

Alternating current is most commonly used on the FE exam and in most power applications. Alternating current describes the alternating directions of flow in a circuit. Current quickly alternates flow direction from positive to negative many times a second. In the figure below, positive current is shown flowing in a clockwise direction in the figure on the right and this flow direction corresponds to the positive portions of the graph on the left.

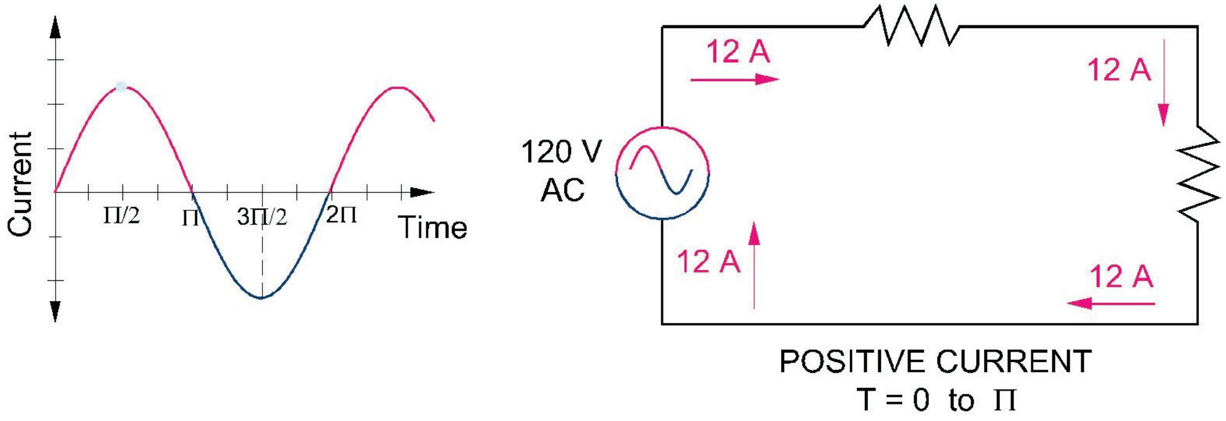

Current
Π/2
Π
3Π/2
2Π
Time
12 A
12 A
120 V
AC
12 A
12 A
POSITIVE CURRENT
T = 0 to Π

Figure 23: Alternating current consists of positive and negative flowing current. This figure shows positive current flow.

In this second figure, negative current is shown flowing in a counter-clockwise direction. The current flow in the figure on the right corresponds to the negative portions of the graph on the left.

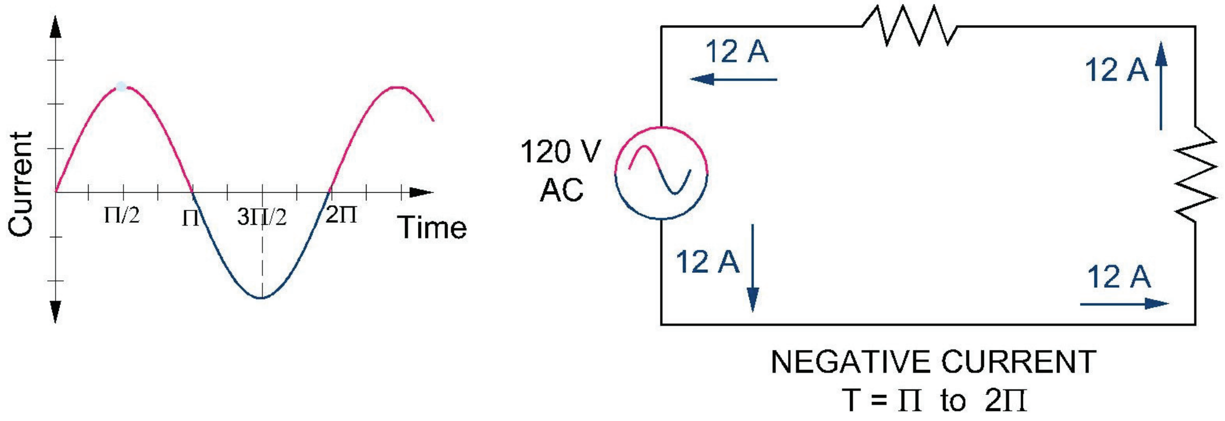

Current
Π/2
Π
3Π/2
2Π
Time
12 A
12 A
120 V
AC
12 A
12 A
NEGATIVE CURRENT
T = Π to 2Π

Figure 24: Alternating current consists of positive and negative flowing current. This figure shows negative current flow.

4.1 FREQUENCY

The frequency of an alternating current wave is the number of cycles completed in one second and is given in terms of hertz. For example, in the USA, the standard frequency for alternating current is 60 hertz (HZ). This means that 60 cycles are completed in one second. In Europe, the standard is 50 hertz (HZ). In the previous figures, 1 second would correspond to the 2π value, where the one cycle is completed.

$$frequency\ f\ (HZ) = \frac{cycles}{second}$$

Another term that is closely related to frequency is angular frequency. Angular frequency is the rotational frequency of alternating current and is given in terms of radians per second. For example, if one rotation is completed in one second, then the angular frequency is 2π. Angular frequency is related to physical frequency through the following equation.

$$angular\ frequency\ \ \omega\ (radians/sec) = 2\pi f$$

4.2 RMS AND MAX

In an AC circuit, the voltage and current are constantly varying in magnitude over time. The RMS method is used to find the average or effective value of a constantly varying value. You may want to simply calculate the average of the value over time to get the effective value of the current or voltage, but in a simple AC circuit, the average will be zero because there are both positive and negative values. The term root mean square is found by taking the sum of "n" number of points along the waveform, squaring each value, summing all "n" values and then taking the square root of the sum.

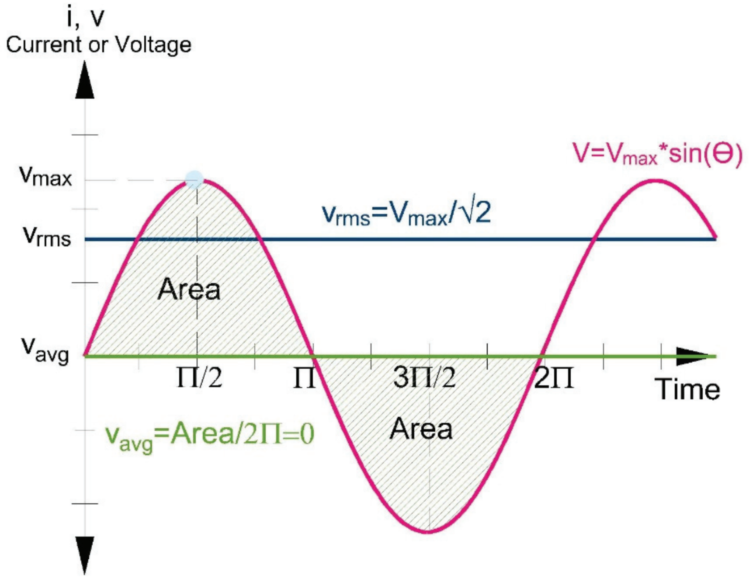


Figure 25: This figure shows the relationship between average, root mean square and maximum values in a sine wave.

The fundamental equation for calculating RMS is shown below. The variable "V" is used for voltage but this variable can also be exchanged for "I" for a current waveform.

$$RMS = \frac{\sqrt{V_1^2 + V_2^2 + V_3^2 \dots + V_n^2}}{n}$$

In a basic sinusoidal waveform, the RMS value can be mathematically determined for an infinite number of "n" points. This value is shown as 0.707 times the peak value of the sinusoidal graph.

$$RMS_{sine\ wave} = \frac{V_{max}}{\sqrt{2}}$$

4.3 COMPLEX NUMBERS

Complex numbers are the best way to represent AC circuits. Complex numbers consist of a real component and an imaginary component. These numbers are used to represent phasors in the electrical engineering field. For example, in terms of power, the real component represents real power and the imaginary component represents reactive power. The phasors are also used to represent resistance, inductance and capacitance.

First, you need to understand the two forms of representing complex numbers, (1) Rectangular form and (2) Polar form. Then you must learn the rules necessary to add/subtract and multiply/divide complex numbers quickly for the FE exam.

4.3.1 Rectangular Form

The rectangular form of a complex number is in the form, real plus the imaginary component. The imaginary component is multiplied by j, which is defined as the square root of negative 1.

$$Rectangular\ form:\ x + jy$$

$$x = real\ component; y = imaginary\ component$$

$$Example\ A\ \rightarrow 5 + j2\ or\ Example\ B \rightarrow\ -1 - j5$$

The rectangular form is used when adding and subtracting phasors and follows the same rules as normal addition and subtraction, where only like terms can be added and subtracted. For example, "Example A" plus "Example B", is solved with the following process.

$$Example\ A + Example\ B = (5 + j2) + (-1 - j5)$$

$$(5 - 1) + (j2 - j5) \rightarrow 4 - j3$$

The rectangular form can also be understood via graphical format, where the x-axis represents the real component and the y-axis represents the imaginary component.

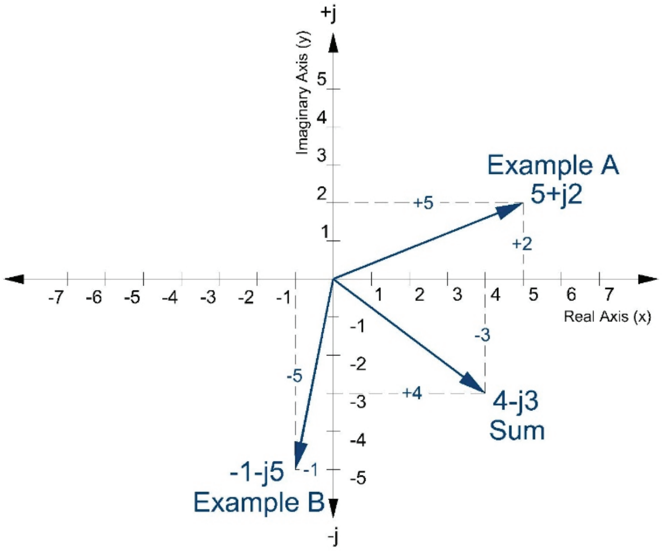


Figure 26: Example A phasor, example B phasor and the sum of the two phasors is shown in the above graph.

4.3.2 Polar Form

The polar form is best understood in its graphical format. The format consists of a phasor magnitude at a phasor angle relative to the x-axis.

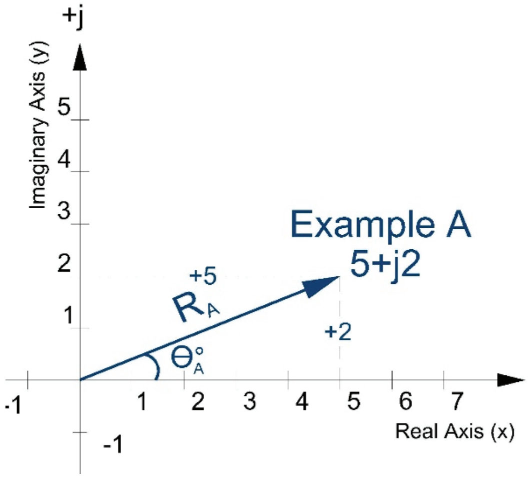


Figure 27: A phasor shown in both polar and rectangular form.

$$Polar\ form\colon\ R\angle\theta°\ ;\ R = phasor\ magnitude;\ \theta = phasor\ angle\ relative\ to\ x - axis$$

$$Example\ A\colon\ R_A = \sqrt{5^2 + 2^2} = 5.4;\ \ \theta_A = \tan^{-1}\left(\frac{2}{5}\right) = 21.8°$$

In the above example, the polar form is converted from the rectangular form by using the Pythagorean Theorem to find the radius (i.e. the magnitude) and the inverse tangent to find the angle. The polar form is not typically used for adding or subtracting, but it is used for multiplication and division. When multiplying or dividing two polar forms, you must multiple/divide the radiuses and add or subtract the angles. If the polar forms are being multiplied, then you must add the angles and if you are dividing one polar form from another then you subtract the divisor from the dividend.

$$5.5\angle 45^\circ \div 1.1\angle 85^\circ \rightarrow \left(\frac{5.5}{1.1}\right)\angle(45-85^\circ) \rightarrow 5\angle-40^\circ$$

4.4 RESISTANCE, INDUCTANCE, CAPACITANCE AND IMPEDANCE

In the DC topic, you learned about resistors which resist current flow. In this topic, you will learn about additional devices that also resist current flow, inductors and capacitors. But first resistors shall be covered again with the context of rectangular and polar forms.

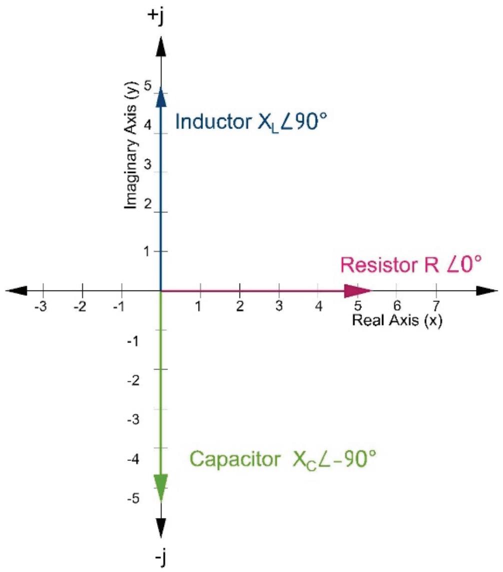

+j
Imaginary Axis (y)
5
4
3
2
1
Inductor $X_L\angle 90^\circ$
Resistor R $\angle 0^\circ$
-3
-2
-1
1
2
3
4
5
6
7
Real Axis (x)
-1
-2
-3
-4
-5
Capacitor $X_C\angle -90^\circ$
-j

Figure 28: A resistor, inductor and capacitor on a phasor diagram

4.4.1 Resistance (Resistors)

Resistance describes the resistors ratio of voltage to current in units of ohms. In the Power Engineering field, resistors are used to describe any electrical load that does useful work. A resistive load can be any light (incandescent, fluorescent, LED, etc.), any motor, any electronic device (computers, cellphone, printers, copy machines) and much more. These resistive loads are true power users because current is dissipated at the load and results in actual work (i.e. motor spins a rotor and a light illuminates). The motor load also has inductive loads, which will be described in the next set of paragraphs.

$$R = Resistance\ (ohms) = \frac{V}{I}$$

In the previous figure, the phasor diagram shows resistance as $R\angle 0°$, which is characterized as a positive vector on the real axis.

4.4.2 Inductors

An inductor is made up of a wire wound about a coil. As the current flows through the wound coil, the current builds a magnetic field. Inductive loads or inductive reactance are any loads that create a magnetic field. This load is not deemed useful since the current is not dissipated. The current builds the magnetic field and when the current is shut off, then the magnetic field dissipates. The inductive reactance, X_L, is found from the equation below.

$$X_L = wL = 2\pi f L$$

$$Z_L = jX_L$$

Where, ω (rad/s) = angular frequency, L (Henry) = Inductance, and f (Hz) =frequency

In the above figure, the phasor diagram shows inductive reactance as $X_L\angle 90°$, which is characterized as a positive vector on the imaginary axis.

4.4.3 Capacitors

A capacitor is made up of two conductors separated by a dielectric. As current is supplied to the capacitor on the positive side, charge is built up, which restricts current. Then the current alternates and charge is built up on the negative side, which also restricts current. It takes time to fill the capacitor with charge, which delays any sudden inrush of voltage. When charged, a capacitor acts similar to a battery. The capacitive reactance, X_C, is found from the equation below.

$$X_C = -\frac{1}{wC} = -\frac{1}{2\pi f C}$$

$$Z_c = jX_C$$

Where, ω (rad/s) = angular frequency, C (Farads) = Capacitance, and f (Hz) =frequency.

In the above figure, the phasor diagram shows capacitive reactance as $X_C\angle -90°$, which is characterized as a negative vector on the imaginary axis.

4.4.4 Impedance

The term impedance combines inductive, capacitive and resistive loads into a single term. Impedance is defined as the complex resistance or the resistance with both a magnitude and angle term in the polar format. In the equation below, impedance, "Z" is given as a function of the resistance, "R" and the reactance, "X".

$$Complex\ Impedance \rightarrow\ Z = R + jX$$

Impedance can also be shown in its phasor form.

$$Complex\ Impedance \rightarrow\ Z\angle\theta°$$

Now you will also have new equations for power and for Ohm's law, since resistance is no longer, "R", it is now replaced with impedance. When you have complex loads, remember to use these equations.

$$Ohm's\ Law \rightarrow\ V = IZ$$

$$Power \rightarrow P = I^2R\ ;\quad P = \frac{V^2}{R}$$

The following example solves for the equivalent impedance of an RLC circuit. Assume a frequency of 60 Hz.

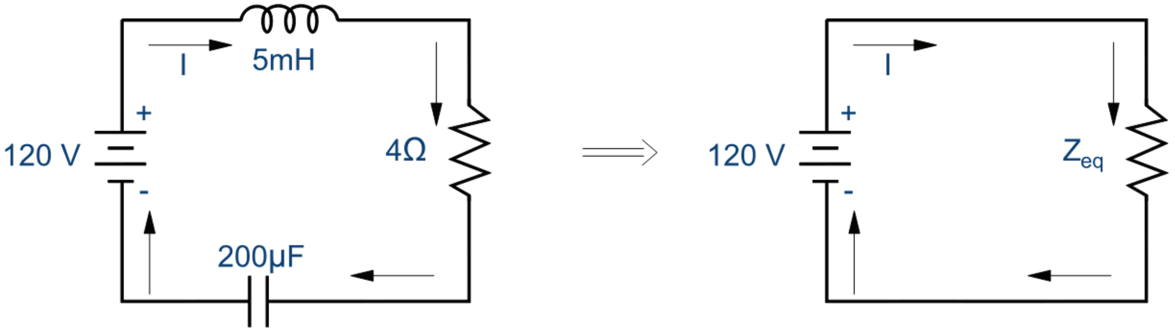


Figure 29: Example for Finding Equivalent Impedance

$$Z_{eq} = X_L + R + X_C = j2\pi fL + R - \frac{j}{2\pi fC}$$

$$Z_{eq} = (j2\pi * 60 * .005) + 4 - \frac{j}{2\pi * 60 * 6 * 10^{-6}} = (4 - 11j)\Omega$$

4.4.5 Apparent Power, Real Power, and Reactive Power

Real power is the power that is used for useful, actual work. This could be used to turn a motor, turn on a light or to power a computer. Real power is given in units of kilowatts (kW), watts (W) or megawatts (MW) and is designated by the variable, "P". Real power is consumed at any resistor, X_R. On a phasor diagram, real power is the power shown on the real axis. On a waveform diagram, real power is the instantaneous current multiplied by the instantaneous voltage at that same point in time. On the exam, you will not need to plot the waveforms to find power at a certain point in time, but you will have to find the effective real power, which will just be called real power through the following equation.

$$P = I * V * \cos(\theta)$$

$$I = RMS\ of\ current\ waveform; V = RMS\ of\ voltage\ waveform;\ \theta = power\ factor\ angle$$

Reactive power is developed by any reactor or inductor, X_L. Reactive power is not consumed, it moves back and forth in the circuit. Reactive power is given the variable, "Q" and is shown in units of kilovar (kVAR), megavar (mVAR) or var (VAR). Similar to real power, you will not need to plot the waveform of reactive power, but you will need to find the effective reactive power, which will just be called reactive power through the following equation.

$$Q = I * V * \sin(\theta)$$

$$I = RMS\ of\ current\ waveform; V = RMS\ of\ voltage\ waveform;\ \theta = power\ factor\ angle$$

The following diagrams are provided for you to use during the exam. The diagrams will allow you to quickly access any equations and visually see the concepts that were previously discussed. The first diagram includes the equations for apparent, real and reactive power.

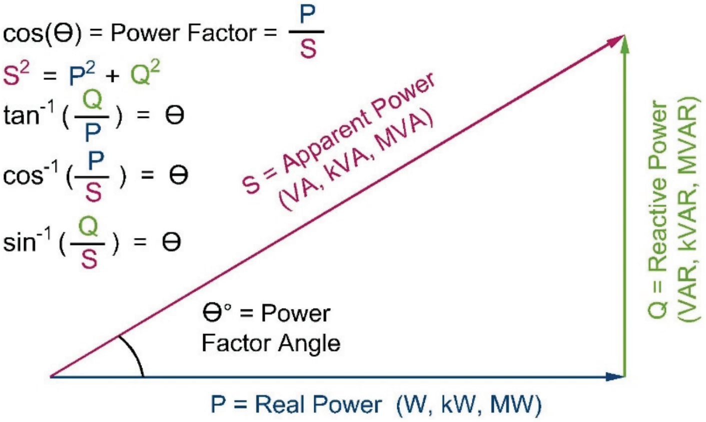

cos(Θ) = Power Factor = P/S
S² = P² + Q²
tan⁻¹ (Q/P) = Θ
cos⁻¹ (P/S) = Θ
sin⁻¹ (Q/S) = Θ
S = Apparent Power (VA, kVA, MVA)
Q = Reactive Power (VAR, kVAR, MVAR)
Θ° = Power Factor Angle
P = Real Power (W, kW, MW)

Figure 30: Power factor triangle

The second diagram shows the power factor triangle with leading and lagging power factors. It also includes the inductor and capacitor effect.

4.5 POWER FACTOR

The term power factor describes the relationship between the apparent power and real power. Power factor describes how much of the apparent power provided is being used for real power. In the equation below, P is real power, S is apparent power, and θ is the angle between the apparent power and real power vectors, also known as the power factor angle.

$$Power\ Factor = \frac{P}{S} = \cos(\theta)$$

A power factor of 1 means that the power is 100% real, while a power factor of 0 is 100% reactive. Good power factors are as close to 1 as possible.

5.0 MOTORS AND GENERATORS

On the FE exam, there is a topic titled motors and generators within the Electricity & Magnetism section. The equations in the *NCEES FE Reference Handbook* on this topic of motors are geared towards finding the motor size that is needed to power mechanical equipment like a fan or pump. The topic of generators is used to generate power, through the use of a turbine. The turbine is discussed in *Section 11.0 Thermodynamics*, but it is briefly introduced in this section too. The primary focus of this topic is on motors and finding the motor size that is necessary to power a fan or pump.

A generator uses the mechanical energy from rotation to produce alternating current electrical energy. Typically a turbine is used to provide the mechanical energy. Motors use alternating current electrical energy to produce mechanical energy in the form of rotation.

When selecting mechanical equipment, the mechanical engineer must coordinate the power requirements with the electrical engineer. This is done through the following four steps: (1) Determine Mechanical Horsepower, (2) Determine Fan/Pump Brake horsepower, (3) Determine Motor Horsepower and finally (4) Determine Electrical Power.

Additional discussion on calculating fan and pump power is provided in the Fluid Mechanics section. The main objective of the following explanations is understanding how to derive the electrical power delivered to the motor. The following equations are provided in English units with embedded conversion factors. For equations in SI units, see the Fluid Mechanics section.

5.1 MECHANICAL HORSEPOWER (HP)

Mechanical HP is defined as the amount of power required to meet the needs of the system in question. For example, if the mechanical equipment was a pump, then the mechanical HP would be the amount of power generated by a certain GPM at a certain pressure. For a fan, the mechanical work would be the amount of power generated by a certain amount of CFM at a certain pressure. These equations can be found below.

$$\boldsymbol{P_{mech\,work,pump[HP]} = \frac{h_{ft} * Q_{gpm} * (SG)}{3956}};$$

$$h = hydraulic\ head[ft], Q = flow\ rate\ [gpm], SG = specific\ gravity$$

$$\boldsymbol{P_{mech\,work,fan[HP]} = \frac{Q_{cfm} * TP_{in\,wg}}{6356}};$$

$$TP = total\ pressure\ [in\ wg], Q = flow\ rate\ [cfm]$$

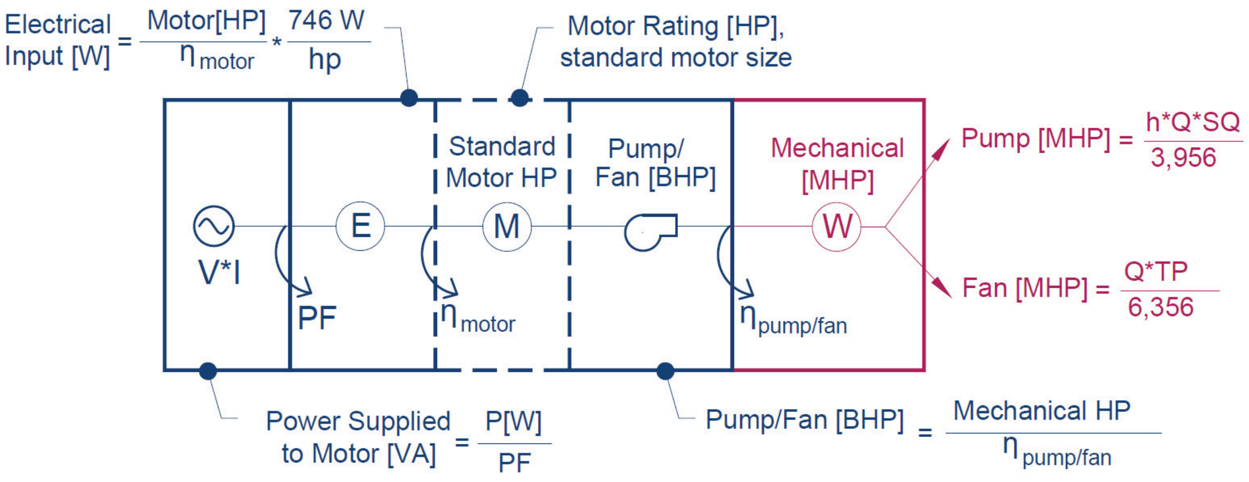

Electrical Input [W] = Motor[HP] / η motor * 746 W / hp
Motor Rating [HP], standard motor size
Standard Motor HP
Pump/ Fan [BHP]
Mechanical [MHP]
Pump [MHP] = h*Q*SQ / 3,956
Fan [MHP] = Q*TP / 6,356
V*I
E
M
W
PF
η motor
η pump/fan
Power Supplied to Motor [VA] = P[W] / PF
Pump/Fan [BHP] = Mechanical HP / η pump/fan

Figure 31: Determining mechanical horsepower

In the figure above, mechanical horsepower is shown on the right in red. Two equations are shown for solving for the mechanical horsepower for a fan and pump, based on flow and pressure. *There are other Mechanical horsepower formulas presented in the NCEES Mechanical PE Reference Handbook based on other given data. You should have these equations ready for the exam.*

5.2 FAN/PUMP HORSEPOWER.

In order to blow air or pump a fluid, a pump or fan is required. This pump/fan is not perfect and cannot supply the mechanical HP required without losing energy due to friction and inefficiencies in the equipment design. Thus the Pump/Fan Horsepower is found by dividing the Mechanical HP by the efficiency of the Pump/Fan. This is the size of the pump or fan required. Often times the fan/pump horsepower is called the brake horsepower or BHP.

$$\boldsymbol{P_{fan/pump[BHP]} = \frac{P_{mech\ work[MHP]}}{\eta_{fan/pump}}};$$

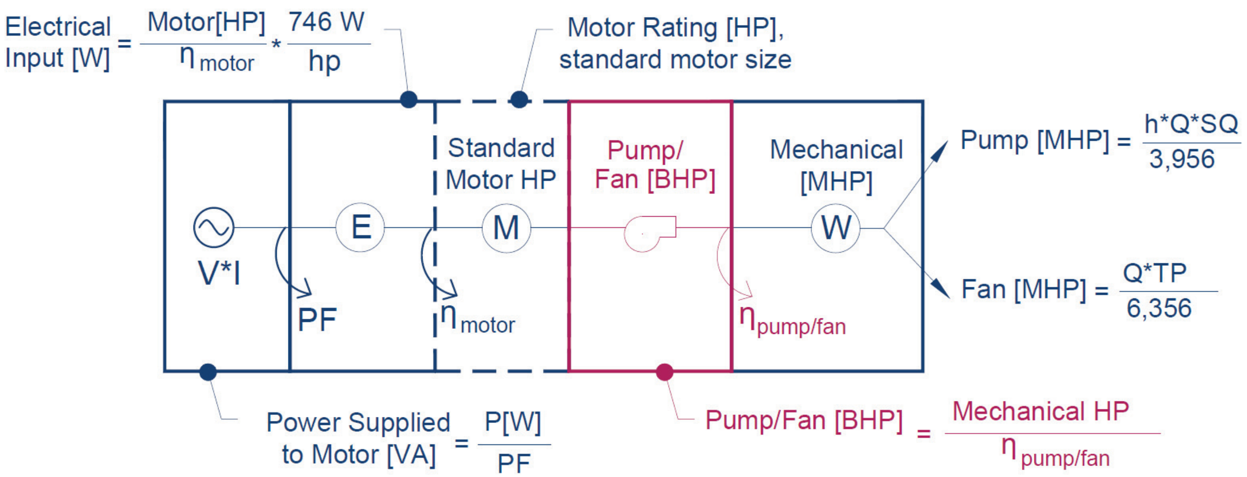

Electrical Input [W] = Motor[HP] / η motor * 746 W / hp
Motor Rating [HP], standard motor size
Standard Motor HP
Pump/ Fan [BHP]
Mechanical [MHP]
Pump [MHP] = h*Q*SQ / 3,956
Fan [MHP] = Q*TP / 6,356
E
M
W
V*I
PF
η motor
η pump/fan
Power Supplied to Motor [VA] = P[W] / PF
Pump/Fan [BHP] = Mechanical HP / η pump/fan

Figure 32: Determining Pump/Fan Horsepower

In the figure above, pump/fan horsepower is shown in the middle in red. Given the mechanical horsepower required, the pump/fan horsepower is found by dividing the Mechanical Horsepower by the efficiency of the Pump/Fan. Typical efficiencies for pumps and fans can range from 60% to 90%. The efficiencies depend on the design of the equipment and the operating point.

5.3 (STANDARD) MOTOR HORSEPOWER RATING

In order to power the pump/fan, a motor is required to supply the power to the equipment. The horsepower rating of the motor is based on the available motor sizes from the manufacturer, which are given in standard increments. These standard motor ratings are discussed in the next section, "Motor Ratings." If the question does not explicitly ask for the standard horsepower rating, you may typically skip this step.

- **Standard Rating Excluded**: If the standard motor horsepower rating is not accounted for, the power calculated is the actual power consumed during operation. This is typically done for energy calculations, where the engineer is concerned about actual electrical consumption, and not rated criteria.
- **Standard Rating Included**: If the standard motor horsepower rating is accounted for, the question is typically asking for a motor selection. Standard horsepower ratings are also used in cooling load calculations when usage and load factors are given, see *Heating & Cooling Loads* chapter of this book.

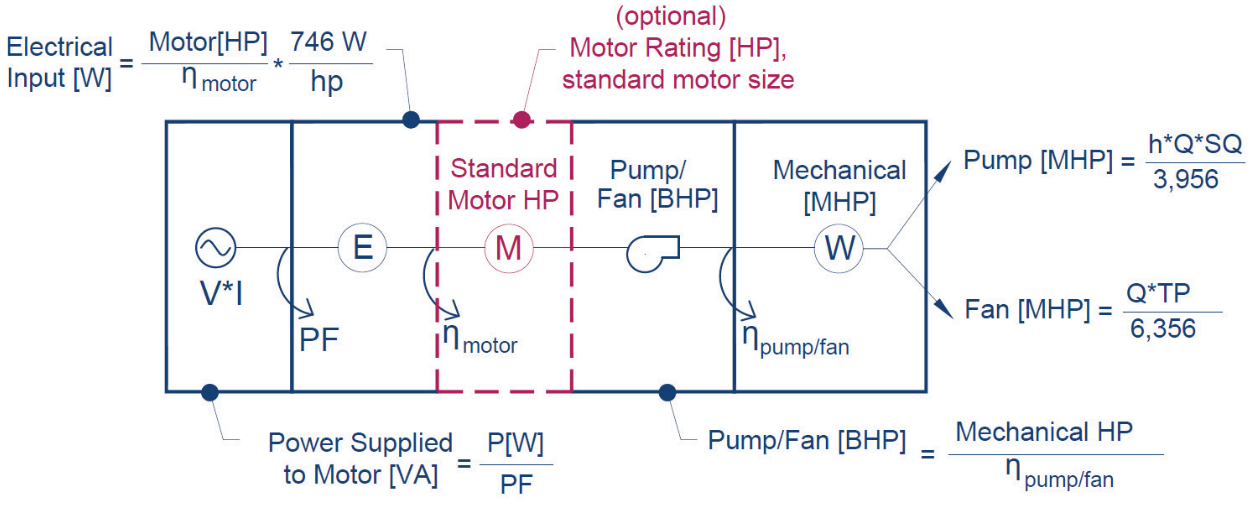

Electrical Input [W] = Motor[HP] / η motor * 746 W / hp
(optional)
Motor Rating [HP],
standard motor size
Standard Motor HP
Pump/ Fan [BHP]
Mechanical [MHP]
V*I
E
M
W
PF
η motor
η pump/fan
Pump [MHP] = h*Q*SQ / 3,956
Fan [MHP] = Q*TP / 6,356
Power Supplied to Motor [VA] = P[W] / PF
Pump/Fan [BHP] = Mechanical HP / η pump/fan

Figure 33: Determining Standard Motor Horsepower

5.4 ELECTRICAL INPUT (WATTS)

The motor is not perfect, similar to the pump/fan and some power is lost due to friction. The electrical power delivered to the motor is found by dividing the standard motor horsepower (or the Pump/Fan brake horsepower if the standard rating is not used) by the efficiency of the motor. This is also known as the real, or usable, electrical power. Real electrical power is typically given in watts, typical efficiencies of motors are around 90%.

$$\boldsymbol{P_{electrical[W]}} = \frac{\boldsymbol{P_{motor[HP]}}}{\boldsymbol{\eta_{motor}}} * \frac{\boldsymbol{746W}}{\boldsymbol{hp}} [standard\ motor\ HP\ used]$$

$$\boldsymbol{P_{electrical[W]}} = \frac{\boldsymbol{P_{fan/pump[BHP]}}}{\boldsymbol{\eta_{motor}}} * \frac{\boldsymbol{746W}}{\boldsymbol{hp}}\ [standard\ motor\ HP\ not\ used]$$

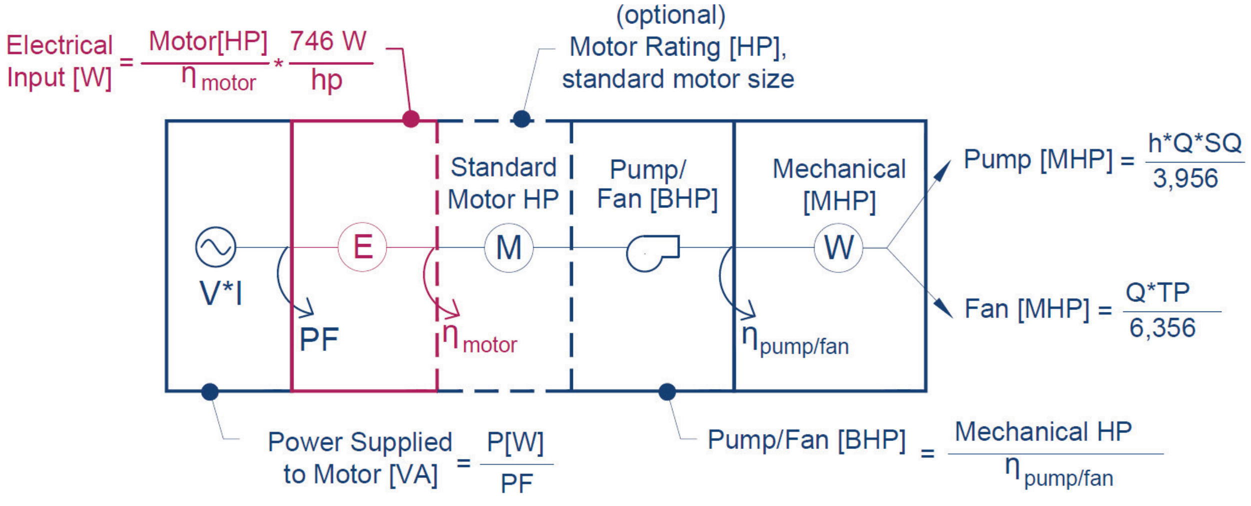
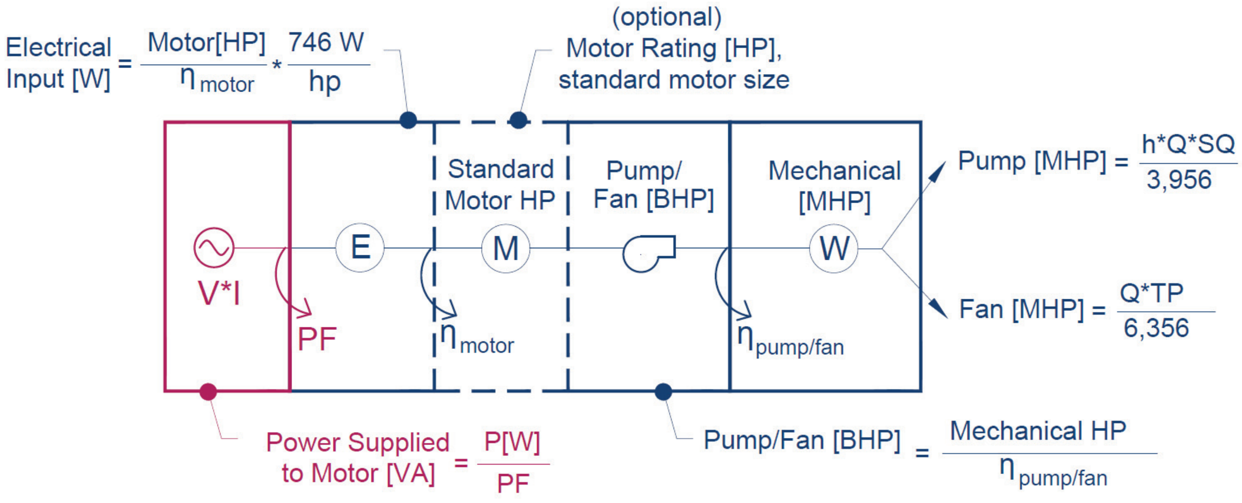

Electrical Input [W] = Motor[HP] / η motor * 746 W / hp
(optional)
Motor Rating [HP],
standard motor size
Standard Motor HP
Pump/ Fan [BHP]
Mechanical [MHP]
V*I
E
M
W
PF
η motor
η pump/fan
Pump [MHP] = h*Q*SQ / 3,956
Fan [MHP] = Q*TP / 6,356
Power Supplied to Motor [VA] = P[W] / PF
Pump/Fan [BHP] = Mechanical HP / η pump/fan

Figure 34: Determining (Real) Electrical Input Power

5.5 ELECTRICAL POWER

In order provide power to the motor, electrical wiring is connected to the motor from a power source. This power source provides the necessary current at the correct voltage of the motor. However, as mentioned in a previous section, the current and voltage supplied to the equipment is not completely in phase, so not all of the power supplied is useful (real power). The amount of electrical apparent power required is found through the use of the power factor. Apparent power is given the variable, "S" and units VA. Real power is given the variable, "P".

$$\boldsymbol{Apparent\ Power \rightarrow S_{supplied\ to\ motor[VA]} = \frac{P_{electrical[W]}}{PF}}$$



Electrical Input [W] = Motor[HP] / η motor * 746 W / hp
(optional) Motor Rating [HP], standard motor size
Standard Motor HP
Pump/ Fan [BHP]
Mechanical [MHP]
Pump [MHP] = h*Q*SQ / 3,956
Fan [MHP] = Q*TP / 6,356
V*I
PF
E
M
W
η motor
η pump/fan
Power Supplied to Motor [VA] = P[W] / PF
Pump/Fan [BHP] = Mechanical HP / η pump/fan

Figure 35: Determining electrical power.

In the previous figure, the power supplied to the motor is shown on the left in red. The power supplied to the motor is greater than the "real" electrical input because of the power factor term. Not all of the power supplied to the motor is useful, thus additional power must be provided. The actual power supplied to the motor is found by dividing the electrical input by the power factor in order to get the apparent power value.

The electrical amperage can be determined from the following equations, depending on whether or not the problem indicates you have a 3-phase or 1-phase motor.

$$1 - Phase\ Motor \rightarrow S\ (VA) = \frac{P(W)}{PF} = Amperage\ (A) * Voltage(V)$$

$$3 - Phase\ Motor \rightarrow S\ (VA) = \frac{P(W)}{PF} = Amperage\ (A) * Voltage(V) * \sqrt{3}$$

6.0 PRACTICE EXAM PROBLEMS

6.1 PRACTICE PROBLEM 1 - MOTORS

Background: A 460 V, 1 phase, 60 HZ, 20 BHP pump. The motor has an efficiency of 75%. There is a power factor of 0.85.

Problem: What is the current in amperes supplied to the motor?

(a) 29

(b) -38

(c) 44

(d) 51

6.2 PRACTICE PROBLEM 2 - MOTORS

Background: A 10 BHP fan operates for 4000 hours in the year. The motor is 85% efficient and the power factor is 0.85. Energy cost is $0.25 per kilowatt-hour.

Problem: How much does it cost to operate the fan in one year?

(a) $7,460

(b) $8,770

(c) $10,320

(d) $12,140

6.3 PRACTICE PROBLEM 3 – SERIES CIRCUIT

Background: A new electrical circuit has 3 resistors in series. Each resistor has a resistance of 4 Ω. The circuit is powered by a 12 V battery.

Problem: What is the current through the circuit?

(a) 0.5 amps

(b) 1.0 amps

(c) 2.0 amps

(d) 3.0 amps

6.4 PRACTICE PROBLEM 4 – PARALLEL CIRCUIT

Background: A new electrical circuit has 2 resistors in parallel. One resistor has a resistance of 4 Ω. The resistance of the other resistor is unknown. The circuit is powered by a 12 V battery.

Problem: If the total current through the circuit is 8 amps, what is the resistance of the 2nd resistor?

(a) 1.2

(b) 1.6

(c) 2.0

(d) 2.4

6.5 PRACTICE PROBLEM 5 – AC CIRCUIT

There is a 480V and 20 A, single phase, load. What is the real power of this load, if the power factor is 0.85? Assume the voltages and currents are root mean square values.

(A) 5.1 kW

(B) 8.2 kW

(C) 11.9 kW

(D) 25.2 kW

7.0 SOLUTIONS

7.1 SOLUTION 1 - MOTORS

Background: A 460 V, 1 phase, 60 HZ, 20 BHP pump. The motor has an efficiency of 75%. There is a power factor of 0.85.

Problem: What is the current in amperes supplied to the motor?

Convert pump brake horsepower to motor horsepower.

$$P_{motor[HP]} = \frac{P_{pump[BHP]}}{\varepsilon_{motor}}$$

$$P_{motor[HP]} = \frac{20\ BHP}{0.75} = 26.7\ HP$$

Determine the amount of power supplied to the motor, use power factor.

$$S_{supplied\ to\ motor} = \frac{P_{motor[HP]}}{PF} = 26.7\ HP * \frac{0.7457\ KW}{HP} * \frac{1}{0.85} = \ 23.42\ KVA$$

Find the current supplied to the motor through the below equation.

$$S_{supplied\ to\ motor,watts} = I * V$$

Make sure to convert horsepower to watts.

$$S_{supplied\ to\ motor,watts} = \ 23.42\ KVA * \frac{1000\ volt - amperes}{KW} = 23{,}423\ VA$$

$$23{,}423\ VA = I * 460\ V$$

$$50.9\ amps = I$$

Correct Answer: (D) 51 Amperes.

7.2 SOLUTION 2 - MOTORS

Background: A 10 BHP fan operates for 4000 hours in the year. The motor is 85% efficient and the power factor is 0.85. Energy cost is $0.25 per kilowatt-hour.

Problem: How much does it cost to operate the fan in one year?

Convert fan brake horsepower to motor horsepower.

$$P_{motor[HP]} = \frac{P_{pump[BHP]}}{\varepsilon_{motor}}$$

$$P_{motor[HP]} = \frac{10\ BHP}{0.85} = 11.8\ HP$$

Determine the amount of power supplied to the motor.

Find the kilowatt-hours (kwh) consumed.

$$P_{supplied\ to\ motor,watts} = I * V$$

$$11.8\ HP * \frac{0.7457\ KW}{HP} * 4{,}000 \frac{hrs}{year} = 35{,}092\ kwh$$

Find the cost (\$) with the electricity cost.

$$35{,}092\ kwh * \frac{\$0.25}{kwh} = \$8{,}772$$

Correct Answer: (B) $8,770

7.3 SOLUTION 3 – SERIES CIRCUIT

Background: A new electrical circuit has 3 resistors in series. Each resistor has a resistance of 4 Ω. The circuit is powered by a 12 V battery.

Problem: What is the current through the circuit?

$$R_{eq,series} = 4 + 4 + 4 = 12$$

$$I = \frac{12}{12} = 1\ amp$$

Correct Answer: (B), 1 amp

7.4 SOLUTION 4 – PARALLEL CIRCUIT

Background: A new electrical circuit has 2 resistors in parallel. One resistor has a resistance of 4 Ω. The resistance of the other resistor is unknown. The circuit is powered by a 12 V battery.

Problem: If the total current through the circuit is 8 amps, what is the resistance of the 2nd resistor?

Determine the current through the 1st resistor.

$$I = \frac{12}{4} = 3\ amp$$

Determine the current through the 2nd resistor.

$$Total\ current - current\ through\ 1st\ resistor = current\ through\ 2nd\ resistor$$

$$8\ amps - 3\ amps = 5\ amps$$

Determine the resistance of the 2nd resistor.

$$R = \frac{12}{5} = 2.4$$

(a) 1.2

(b) 1.6

(c) 2.0

(d) 2.4

Correct Answer: (D) 2.4 Ω

7.5 SOLUTION 5 - AC CIRCUIT

There is a 480V and 20 A, single phase, load. What is the real power of this load, if the power factor is 0.85? Assume the voltages and currents are root mean square values.

For this question, you must use the real power equation which is shown below.

$$P = PF * V_{rms} * I_{rms}$$

$$P = 0.85 * 480\ V * 20\ A$$

$$P = 8{,}160\ W$$

The correct answer is most nearly, **(B) 8.2 kW**.

6 - Statics

MECHANICAL FE TEXTBOOK

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Section 6.0 – Statics

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1.0 INTRODUCTION

Statics accounts for approximately 9 to 14 questions on the Mechanical FE exam. These questions can cover statics, but not dynamics. The statics topic on the NCEES exam is similar to a common statics college engineering class. Statics is the study of components at equilibrium, which means the components are at rest or at zero acceleration. This topic includes vectors, free body diagrams, moments, reaction forces, first moment of area, static friction and second moment of area. These concepts and skills are used to solve problems on pulleys, cables, springs, beams, trusses, frames, etc.

The *NCEES FE Reference Handbook* Statics section has some basic equations for the topics below, but it does not explain the skills and concepts necessary to use these equations. You should learn the skills and concepts presented in this section and go through the handbook to confirm that you know how to use the basic equations. You may also need to know some of the Mathematics equations like law of cosines and other trigonometry equations presented in the Mathematics section. The handbook also presents screw threads but this is covered more in *Section 14.0 Mechanical Design and Analysis* in this book.

Section 6.0 Statics (9 to 14 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
6A	Section 2.0	Resultants of Force Systems
6B	Section 3.0	Concurrent Force Systems
6C	Section 4.0	Equilibrium of Rigid Bodies
6D	Section 5.0	Frames and Trusses
6E	Section 6.0 & 7.0	Centroids & Moments of Inertia
6F	Section 8.0	Static Friction
	Section 9.0	Practice Exam Problems

2.0 RESULTANTS OF FORCE SYSTEMS

As previously stated, statics is the study of mechanical components at equilibrium, which means the components have zero acceleration or are at rest. The material presented on statics focuses on the key equations and skills necessary to complete the possible problems within this topic on the FE exam.

2.1 VECTORS

Vectors are used to visualize forces and moments in this section and in *Section 8.0 Mechanics of Materials* and they are used to visualize movement in *Section 7.0 Dynamics, Kinematics and*

Vibration. This section will focus on vectors being used to describe the magnitude and direction of force or moment. On the FE exam you must be able to translate words or diagrams into force/moment vectors and you must be able to add/subtract vectors and multiply/divide vectors by scalars.

Vectors can either be represented in a rectangular form or a polar form. The rectangular form consists of an x-component and a y-component. These values are used to represent the magnitude in the x and y directions. In real applications, there will also be a z-component, but for the purposes of the exam you most likely will only need the x and y components. The polar form is shown as a magnitude and an angle. The magnitude describes the length of the vector while the angle determines the direction.

Vector forms are discussed in the Mathematics section of this book, but is repeated here for completeness.

2.1.1 Rectangular Form

The rectangular form is shown as, x plus the y component. The y component is shown as j and the x component is shown as i.

$$Rectangular\ form\colon\ xi + yj$$

$$x = x -\ component; y = y -\ component$$

$$Example\ A\ \rightarrow 5i + 2j\ or\ Example\ B \rightarrow\ -1i - 5j$$

The rectangular form is used when adding and subtracting vectors and follows the same rules as normal addition and subtraction, where only like terms can be added and subtracted. For example, “Example A” plus “Example B”, is solved with the following process.

$$Example\ A + Example\ B\ = (5i + 2j) + (-1i - 5j)$$

$$(5i - 1i) + (2j - 5j) \rightarrow 4i - 3j$$

The rectangular form can also be understood via a graphical format, where the x-axis represents the real component and the y-axis represents the imaginary component.

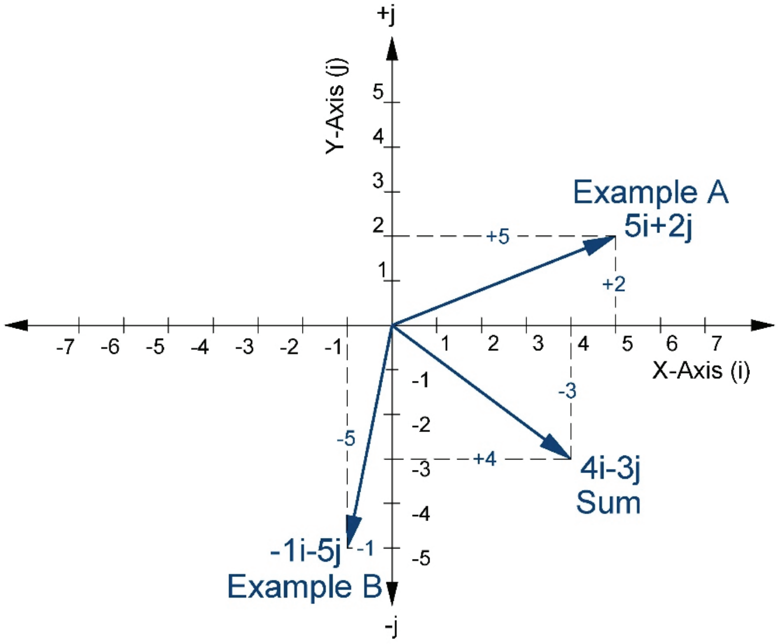

+j
Y-Axis (j)
5
4
3
2
1
-1
-2
-3
-4
-5
-j
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
X-Axis (i)
Example A
5i+2j
+5
+2
-3
+4
4i-3j
Sum
-5
-1
-1i-5j
Example B

Figure 1: Example "A" vector, example "B" vector and the sum of the two vectors is shown in the above graph.

2.1.2 Polar Form

The polar form is best understood in its graphical format. The format consists of a phasor magnitude at a phasor angle relative to the x-axis.

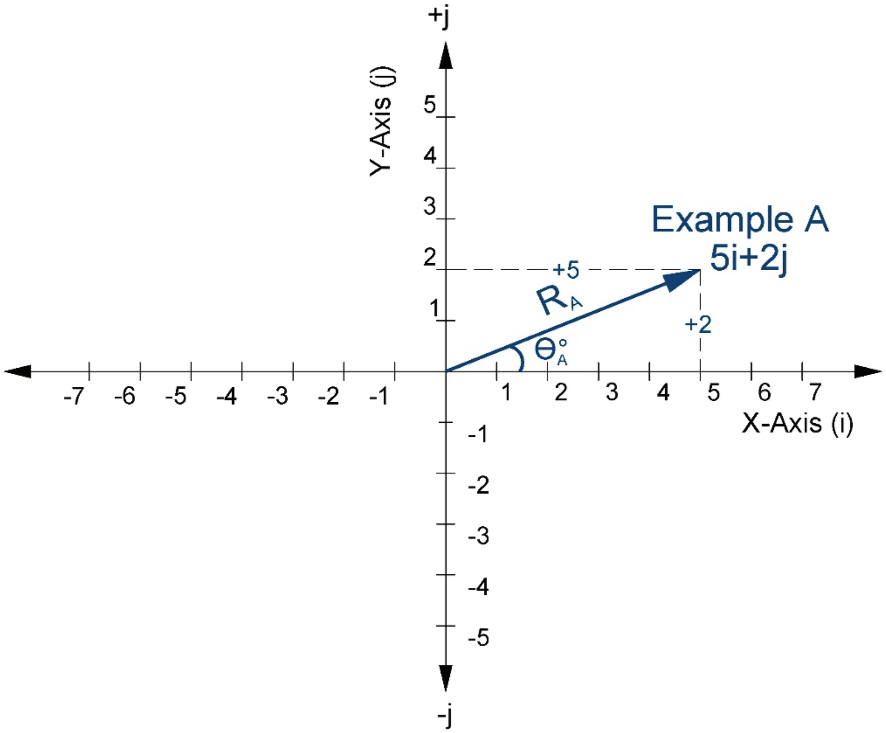


Figure 2: A phasor shown in both polar and rectangular form.

$$Polar\ form:\ R\angle\theta^\circ\ ;\ R = phasor\ magnitude;\ \theta = phasor\ angle\ relative\ to\ x-axis$$

$$Example\ A:\ R_A = \sqrt{5^2 + 2^2} = 5.4;\ \ \theta_A = \tan^{-1}\left(\frac{2}{5}\right) = 21.8^\circ$$

In the above example, the polar form is converted from the rectangular form by using the Pythagorean Theorem to find the radius (i.e. the magnitude) and the inverse tangent to find the angle. The polar form is not typically used for adding or subtracting, but it is used for multiplication and division. When multiplying or dividing two polar forms, you must multiple/divide the radiuses and add or subtract the angles. If the polar forms are being multiplied, then you must add the angles and if you are dividing one polar form from another then you subtract the divisor from the dividend.

$$5.5\angle 45^\circ\ \div\ 1.1\angle 85^\circ\ \ \rightarrow\ \ \left(\frac{5.5}{1.1}\right)\angle(45-85^\circ)\ \ \rightarrow\ \ 5\angle -40^\circ$$

2.1.3 Converting Polar and Rectangular Forms - Calculator

During the exam, you will need to convert from polar form to rectangular form and vice versa. You will need to convert between the two forms in order to carry out multiplication/division or addition/subtraction. You should be able to quickly convert between the forms with your calculator. This will help to save you time for more difficult tasks during the exam.

Discussion on using your calculator to convert polar and rectangular forms is included in the Mathematics section of this book.

2.1.4 Vector Mathematical Operations

Discussion of adding, subtracting, multiplying, and dividing vectors are provided in the Mathematics section of this book.

2.2 FORCE

Force is the action that pushes or pulls an object. The force can be invisible like gravity, electrical or magnetic. In mechanical design, the only invisible force that is tested is gravity. Visible forces due to pulleys, gears, hydraulics and much more are tested on the Mechanical FE exam.

One of the first set of concepts you learn in Statics are Newton's Three Laws of Motion. The application of these laws is tested on the exam and it is assumed that you learned these laws and other fundamental Statics concepts in college/university.

First Law: The first law states that an object at rest or at constant velocity (zero acceleration) will remain that way, until an unbalanced force acts upon the object.

Second Law: The second law states that an object subject to an unbalanced force will be subject to an acceleration that is directly proportional to the unbalanced force and inversely proportional to the object's mass.

$$F = ma$$

Third Law: The third law states that objects that are in contact with each other will experience a force opposite the subject force acting upon the object. This force is called the reaction force. This law is useful in the beam type problems.

The units of force may be given in US customary system (USCS) or the International System (SI) units for the FE exam. This means that force is measured in pounds (lbf) for USCS. Mass is given in terms of slugs $\left(\frac{lbf*s^2}{ft}\right)$ or in pound-mass (lbm). See the Fluid Mechanics section for discussions on pound-force and pound-mass. For SI units, force is measured in Newtons $\left(\frac{kg*m}{s^2}\right)$ and mass is given in terms of kilograms (kg). Make sure you know where to find the conversions between the two sets of units in your *NCEES FE Reference Handbook*.

3.0 CONCURRENT FORCE SYSTEMS

A concurrent force systems is a collection of force vectors acting upon a single point. A concurrent force system is assumed in the next topics, Equilibrium of Rigid Bodies and Frames & Trusses.

4.0 EQUILIBRIUM OF RIGID BODIES

Equilibrium of rigid bodies describes the problems that are at rest and all its moments and forces are balanced such that the sum of the moments and forces acting upon a rigid body is zero. A rigid body is defined as a mass that does not deform when a force acts upon it. This makes all the following equations simpler because you do not need to account for forces that are internalized within a mass.

$$\sum M = 0; \sum F_x = 0; \sum F_y = 0; \sum F_z = 0;$$

In order to determine if the forces and moments balance out, you will need to be able to draw a free body diagram to visualize the forces and moments acting upon a rigid body.

4.1 FREE BODY DIAGRAM

On the exam, there will be questions with diagrams and sometimes without diagrams. But one thing that will not be provided at any time on the exam will be the forces and directions of these forces. You will need to be able to draw a free body diagram of the forces acting upon an object in order to create the equilibrium equations that you will need to solve the problems. The first step is to assign a frame of reference and assign which direction is positive and negative. Typically, forces acting in the left direction (x-axis) and down direction (y-axis) and out of the page (z-axis) are assigned a negative force. Forces acting in the right direction (x-axis), up direction (y-axis) and into the page (z-axis) are assigned a positive force.

4.2 EQUILIBRIUM

If an object is not moving, then the object is in equilibrium. This means that the forces acting upon an object in the x, y and z directions will sum to equal zero. On the exam, you can use these equilibrium equations to solve for the unknowns.

$$\sum F_x = 0; \sum F_y = 0; \sum F_z = 0$$

4.3 SPRINGS

A good example of equilibrium equations are the equations developed for spring applications. When an external force acts upon a screen, there is an equal and opposite reaction produced by the spring. The spring force is a function of the stiffness constant of the spring and the displacement produced within the spring.

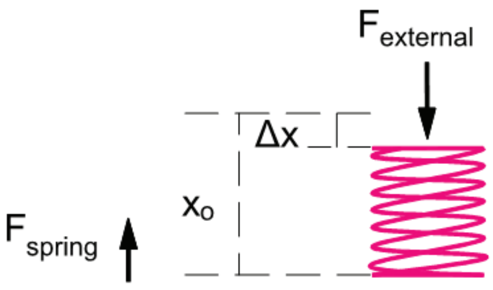


Figure 3: If the spring is not moving, then the external force will be equal to the spring force. The spring force will be equal to the spring's stiffness value multiplied by the displacement.

$$F_{spring} = F_{external}$$

$$k * \Delta x = F_{external};\ x_0 = original\ length$$

$$k = spring\ stiffness\ \left(\frac{lbf}{in}\quad or\quad \frac{N}{m}\right);\ \Delta x = displacement\ (in\quad or\quad m);$$
$$F_{external} = external\ force\ (lbf\quad or\quad N)$$

4.4 PULLEYS AND CABLES

Another example of developing equilibrium equations can be shown for pulleys and cables. A pulley is a wheel on an axis that is securely attached to a surface. The pulley is used to change the direction of the cable's force. A cable is used to transmit force (tension) from one end to another end of the cable. The important thing to remember is that the tension in a cable is the same at any location along the cable.

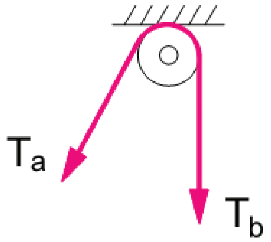


Figure 4: The tension in a cable is constant. T_a equals T_b.

At first glance, the downward forces do not balance out and it would appear that the cable and pulley are not in equilibrium. However, the pulley is securely attached to the upper surface. This produces reactionary forces that balance out the downward forces as shown in the free body diagrams below.

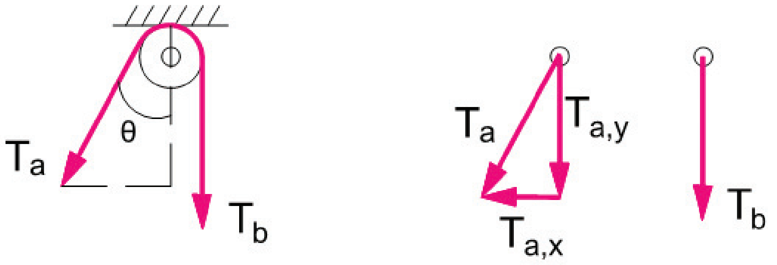


Figure 5: In this figure the downward forces acting on the pulley are due to T_a and T_b. T_b is purely in the y-direction, but Ta can be split up into its x and y directions.

The reactionary force that acts upon the pulley must be equal and opposite to these downward forces. This reactionary force is called “R” in the diagram below.

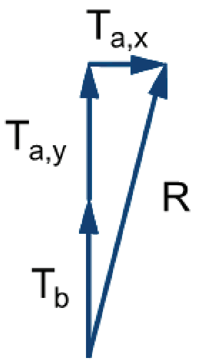


Figure 6: The reactionary force value acting in the upward y-direction is equal to the sum of T_b, $T_{a,y}$. The reactionary force value acting in the right x-direction is equal to $T_{a,x}$.

The spring and pulley/cable examples should give you an idea of the type of questions that will be asked on the FE exam with respect to static equilibrium when only forces are involved. The next section will introduce moment equilibrium equations.

4.5 REACTIONS

The following reaction diagrams visualizations can help you construct your free body diagrams. When you see these types of supports on the exam you should be able to determine what types of reaction forces will occur at the support. The first set of diagrams are in the isometric view to show that there can be x, y and z reaction forces, but for the purposes of the exam you will most likely only need to know the second set of reaction diagrams in two dimensions.

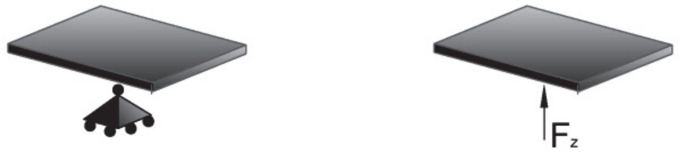


Figure 7: This is a simple or roller support that only provides reaction forces in the upward direction.

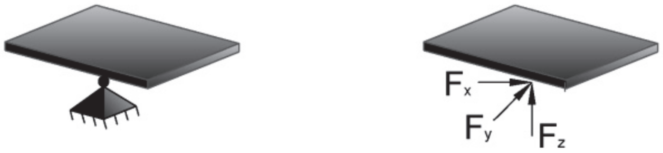


Figure 8: This is a pinned support that provides reaction forces in all directions.

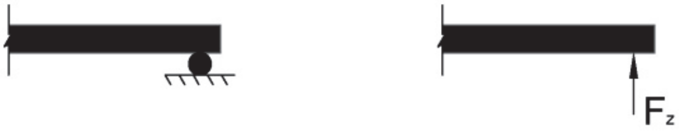


Figure 9: This is another view of the roller support that only provides reaction forces in one direction.

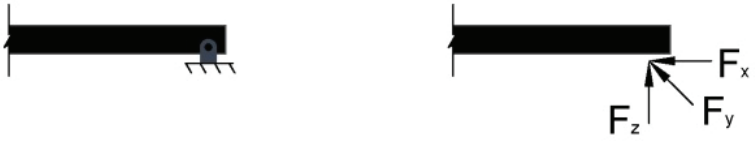


Figure 10: This is a pinned support that provides reaction forces in all directions. There is no reaction moment.

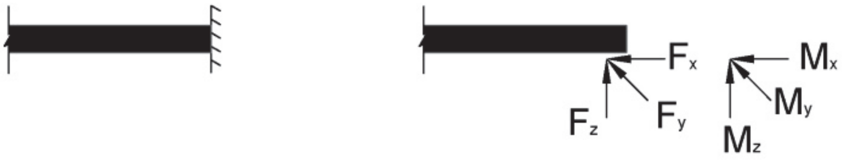


Figure 11: The fixed support has reaction forces and moments in all directions.

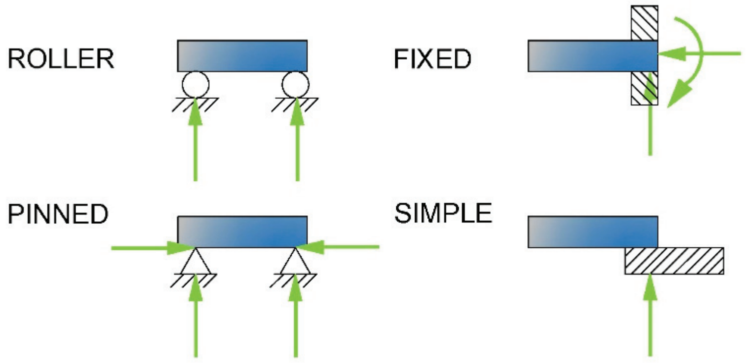


Figure 12: Two dimensional view of the reaction forces for the most common types of supports.

5.0 FRAMES AND TRUSSES

Frames and trusses are constructed of members, typically steel members in mechanical engineering. Trusses are joined by pins and are two force members, while frames are typically welded and are multi-force members. This difference will be discussed in more detail in the individual paragraphs on frames and trusses.

5.1 TRUSSES

In a trust the joints are pinned together. If you remember from the previous reactions topic, pinned joints cannot transfer moments. This means that the members are free to rotate. This also results in each member of a truss to be subject to either a tensile force that wants to pull apart the member of a compressive force that wants to push a member together.

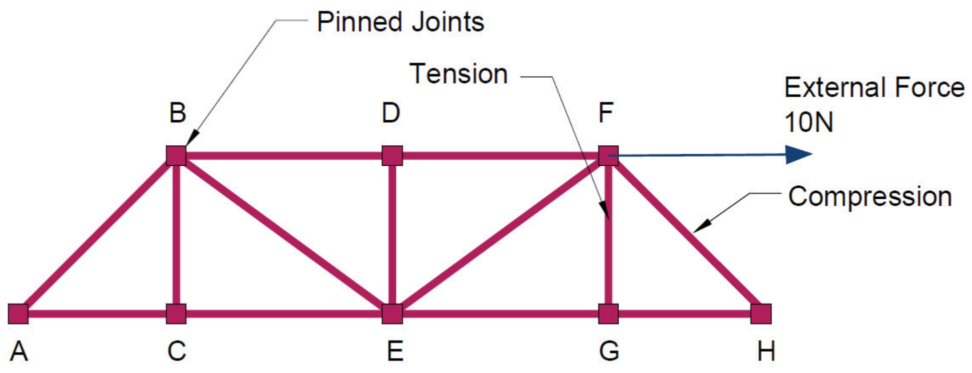


Figure 13: A truss consists of members that are connected via pinned joints. A truss is typically constructed of triangular shapes. In this figure an external force is applied that pulls the truss to the right. If everything is pinned such that the joints do not move. There will be a compressive force in member F-H and a tensile force in F-G. You can visualize compressive and tensile forces in the other members too, but they are not shown in this figure.

There are two methods that are used to calculate the forces within the members of a truss, (1) Method of Joints and (2) Method of Sections. You should be familiar with both methods and when to use each method.

5.2.1 Method of Joints

The method of joints is one method used to calculate the forces within the members of a truss. In this method, you look at the forces acting on a joint and since the joint is not moving, then the net force acting upon the joint must be equal to zero. Most problems will be coplanar, meaning that most problems will only involve the X and Y coordinates. In this method you must balance the forces acting in the x and y-directions. It is important to be able to draw an accurate free body diagram.

The following figure will be used as an example to show you how to use the Method of Joints.

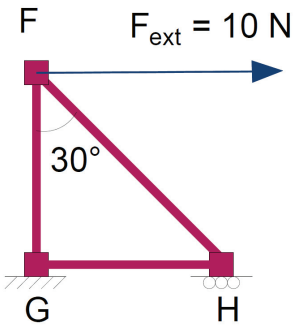


Figure 14: This example will be used to show the method of joints. Assume a right angle at joint G. FG forms a right angle with member GH. Please note that support H does not provide any reactive force in the X-direction.

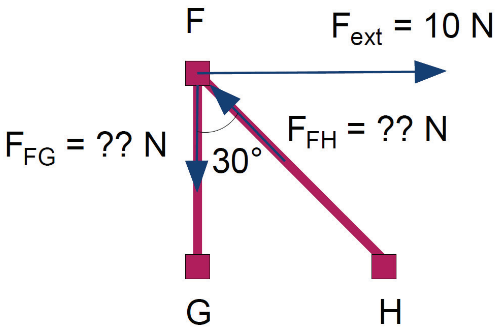


Figure 15: This figure shows a free body diagram of the previous figure. It focuses on the forces acting upon only joint F. At joint F, there is an external force of 10 N acting in the right x-direction. This external force must be balanced by the forces within members FH and FH.

Once you have the free body diagram, then you need to balance the forces in the x and y-directions.

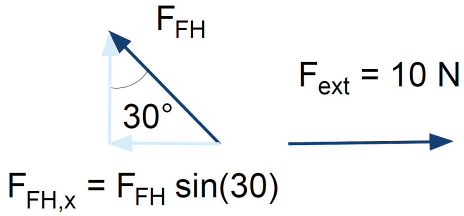


Figure 16: In this figure the forces acting in the x-direction are shown. The x-component of force F_{FH} must be equal and opposite as compared to the external force value.

$$\sum F_x = 0$$

$$F_{FH,x} + F_{ext} = 0$$

$$F_{FH} * \sin(30) + 10N = 0$$

$$F_{FH} = -20\ N$$

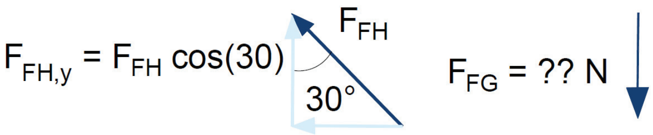


Figure 17: In this figure the forces acting in the y-direction are shown. The y-component of force F_{FH} must be equal and opposite as compared to the vertical force in member FG.

$$\sum F_y = 0$$

$$F_{FH,y} + F_{FG} = 0$$

$$20 * \cos(30) + F_{FG} = 0$$

$$F_{FG} = -17.3\ N$$

Joint H can then be analyzed using the same method.

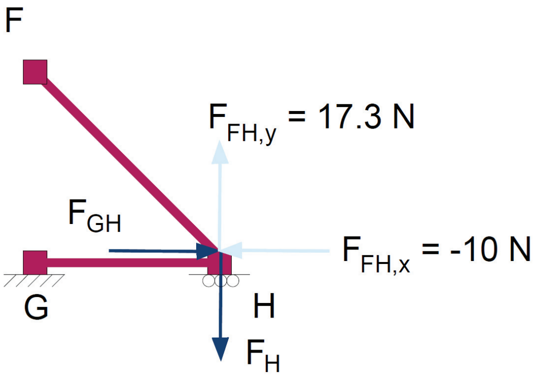


Figure 18: This figure shows that the horizontal force within member GH will oppose the x-component and the reactive force at joint H will oppose the vertical, y-component from member FH.

$$F_{GH} = 10\ N;\ F_{H,y} = 17.3\ N$$

Joint G is finally analyzed using the same method.

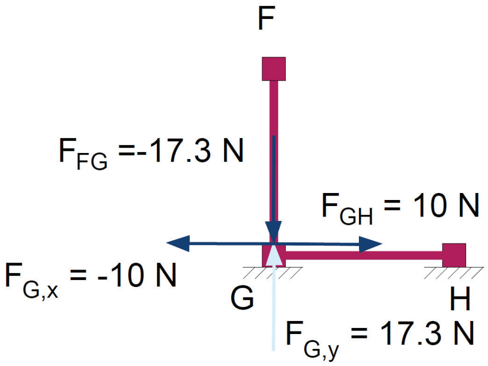


Figure 19: In this figure, the reactive forces at support G counteract the forces in members FG and GH.

$$F_{G,x} = -10\ N;\ F_{g,y} = 17.3\ N$$

5.2.2 Method of Sections

The second method that you can use to solve truss equations is the method of sections. In this method, you will cut through a section of the truss and draw a free body diagram of the forces that act upon one side of that section. The idea is that if a truss is in equilibrium then any section within that truss must also be in equilibrium

One important distinction between this method and the previous method is that you typically have to find the reactive forces at the supports first.

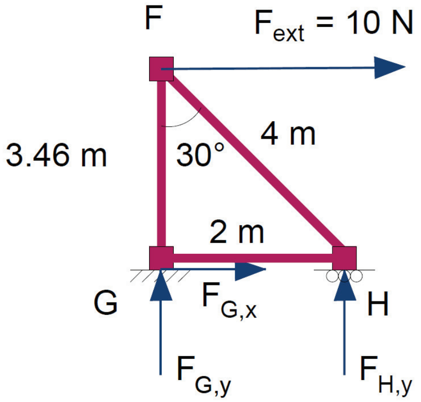


Figure 20: The same example will be used to show the difference between the method of sections and the method of joints. The dimensions of each member are added, but you can complete this example without the dimensions, since you have the angles within the triangle.

In order to find the y-component of the reactive force at joint H, you can use the moment equation for joint G. At joint G, the only forces that produce a moment are the external force and the y-component of the reactive force at joint H.

$$\sum M_G = 0$$

$$F_{ext} * 3.46\ m - F_{H,y} * 2\ m = 0$$

$$F_{H,y} = 17.3\ N$$

Then you can find the y-component at support G, since there are no other vertical components acting upon the entire truss. The vertical component at support G must be equal and opposite of the vertical component at support H.

$$\sum F_y = 0$$

$$F_{G,y} + F_{H,y} = 0$$

$$F_{G,y} = -17.3\ N$$

Finally, you can calculate the support resistance in the x-direction at point G. This is done by summing the forces in the x-direction.

$$\sum F_x = 0$$

$$F_{G,x} + F_{H,x} + F_{ext} = 0$$

$$F_{G,x} = -10\ N$$

Now that you have the reactive forces you can calculate the forces within the members by cutting a section through a member or multiple members. The section cut is made between FH and GH and is shown below.

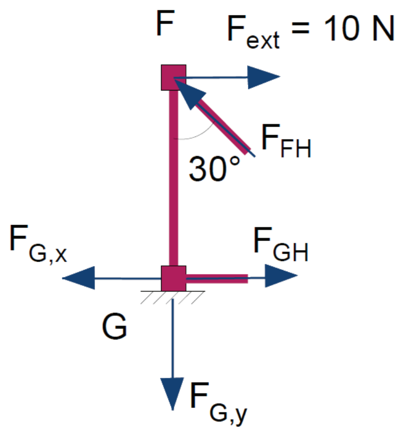


Figure 21: This figure shows the section cut through members FH and GH. Although it looks a little complex, you can simplify this figure through the following two equations.

First, sum the moments around point G, such that the moments equal zero.

$$\sum M_G = 0$$

$$F_{ext} * 3.46\ m + F_{FH} * 1.73 = 0$$

$$F_{FH} = 20\ N$$

The difficult part about finding the moment about G is finding the perpendicular distance to force in member FH. A figure is shown below to help you to visualize how to find the perpendicular distance to a vector from a point.

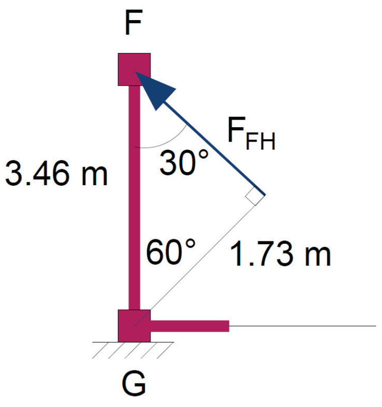


Figure 22: This figure shows how to find the perpendicular distance from point G to vector FH. First, draw a line perpendicular from G to FH. This will create a 90 degree angle and also form

a 90 degree triangle. Then you can find the perpendicular distance by using the cosine or sine equation.

$$\cos(60°) = \frac{adjacent}{hypotenuse} = \frac{perpendicular\ distance}{member\ FH\ length} = \frac{1.73\ m}{3.46\ m}$$

$$\sin(30°) = \frac{opposite}{hypotenuse} = \frac{perpendicular\ distance}{member\ FH\ length} = \frac{1.73\ m}{3.46\ m}$$

In order to find the force within member FG, you can balance the moments about point F.

$$\sum H = 0$$

$$F_{G,x} * 3.46\ m + F_{GH} * 3.46\ m = 0$$

$$F_{GH} = -F_{G,x} = 10\ N$$

You will be unable to calculate the force in FG, unless you cut a section through FG. This is important to remember that in order to get the force in a member you need to analyze the section or the joint. Only the reactive forces and external force can be found by completing an overall free body diagram that balances overall moments and forces.

$$\sum F_y = 0$$

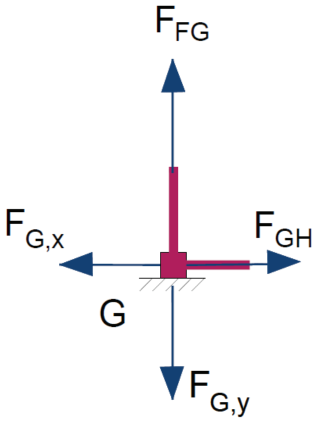


Figure 23: This figure cuts a section through members FG and GH.

$$F_{G,y} + F_{FG} = 0$$

$$-17.3\ N + F_{FG} = 0$$

$$F_{FG} = 17.3\ N$$

5.2 FRAMES

In a frame members are joined rigidly, most commonly the joints are welded. This is similar to a fixed reaction point, where there are reactive forces and reactive moments. Frames also are consist of members with multi-forces, more than just tension or compression. This makes these problems very difficult with multiple equations and multiple unknowns. These types of questions will take more than a few minutes to solve, so it is most likely that these problems will not be included on the FE exam. However, you should know generally the difference between frames and trusses.

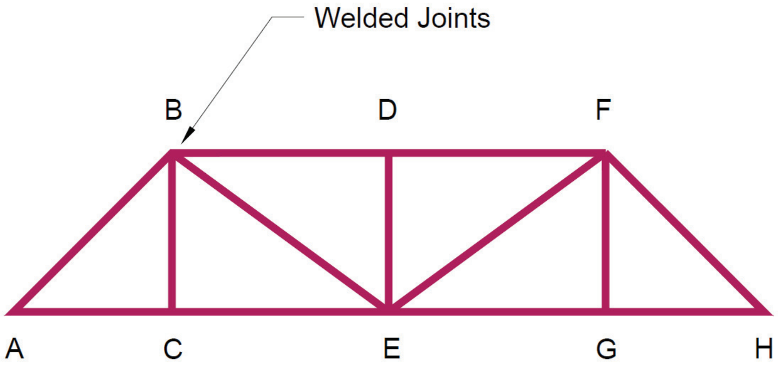


Figure 24: In a frame, the joints are typically welded together as opposed to pinned. However, the main distinguishing factor between frames and trusses is that frames have multi-force members. Members that have more than just tension and/or compression.

6.0 CENTROIDS

Centroids are the center of an area and are typically used to find the center of mass of an object with uniform density. The centroid of an area may sometimes be outside of an area, like in the most common example of a ring. The centroid is found by dividing the first moment of area by the area. So first, you will learn how to calculate the first moment of area.

6.1 FIRST MOMENT OF AREA

The first moment of area should not be confused with the moment of inertia which will be discussed in the following section. The first moment of area is equal to the sum of the area multiplied by the distance to an axis. The axis is the centroid of the area. There are two first moments of area, one about the x-axis and the other about the y-axis for a 2-dimensional object.

$$S_x = A\bar{y} = \int_A y * dA$$

$$S_y = A\bar{x} = \int_A x * dA$$

In the equations above, “y” is the distance from the centroid in the y-direction and “x” is the distance from the centroid in the x-direction. If the axis is changed from the centroid, to another axis, then you must use the parallel axis theorem to convert the first moment of area to another axis. The parallel axis theorem will be discussed in the Moments of Inertia topic.

6.2 CENTROID CALCULATION

Once you have the first moment of area, then you can calculate the centroid. On the FE exam, you most likely will be only dealing with x and y-directions. So you can calculate the location of the centroid with only two equations, one for the x-direction and one for the y-direction, as shown below.

$$centroid\ location = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\int_A (x^*)dA}{\int_A dA}, \bar{y} = \frac{\int_A (y^*)dA}{\int_A dA}$$

In the above equations you are breaking up the area into little slices. Then you are calculating the area and multiplying that area by the distance away from the x or y-axis. This means that for certain distances with larger areas, their distance from the x or y-axis will be weighted more. Then you are summing up those products and dividing by the total area. This will leave you with just the composite center in the x or y direction of the area.

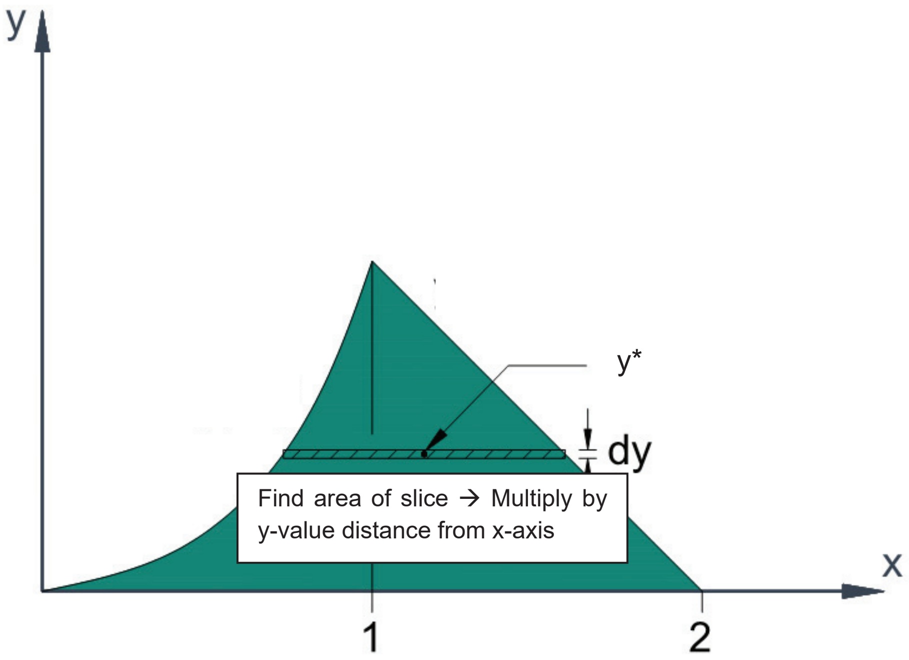

y
y*
dy
Find area of slice → Multiply by y-value distance from x-axis
x
1
2

Figure 25: In this example, you have a tiny slice of the area. That area is multiplied by the y value which is the distance from the y-axis. This method is also completed for the x-direction. Instead of vertical slices, you have horizontal slices. Instead of measuring the distance of each slice from the y-axis, you measure the x-distance away from the y-axis.*

7.0 MOMENTS

Moment describes the force on a body that causes the body to rotate about an axis. Moment can also be called Torque. The most difficult part in completing problems with Moment or Torque is drawing the free body diagram. You must be able to imagine forces being applied to a body's axis at different distances from the axis in such a way to rotate the body about its axis.

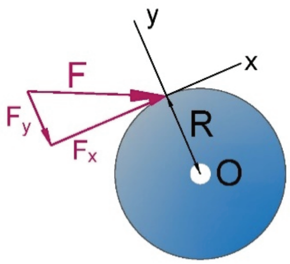


Figure 26: The moment is the multiplication of the tangential force times the perpendicular distance to the rotational axis.

$$M = F_x * R$$

$$M = Moment\ (lbf \cdot ft \quad or \quad N \cdot m);\ \ F = Force\ (lbf \quad or \quad N);\ \ R = Distance\ (ft \quad or \quad m)$$

7.1 FREE BODY DIAGRAM

On the FE exam you will be given scenarios and diagrams without forces and moments shown. It will be up to you to draw the moments acting upon the objects.

The magnitude of the moment of a force at the circumference of the above circle, is equal to the perpendicular distance R, multiplied by the magnitude of the force: M = F*R.

7.2 EQUILIBRIUM

If an object is not rotating, then the object is in equilibrium. This means that the moments at a point will sum to equal zero. On the exam, you can use these equilibrium equations to solve for the unknowns. When constructing an equilibrium equation you need to assign positive and negative moments, similar to positive and negative forces. For example, it is most commonly assumed that any moment acting in the clockwise direction is assigned a positive value and a negative value for counterclockwise moments.

$$\sum M_x^{axis\ 1} = 0;\ \sum M_y^{axis\ 1} = 0;\ \sum M_z^{axis\ 1} = 0$$

7.3 CALCULATING MOMENTS OF INERTIA

Moment of inertia describes an object's resistance to rotation. It can be classified either as area moment of inertia or mass moment of inertia. Area moments of inertia, also known as the second moment of area, are associated with cross sectional areas, while mass moments of inertia are associated with objects undergoing rotational or angular movement. Area moment of inertia is discussed here and the mass moment of inertia is discussed in the dynamics section. Area moment of inertia tables for various shapes are found under the Statics section of the *NCEES FE Exam Handbook*, while the mass moment of inertia tables are found under the Dynamics section.

7.3.1 Second Moment of Inertia (Area Moment of Inertia)

Area moment of inertia or second moment of area is used to convey the situation when forces act upon a three dimensional object, the internal forces within the object will vary based on the distance from the forces and the cross section of the object. For example, the area moment of inertia helps to characterize a beams resistance to deflection or bending around an axis located within a cross section of the beam. The stresses within an object will vary throughout the thickness of the object. The previous topic on beams showed the moments, based on the assumption that the object was a uniform beam.

Mathematically, the second moment is found by multiplying each unit area of an object by the square of the distance from a specific axis. Since the second moment is taken from an axis, there will be multiple second moments for each axis. The equations shown below show the different moment of inertia equations.

$$I_x = \int y^2 dA\,;\ I_y = \int x^2 dA\,;$$

$$I_x\ or\ I_y = \ Area\ Moment\ of\ Inertia\ \ (m^4\ \ or\ \ in^4)$$

There is no "z" second moment, because the second moment is only based on the 2-dimensional cross section of the object.

On the exam, you should not be required to integrate to find the second moment. You will most likely need to use one of the second moment equations for various cross sections.

Moments of inertia are useful in completing beam type problems, but first you should develop an understanding of moments of inertia by looking at the below examples.

The moment of inertia of the rectangle below is found by integrating the area multiplied by the square of the distance from the axis. First, the area moment of inertia of the x-axis can be found. The integral shall be taken from 0 to b, on the variable "y". The area is simply "a" times the "dy", where "dy" is a tiny strip of y, integrated from 0 to b. You should use the similar method to find the area moment of inertia about the y-axis.

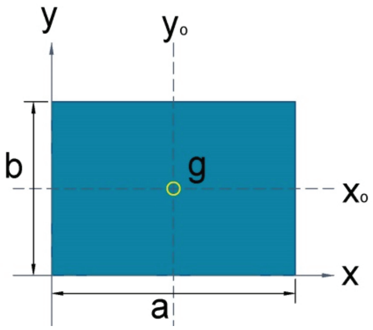


Figure 27: The area moment of inertia about the x-axis and y-axis can be found with the equations below.

$$I_X = \int_0^b y^2(a * dy) = \frac{1}{3}ab^3;\ I_y = \int_0^a x^2(b * dx) = \frac{1}{3}ba^3$$

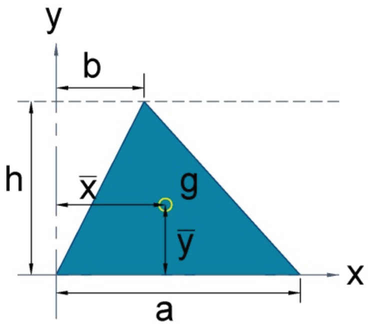


Figure 28: The area moment of inertia for this triangular cross section can be found using the same method, previously discussed.

$$I_X = \int_0^h y^2 dA = \frac{1}{12} a h^3$$

Locate the table of area moments of inertia for additional shapes in the *NCEES FE Reference Handbook*.

The moment of inertia can also be calculated for polar situations, about a z-axis. This is called the polar moment of inertia.

$$J = \int_A r^2 dA\,;$$

$$r = radius\ of\ gyration, radial\ distance\ from\ z - axis$$

$$J = \ Polar\ Area\ Moment\ of\ Inertia\ \ (m^4\ \ or\ \ in^4)$$

7.4 PARALLEL AXIS THEOREM

The parallel axis theorem is used to shift the axes that were used to calculate the moments of inertia. The moment of inertia is calculated through the centroid, but if you want the moment of inertia about another axis then you can use this formula. This is used for complex areas that are constructed of multiple distinct areas, like in an I-beam. In an I-beam, you have the top, middle and bottom rectangular areas. You can calculate the moment of inertia for each rectangular area about its own centroid. Then you can combine them through the parallel axis theorem.

$$I_{composite,x} = I_x + A * y^2; where\ y = vertical\ distance\ between\ centroids\ x - axes$$

$$I_{composite,y} = I_y + A * x^2; where\ x = horizontal\ distance\ between\ centroids\ y - axes$$

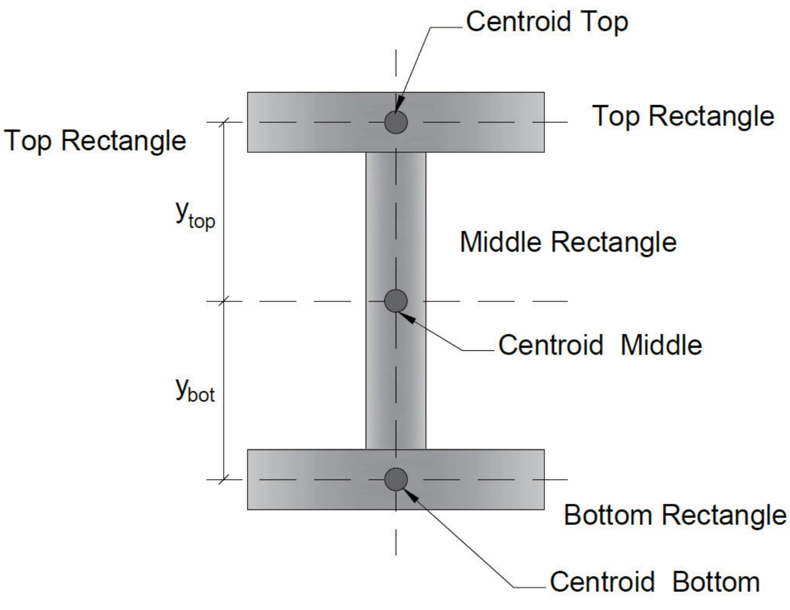


Figure 29: The parallel axis theorem is often used to combine the moments of inertia for multiple areas. In this figure, the vertical distance between the centroid's x-axes are shown as y_{top} and y_{bot}. The area of the top, bottom and middle rectangles are given as A_{top}, A_{bot} and A_{mid}.

$$I_{beam,x} = I_{middle,x} + (I_{top,x} + A_{top} * y_{top,}^2) + (I_{bot,x} + A_{bot} * y_{bot}^2)$$

Since the horizontal distance between the centroids for the top, middle and bottom rectangles are the same, the composite beam moment of inertia in the y-direction can be simply added together.

$$I_{beam,y} = I_{middle,y} + (I_{top,y} + A_{top} * 0) + (I_{bot,y} + A_{bot} * 0)$$

$$I_{beam,y} = I_{middle,y} + I_{top,y} + I_{bot,y}$$

7.5 RADIUS OF GYRATION

The radius of gyration is also mentioned in the *NCEES FE Reference Handbook* under the statics section. This term describes the radial distribution of a column's area about its centroid axis. This term will be discussed more in the column section, since it is used to characterize the stiffness of a column. The equation for radius of gyration is shown below.

$$k_x = \sqrt{\frac{I_x}{A}}; radius\ of\ gyration\ about\ x - axis$$

$$k_y = \sqrt{\frac{I_y}{A}}; radius\ of\ gyration\ about\ y - axis$$

$$k_{polar} = \sqrt{\frac{J}{A}}; radius\ of\ gyration\ for\ polar\ moment\ of\ inertia$$

8.0 STATIC FRICTION

Friction is the force that resists the movement of surfaces that slide upon one another. Most problems on the FE exam will assume that the surfaces are frictionless. If a friction factor or frictional force is not provided, then it is safe to assume that the surfaces are frictionless. However, there should be a few problems that will incorporate friction into the problem. For these few problems it is important to understand two main concepts.

First, the object in question will have a normal force that is perpendicular to the contacted surface. Secondly, there will be a force that is tangent to the surface that is opposite to the movement of the object. If the object is not moving, then the force is opposite the direction that the object is approaching movement. For example, if the object is on a slope, then the impeding motion is downward and the frictional force will be opposite this motion.

There are two types of friction, static friction and kinetic friction. When an object is at rest, there will be no motion until the force to cause movement is greater than the maximum force due to static friction. Once the movement force is greater than this value, then the object will move. Once the object is moving, the friction force that opposes movement will now be equal to the equation for kinetic friction, shown below.

$$Static\ Friction \rightarrow F_{max} = \mu_s * N$$

$$Kinetic\ Friction \rightarrow F_k = \mu_k * N$$

There are three types of problems involving static friction that can be on the FE exam. These problems are for a block on an angled surface and pulleys.

8.1 ANGLED SLIDING BLOCK WITHOUT EXTERNAL FORCE

In this scenario a block is placed on an angled surface. The block is subject to both gravitational force and frictional force. However, only a portion of the gravitational force is applied in the x direction, since the other portion is applied in the y-direction. Also in this angled scenario, the x-axis and y-axis are rotated. The x-axis is in the direction of the angled surface and the y-axis is perpendicular to this surface.

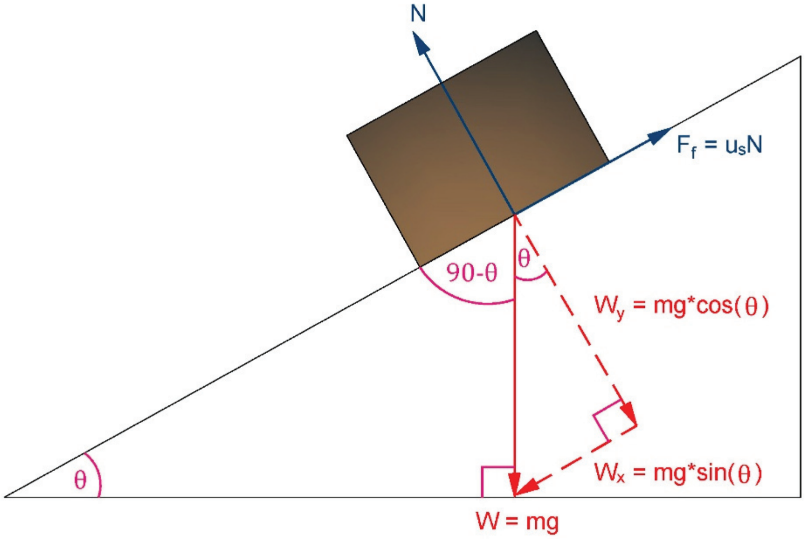


Figure 30: A block sliding down an incline is subject to frictional forces in the opposite direction of movement.

Similar to the first example, the forces are summed for both the x and y directions.

X-Direction (F_x)	Y-Direction (F_y)
$Impending\ Motion$ $\mu_{static}N = mg * sin(\theta);$	$Impending\ Motion$ $N = mg * cos(\theta);$

8.2 ANGLED SLIDING BLOCK WITH EXTERNAL FORCE

This scenario is similar to scenario B. The block is subject to both gravitational force and frictional force and only a portion of the gravitational force is applied in the x direction, since the other portion is applied in the y-direction. The main difference in this scenario is that there is an external force applied to the block shown as push/pull. The external force can either be directed upwards or downward. In addition, the problem can be complicated if the direction of the external force is not in plane with the frictional force. If this is the case then the external force must be broken into its x and y components, similar to how the weight is broken into its x and y components.

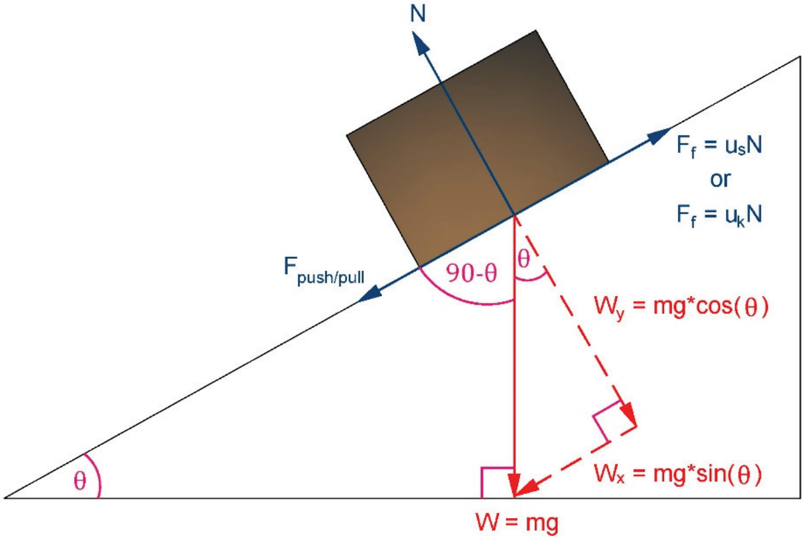
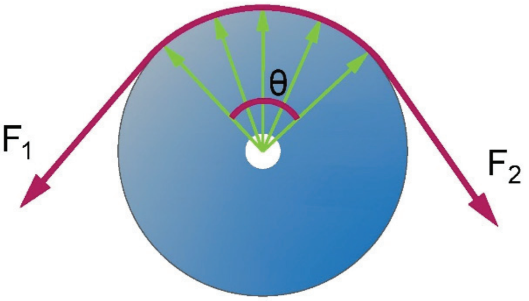


Figure 31: A sliding block with an external force acting upon the block

X-Direction (F_x)	Y-Direction (F_y)
$Impending\ Motion$ $\mu_{static} N = mg * sin(\theta) + F_{push\ or\ pull};$	$Impending\ Motion$ $N = mg * cos(\theta);$

8.3 BELTS AND PULLEYS FRICTION

Unlike the previous discussions of friction, the friction at belts and pulleys is dependent on the amount of contact with the rotating surface. At each point of contact, there will be a normal force from the center of the rotating part to the point of contact. In addition there will be a frictional force at that point of contact, perpendicular to the normal force. A simple moment or force balance equation will not suffice for this situation. You must use the following formulas to solve these types of problems.



$$Pulley, Flat\ Belt\ Friction \rightarrow F_2 = F_1 e^{\mu\theta}$$

$$\theta = Contact\ surface;\ \mu = static\ friction$$

You should always make sure that your angle values are in radians, when using this equation.

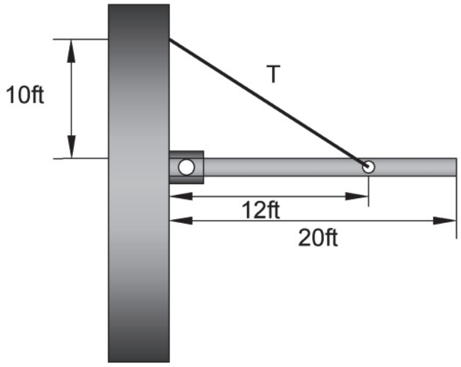
If you encounter a V-belt situation which will be discussed more in Section 15.0 Mechanical Design and Analysis, then you should use the following equation. The alpha term is the angular gap in radians between contact surfaces, since the belt is grooved. See Mechanical Components for more discussion into this topic.

$$V - Belt\ Friction \rightarrow F_2 = F_1 e^{\mu\theta/\sin(\frac{\alpha}{2})}$$

9.0 PRACTICE EXAM PROBLEMS

9.1 PRACTICE PROBLEM 1 – BALANCING MOMENT

A beam of weight 100 lbs (45.36 kg) is supported by a tension cable. The dimensions are shown in the diagram below. What is the tension in the cable?



(A) 100 LBS (45.36 kg)

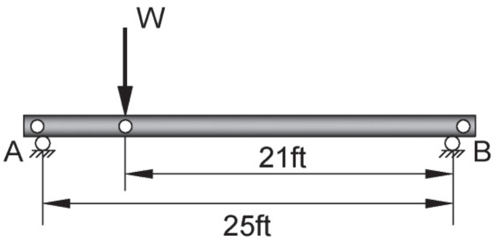
(B) 130 LBS (58.97 kg)

(C) 160 LBS (72.58 kg)

(D) 190 LBS (86.18 kg)

9.2 PRACTICE PROBLEM 2 – BALANCING MOMENT

A beam has a weight of 450 lbs (204.12 kg) and there is a point load of 150 lbs (68.04 kg) as shown in the figure below. What is the reaction force at point A?



(A) 100 LBS (45.36 kg)

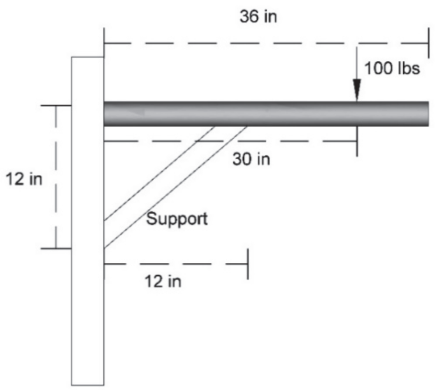
(B) 150 LBS (68.04 kg)

(C) 250 LBS (113.4 kg)

(D) 350 LBS (158.76 kg)

9.3 PRACTICE PROBLEM 3 – BALANCING MOMENT

What is the magnitude of the force in the 45 degree support in the figure below? There is a point force of 100 lbs (45.36 kg) that acts downward. The support is attached to the beam via a rigid pin support. The support and beam are also attached to the side structure via a rigid pin support.



(A) 100 lbs (45.36 kg)

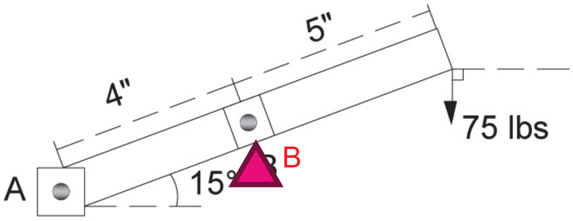
(B) 250 lbs (113.4 kg)

(C) 310 lbs (140.62 kg)

(D) 350 lbs (158.76 kg)

9.4 PRACTICE PROBLEM 4 – BALANCING MOMENT

An external force acts upon the two components below. The components are connected by a bolt at point B. The components are simply supported at point B. The components are rigid. The components are pinned securely at point A and do not move. What is the reactionary force at point B, normal to the component's length?



(A) 149 lbs (67.59 kg)

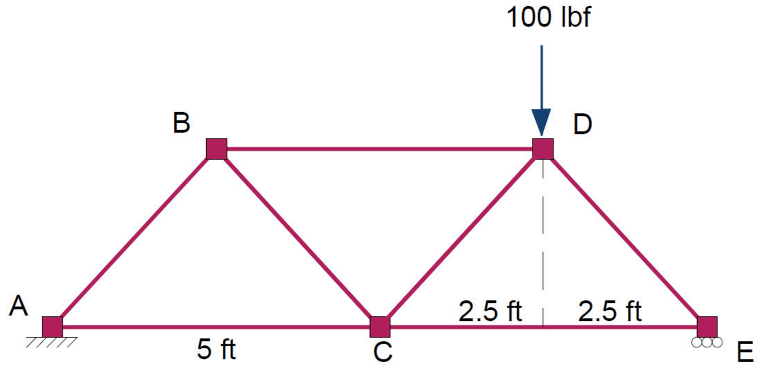
(B) 163 lbs (73.94 kg)

(C) 181 lbs (82.1 kg)

(D) 225 lbs (102.06 kg)

9.5 PRACTICE PROBLEM 5 – TRUSS

What is the force acting within member DE of the following truss? Assume member DE is 4 feet (1.22 m) long. (100 lbf = 444.82 N)


100 lbf
B
D
A
2.5 ft
2.5 ft
5 ft
C
E

(A) 75 lbf (333.62 N)

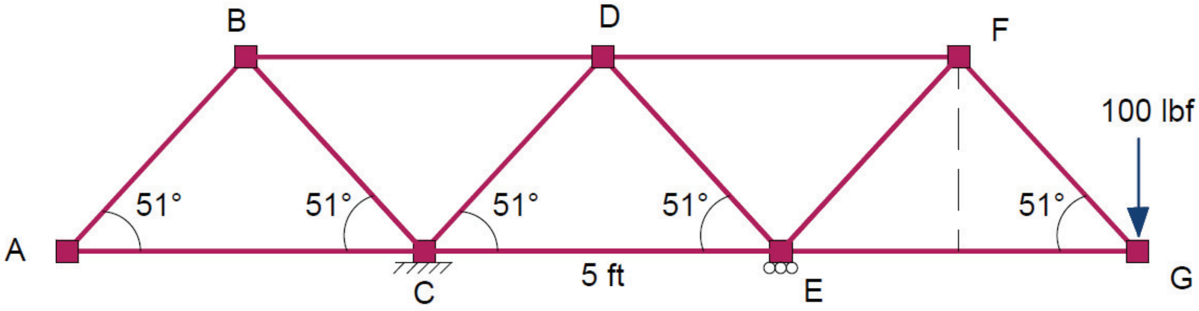
(B) 96 lbf (427.03 N)

(C) 100 lbf (444.82 N)

(D) 141 lbf (627.2 N)

9.6 PRACTICE PROBLEM 6 – TRUSS

What is the force acting in member FG in the below truss? 100 lbf = 444.82 N


B
D
F
100 lbf
51°
51°
51°
51°
51°
A
C
5 ft
E
G

(A) 72 lbf (320.27 N)

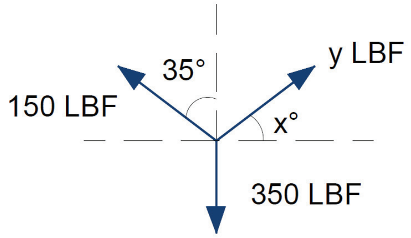
(B) 91 lbf (404.79 N)

(C) 100 lbf (444.82 N)

(D) 128 lbf (569.37 N)

9.7 PRACTICE PROBLEM 7 – CONCURRENT FORCE SYSTEMS

The force vectors shown below are in equilibrium. Solve for “y”. (150 lbf = 667.23 N; 350 lbf = 1,556.88 N)



(A) 71 lbf (315.82 N)

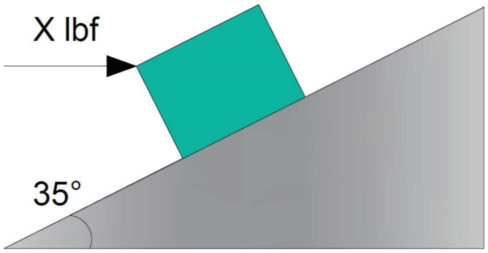
(B) 129 lbf (573.82 N)

(C) 242 lbf (1,076.47 N)

(D) 324 lbf (1,441.22 N)

9.8 PRACTICE PROBLEM 8 – STATIC FRICTION

What is the force, “X” required to keep the block stationary. The block weighs 100 lbs (45.36 kg) and the static friction coefficient is 0.5.



(A) 15 lbf (66.72 N)

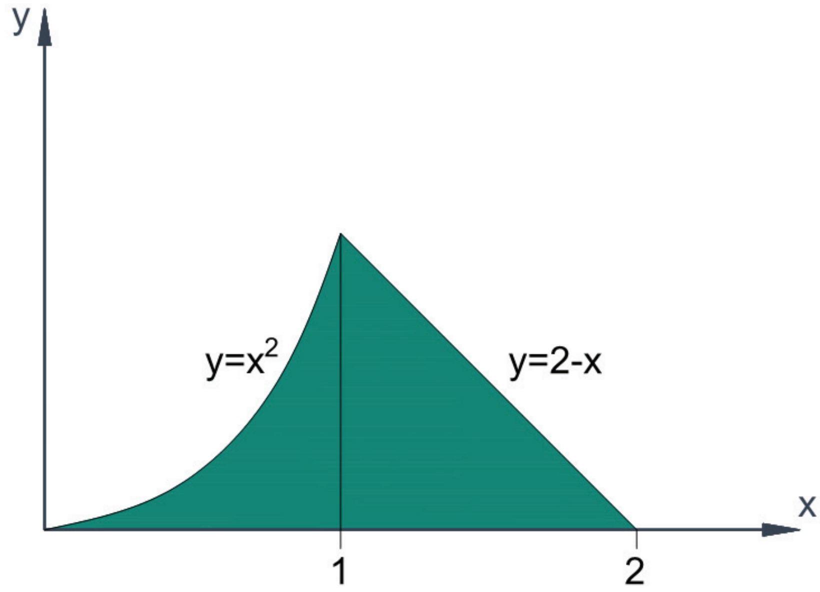
(B) 90 lbf (400.34 N)

(C) 100 lbf (444.82 N)

(D) 150 lbf (667.23 N)

9.9 PRACTICE PROBLEM 9 – CENTROID

Find the expression for the centroid, $\bar{y}$, of the shaded area.



(A) $\bar{y} = \dfrac{\int_0^1 y * (y - 2 - \sqrt{y})dy}{\int_0^1 (y - 2 - \sqrt{y})dy}$

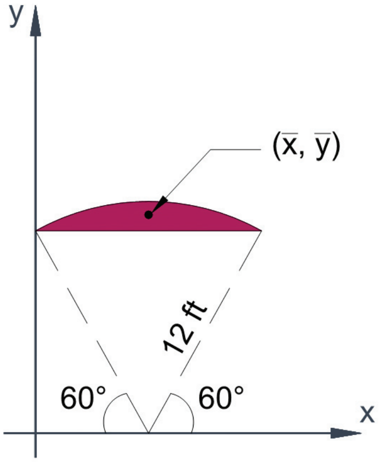
(B) $\bar{y} = \dfrac{\int_0^1 (2y - y^2 - y^{\frac{3}{2}})dy}{\int_0^1 (2 - y - \sqrt{y})dy}$

(C) $\bar{y} = \dfrac{\int_0^2 y * (2 - y - \sqrt{y})dy}{\int_0^2 (2 - y - \sqrt{y})dy}$

(D) $\bar{y} = \dfrac{\int_0^2 (\sqrt{y} + 2 - y)dy}{\int_0^2 y * (2 - y - \sqrt{y})dy}$

9.10 PRACTICE PROBLEM 10 – CENTROID

Find the centroid coordinates, $(\bar{x}, \bar{y})$ for the shaded region of the following figure. (12ft = 3.66m)



(A) (6ft, 11ft) or (1.83m, 3.35m)
(B) (4ft, 10ft) or (1.22m, 3.05m)
(C) (4ft, 11.5ft) or (1.22m, 3.51m)
(D) (5ft, 11ft) or (1.52m, 3.35m)

10.0 SOLUTIONS

10.1 SOLUTION 1 – BALANCING MOMENT

This problem is solved by balancing the moment about the fulcrum. The moment about this point must be equal to zero. The force due to gravity occurs at the center of the beam at 10 ft from the fulcrum. The tension has a vertical component and a horizontal component. The horizontal component is not included in the moment equilibrium equation since it does not cause rotation.

$$\sum M_{fulcrum} = 0$$

$$\sum M\ (lbf - ft) = 0 = -100lbs * \frac{g}{g_c} * 10ft + T\frac{g}{g_c} sin(\theta) * 12ft$$

$$\frac{g}{g_c} = \frac{32.174\frac{ft}{s^2}}{32.174\frac{lbm - ft}{lbf - s^2}}$$

The angle theta is found through the following equation.

$$tan(\theta) = \frac{10\ ft}{12\ ft} = \frac{3.048\ m}{3.6576\ m} \rightarrow 39.8°$$

Insert theta and solve for T.

$$0 = -100lbs * \frac{g}{g_c} * 10ft + T * \frac{g}{g_c} * sin(39.8°) * 12ft$$

$$T = 130\ lbs$$

The correct answer is most nearly, **(B) 130 LBS**.

$$\sum M\ (N - m) = 0 = -45.36\ kg * g\ * 3.048m + Tgsin(\theta) * 3.6576m$$

$$g = 9.807\frac{m}{s^2}$$

$$0 = -45.36\ kg * g * 3.048m + T * g * sin(39.8°) * 3.6576m$$

$$T = 58.968\ kg$$

10.2 SOLUTION 2 – BALANCING MOMENT

First, balance the moment about A. The weight of the beam occurs at the center of the beam.

$$\sum M_A = 0$$

$$\sum M = 0 = -150lbs * \frac{g}{g_c} * 4ft - 450lbs * \frac{g}{g_c} * 12.5ft + R_B * \frac{g}{g_c} * 25ft$$

Solve for reaction at B.

$$R_B = \frac{150lbs * 4ft + 450lbs * 12.5ft}{25\ ft} = 249\ lbs$$

Next balance all the forces in the Y-direction. Since the beam is not moving or accelerating, then the sum of the forces should be equal to zero.

$$0 = R_A + R_B - Weight - Point\ Load$$

$$0 = R_A + 249\ lbs - 450\ lbs - 150\ lbs$$

$$R_A = 351\ lbs$$

The correct answer is most nearly, **(D) 350 LBS**.

$$\sum M = 0 = -68.04\ kg * g * 1.219m - 204.12\ \mathrm{kg} * g * 12.5ft + R_B * g * 7.62m$$

Solve for reaction at B.

$$R_B = \frac{68.04kg * 1.219m + 204.12kg * 3.81m}{7.62m} = 112.95\ kg$$

Next balance all the forces in the Y-direction. Since the beam is not moving or accelerating, then the sum of the forces should be equal to zero.

$$0 = R_A + R_B - Weight - Point\ Load$$

$$0 = R_A + 112.95\ kg - 204.12\ kg - 68.04\ kg$$

$$R_A = 159.21\ kg$$

10.3 Solution 3 – Balancing Moment

The first step is to calculate the moment about the beam connection to the side structure on the left. The moment about this point should be equal to zero.

$$\sum M = 0$$

$$\sum M = 0 = -100\ lbs * \left(\frac{g}{g_c}\right) * (30\ in) + F_{supp,vertical} * (\frac{g}{g_c}) * (12\ in)$$

But first you need to change the vertical support force to be in terms of the overall support force.

$$F_{supp,vertical} * (\frac{g}{g_c}) = F_{supp} * (\frac{g}{g_c}) * \frac{12}{12\sqrt{2}} (\frac{vertical}{hypotenuse})$$

$$0 = -100\ lbs * (30\ in) * (\frac{g}{g_c}) + F_{supp} * \frac{1}{\sqrt{2}} * (12\ in) * (\frac{g}{g_c})$$

$$F_{supp} = 353\ lbs$$

The correct answer is most nearly, **(D) 350 lbs**.

$$\sum M = 0 = -45.36\ kg * (.762m)g + F_{supp,vertical} * (.3048m)g$$

But first you need to change the vertical support force to be in terms of the overall support force.

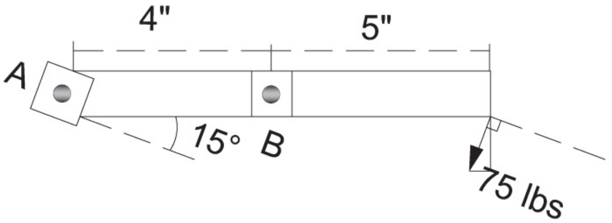
$$F_{supp,vertical} * g = F_{supp} * g * \frac{.3048\ m}{.3048\sqrt{2}\ m} (\frac{vertical}{hypotenuse})$$

$$0 = -45.36\ kg * (.762\ m) + F_{supp} * \frac{1}{\sqrt{2}} * (.3048\ m)$$

$$F_{supp} = 160.35\ kg$$

10.4 SOLUTION 4 – BALANCING MOMENT

Since the component is pinned at “A”, then the moment at point A is equal to zero. Use the moment equation to solve for the normal force at B. The difficulty in this problem is finding the vertical forces that are perpendicular to the component. In order to make it simpler, rotate the components such that they are horizontal as shown below.



$$\sum M = 0 = -75\ lbs * \left(\frac{g}{g_c}\right) \cos(15) * (9\ in) + F_B * \left(\frac{g}{g_c}\right) * (4\ in)$$

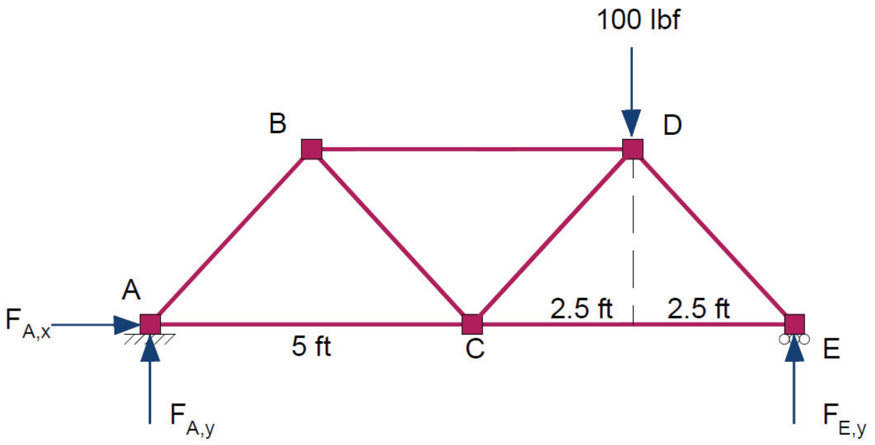
$$\sum M = 0 = -75\ lbs * \left(\frac{g}{g_c}\right) * \cos(15) * (9\ in) + F_B * \left(\frac{g}{g_c}\right) * (4\ in)$$

$$F_B = 163\ lbs$$

The correct answer is most nearly, **(B) 163 lbs**.

10.5 SOLUTION 5 - TRUSS

In this problem, first conduct a free body diagram.


100 lbf
B
D
A
$F_{A,x}$
5 ft
C
2.5 ft
2.5 ft
E
$F_{A,y}$
$F_{E,y}$

Next, solve for the reactive force at E by taking the moment about point A.

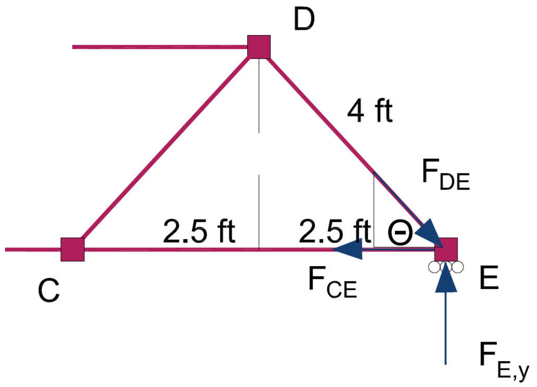
$$\sum M_A = 0$$

$$F_D * 7.5\ ft + F_{E,y} * 10\ ft = 0$$

$$-100\ lbf * 7.5\ ft + F_{E,y} * 10\ ft = 0$$

$$F_{E,y} = 75\ lbf$$

Now, use the method of joints around joint E to find the force in member DE.


D
4 ft
F_{DE}
2.5 ft
2.5 ft
Θ
C
F_{CE}
E
$F_{E,y}$

First, find the theta.

$$\cos^{-1}\left(\frac{2.5}{4}\right) = \theta$$

$$\theta = 51.32°$$

Use the angle to find the vertical component of F_{DE}. This vertical component must be equal and opposite of the vertical reactive force at joint E.

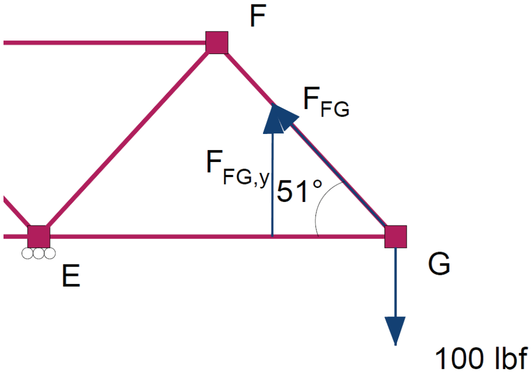
$$\sin(51.32°) * F_{DE} + (75\ lbf) = 0$$

$$F_{DE} = 96.07\ lbf$$

The correct answer is most nearly, (B) 96 lbf.

10.6 SOLUTION 6 - TRUSS

For this problem, you have an external force acting upon joint G, so you can use the method of joints on this joint. The vertical component of member FG will have to counteract the downward force of 100 lbf.



$$\sin(51°) * F_{FG} + (-100\ lbf) = 0$$

$$F_{FG} = 128.1\ lbf$$

The correct answer is most nearly, (D) 128 lbf.

10.7 SOLUTION 7 – CONCURRENT FORCE SYSTEMS

First balance the forces in the y-direction.

$$-350\ lbf + (y\ lbf) * \sin(x°) + 150\ lbf * \cos(35°) = 0$$

$$(y\ lbf) * \sin(x°) - 227.1\ lbf = 0$$

$$(y\ lbf) = \frac{227.1\ lbf}{\sin(x°)}$$

Next balance the forces in the x-direction.

$$-150\ lbf * \sin(35°) + (y\ lbf) * \cos(x°) = 0$$

$$-86\ lbf + (y\ lbf) * \cos(x°) = 0$$

Plug-in the value for "y" into the previous equation and solve for x. Sine divided by cosine is equal to tangent.

$$\frac{227.1\ lbf}{\sin(x°)}) * \cos(x°) = 86\ lbf$$

$$\tan(x°) = 2.64$$

$$x = 69.3°$$

Plug-in the value for x back into the x-direction balanced equation and solve for y.

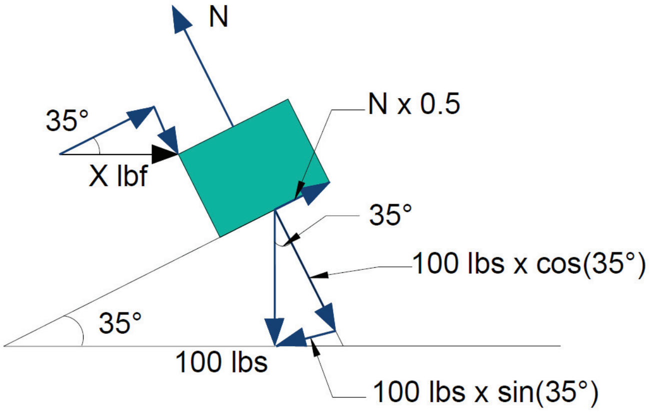
$$(y\ lbf) * \sin(69.3°) - 227.1\ lbf = 0$$

$$y = 242.8\ lbf$$

The correct answer is most nearly, (C) 243 lbf.

10.8 SOLUTION 8 – STATIC FRICTION

In this problem, you first need to calculate the normal force, with the free body diagram shown below.



You can convert 100 pounds to pound force and you can calculate the normal force.

$$F_{weight} = 100\ lbs * \frac{32.2\frac{ft}{s^2}}{32.2\frac{lb-ft}{lbf-s^2}} = 100\ lbf$$

Next sum up the forces acting in the "x-direction", which is pointing parallel to the slope of the ground.

$$\sum F_{x\prime}$$

$$N * \mu_s + X * \cos(35°) - 100\ lbf * \sin(35°) = 0$$

$$N * 0.5 + X * \cos(35°) - 100\ lbf * \sin(35°) = 0$$

$$N = 2 * [57.4 - 0.82 * X] \rightarrow N = 114.8 - 1.64X$$

Next sum up the forces acting in the “y-direction”, which is pointing perpendicular to the slope of the ground.

$$\sum F_{y\prime} = 0$$

$$N - X * \sin(35) - 100\ lbf * \cos(35°) = 0$$

$$N - X * 0.57 - 81.92\ lbf = 0$$

Next, plug in the equation for “N” from above.

$$114.8 - 1.64X - X * 0.57 - 81.92\ lbf = 0$$

$$-2.21 * \mathrm{X} = -32.9\ lbf$$

$$X = 14.9\ lbf$$

$$N = 90.25\ lbf$$

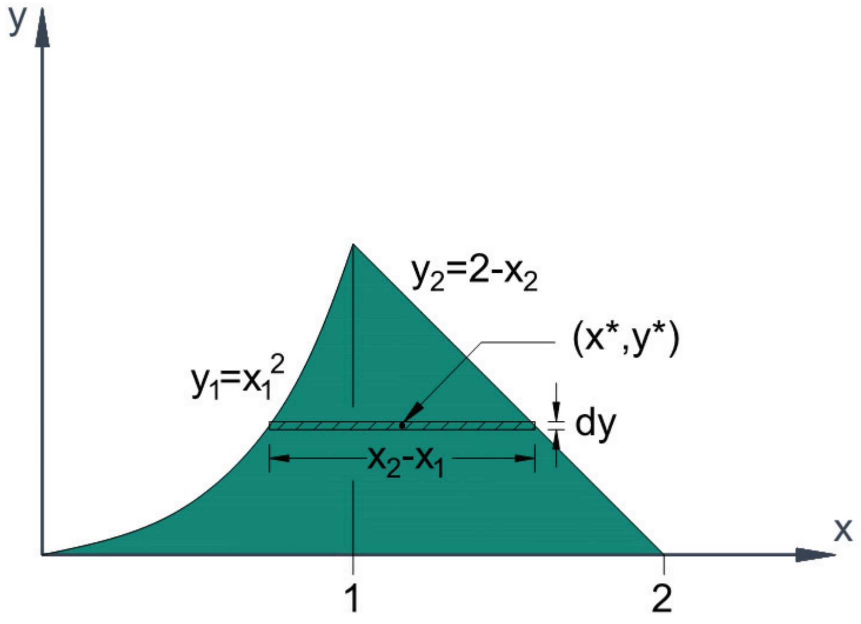
The correct answer is most nearly, (A) 15 lbf.

10.9 SOLUTION 9 - CENTROID

To find the centroid along the y-axis, use the following equation, which is a version of the equation given in the Centroids section of the Statics chapter in the *NCEES FE Reference Handbook.*

$$\bar{y} = \frac{\int_A (y^*) dA}{\int_A dA}$$

This problem can be solved in two ways, either by integrating along the x-axis or the y-axis. Since the solution options indicate integration along the y-axis, dy, then convert the equations that define the area within the shaded region in terms of x.



Then, the two curves become:

$$x_1 = \sqrt{y_1} \qquad and \qquad x_2 = 2 - y_2$$

Since y_1 and y_2 are now the independent variable, then y_1 is equal to y_2. The shaded area between the two curves becomes:

$$x_2 - x_1 = (2 - y) - \sqrt{y}$$

Then, find the incremental area of each strip, dA.

$$dA = [x_2 - x_1] * dy = \left[(2 - y) - \sqrt{y}\right] * dy$$

Next, find the center point of each incremental area dA. For this orientation, where the shaded region is taken along the y axis, dy, the center points are defined as the following:

$$x^* = \frac{x_2 - x_1}{2} \qquad and \qquad y^* = y$$

Remark:: if the shaded region were taken along the x-axis, dx, then $x^* = x$ *and* $y^* = y/2$*.*

Finally, replace the expressions in the initial equation, where the integral along the y-axis should be taken from 0 to 1, as this is the extent of the shaded region.

$$\bar{y} = \frac{\int_A (y^*)dA}{\int_A dA} = \frac{\int_0^1 y * (2 - y - \sqrt{y})dy}{\int_0^1 (2 - y - \sqrt{y})dy} = \frac{\int_0^1 (2y - y^2 - y^{\frac{3}{2}})dy}{\int_0^1 (2 - y - \sqrt{y})dy}$$

The correct answer is (B).

10.10 SOLUTION 10 - CENTROID

Find the equation for the centroid of a circular segment in the Statics section of the *NCEES FE Reference Handbook.* The figure is rotated 90 degrees, so switch the x and y equations and shift the center point of the x centroid. The equations become

$$\bar{x} = \frac{12ft * \cos(60°)}{1} = 6ft$$

$$\bar{y} = \frac{2 * a}{3} \frac{\sin^3 \theta}{\theta - sin\theta * cos\theta}$$

In the above equation, a=12ft and θ = 90°-60° = 30° = π/6.

$$\bar{y} = \frac{2 * 12ft}{3} * \frac{\sin^3(30°)}{\pi/6\ - \sin(30°) * \cos(30°)} = 11.04ft$$

So, the centroid, $(\bar{x}, \bar{y})$ = (6ft, 11ft)

The correct answer is (A) (6ft, 11ft)

7 – Dynamics, Kinematics & Vibrations

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Section 7.0 – Dynamics, Kinematics and Vibrations

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1.0 INTRODUCTION

Dynamics, Kinematics and Vibrations accounts for approximately 9 to 14 questions on the Mechanical FE exam. This section is first broken down into the two topics, (1) particles and (2) rigid bodies. Particles are a single mass subject and rigid bodies consist of a collection of particles into a solid body that does not deform. The kinematics, work, energy, impulse and momentum and kinetics equations first focus on a single mass subject. You may be familiar with problems that had a block moving down a hill or a ball being thrown. These types of problems covered particles. Problems on rigid bodies are those that contain mechanical components like cams where one part of the body moves relative to another part of the body.

Kinematics covers the movement, speed and acceleration of particles and rigid bodies. This includes radial movement and movement due to gravity. Kinetics builds upon kinematics by including force and energy, which also transitions into the work-energy topic. Lastly, this section covers friction and impulse-momentum.

Section 7.0 Dynamics, Kinematics & Vibrations (9 to 14 Problems)		
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2.0 KINEMATICS OF PARTICLES

Kinematics type problems will be centered on finding one of these three variables, (1) the distance traveled by an object, (2) the velocity of an object or (3) the acceleration of an object at any given time or location. If a problem asks for one of these three variables and forces are not involved, then most likely the solution will be found using the equations presented in the kinematics topic.

Within the topic of kinematics you may encounter problems with either linear motion or angular motion. Linear motion is the movement of an object within the x-y-z plane in either a straight line or a curve. Curve type movement is typical of projectiles and straight line movement is typical of vehicles, sliding blocks, pistons and springs. Angular motion is the circular movement of an object about an axis, within the x-y-z plane. This type of movement is typical of gears, pumps, fans and any other equipment that rotates about an axis.

2.1 LINEAR MOTION

Linear motion can be characterized by its displacement, velocity and acceleration equations.

2.1.1 Linear Displacement

The first equation in linear motion is displacement, represented by the variable, *x*, as a function of time, *t*, also expressed as the function, *x(t)*. The initial position is given as x_i and the final position is represented as x_f. If the velocity of a particle is constant, then the acceleration is zero and the final position can be expressed as the following equation.

$$Constant\ velocity \rightarrow x_f = x_i + vt; a = 0;$$

On the FE exam, you can use the following equations to help you solve any kinematics problems for the final position (x_f) when there is constant acceleration.

Solving for Distance with Uniform Acceleration *(a=constant acceleration; V_i=initial velocity)*			
$x_f = x_i + V_i t + \frac{1}{2}at^2$	When $x_i = 0 \Rightarrow$		
	$x_f = \frac{(V_f + V_i)t}{2}$	$x_f = \frac{(V_f^2 - V_i^2)}{2a}$	$x_f = V_f t - \frac{1}{2}at^2$

2.1.2 Linear Velocity

The instantaneous velocity at a time, *t*, is the derivative of the position.

$$v(t) = \frac{dx}{dt};\ Units \rightarrow \frac{m}{s}\ or \frac{ft}{s}$$

The average velocity of an object can be found by dividing the change in position over a specific time interval.

$$Average\ velocity = \frac{\Delta x}{\Delta t}$$

On the FE exam, you may use the following equations to help you solve any kinematics problems for the specific scenario of uniform acceleration.

Solving for Velocity with Uniform Acceleration *(a=constant acceleration; V_i=initial velocity; x_i=0)*			
$V_f = V_i + at$	$V_{avg} = \frac{x_f - x_i}{t}$	$V_f^2 = V_i^2 + 2a(x_f - x_i)$	$V_f = \sqrt{V_i^2 + 2ax_f}$

2.1.3 Linear Acceleration

The instantaneous acceleration at a time, *t*, is the derivative of the velocity or the second derivative of the position.

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}; \;\; Units \rightarrow \left[\frac{m}{s^2}\right] or \left[\frac{ft}{s^2}\right]$$

The average acceleration of an object can be found by dividing the change in velocity over a specific time interval.

$$Average\ acceleration = \frac{\Delta v}{\Delta t}$$

If the object is increasing in speed (accelerating), then acceleration will be positive. If the object is decreasing in speed (decelerating), then acceleration will be negative. If the object is moving at constant velocity then acceleration will be equal to 0.

Solving for Acceleration with Uniform Acceleration *(a=constant acceleration; V_i=initial velocity; x_i=0)*			
$a = \frac{(V_f^2 - V_i^2)}{2x_f}$	$a = \frac{v_f - v_i}{t}$	$a = \frac{2(x_f - V_i t)}{t^2}$	$a = \frac{2(V_f t - x_f)}{t^2}$

2.1.4 Projectiles

When completing these types of problems, realize that motion can occur in the x, y, and z-axis and equations must be produced for each axis. For example, projectiles can move in the x and y-axis at the same time, but acceleration due to gravity only occurs in the y-axis. The previously discussed equations can be re-written for the x and y-axis, based on the assumption that there is no x-acceleration and that the only y-acceleration is downward (negative) due to gravity.

x-axis *(a = 0)*	y-axis *(a = gravity = -g)*
$V_{f,x} = V_{i,x} + \cancel{at}$	$V_{f,y} = V_{i,y} - gt$
$x_{f,x} = x_{f,x} + V_{f,x}t - \cancel{\frac{1}{2}at^2}$	$x_{f,y} = x_{f,y} + V_{f,y}t - \frac{1}{2}gt^2$
$V_{f,x} = \sqrt{V_{i,x}^2 + \cancel{2ax_f}}$	$V_{f,y} = \sqrt{V_{i,y}^2 - 2gx_f}$

2.2 ANGULAR MOTION

Similar to linear motion, angular motion can be characterized by its equations governing displacement, velocity and acceleration.

2.2.1 Angular Displacement

The first variable is angular displacement, or the amount of rotation that has occurred as a function of time, $\theta(t)$. When the angular velocity is constant, the angular acceleration is zero and the final angular position can be expressed as the following equation.

$$Constant\ angular\ velocity \rightarrow \theta_f = \theta_i + \omega t; \alpha = 0;$$

$$where\ \theta_i = initial\ angle;\ \theta_f = final\ angle;\ \omega = angular\ velocity\ (radians\ per\ second);$$
$$\alpha = angular\ acceleration\ (rad/s^2)$$

$$Conversion \rightarrow \quad \omega\left(\frac{rad}{s}\right) = \left(\frac{\pi}{30}\right) * n(rpm)$$

$$n = angular\ velocity\ (revolutions\ per\ minute)$$

On the FE exam, you may use the following equations to help you solve any kinematics problems for the angle in radians (θ_f) when there is constant angular acceleration.

Solving for Angular Rotation with Uniform ***Angular*** Acceleration *(α=constant angular acceleration; ω_i=initial angular velocity)*			
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$\theta_i = 0 \Rightarrow$		
	$\theta_f = \frac{(\omega_f + \omega_i)t}{2}$	$\theta_f = \frac{(\omega_f^2 - \omega_i^2)}{2\alpha}$	$\theta_f = \omega_f t - \frac{1}{2}\alpha t^2$

2.2.2 Angular Velocity

The instantaneous angular velocity at a time, *t*, is the derivative of the position.

$$\omega(t) = \frac{d\theta}{dt};\ \ Units \rightarrow \frac{rad}{s}$$

The average angular velocity of an object can be found by dividing the change in radians over a specific time interval.

$$Average\ angular\ velocity = \frac{\Delta\theta}{\Delta t}$$

On the FE exam, you may use the following equations to help you solve any kinematics problems for the angular velocity with uniform acceleration.

Solving for Angular Velocity with Uniform ***Angular*** Acceleration *(α=constant angular acceleration; ω=angular velocity; θ_i = 0)*			
$\omega_f = \omega_i + \alpha t$	$\omega_f = \frac{2\theta_f}{t} - \omega_i$	$\omega_f = \frac{\theta_f}{t} + \frac{1}{2}\alpha t$	$\omega_f = \sqrt{\omega_i^2 + 2\alpha\theta_f}$

2.2.3 Angular Acceleration

The instantaneous angular acceleration at a time, *t*, is the derivative of the angular velocity or the second derivative of the angular position.

$$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2};\ \ Units \rightarrow \left[\frac{rad}{s^2}\right]$$

The average angular acceleration of an object can be found by dividing the change in angular velocity over a specific time interval.

$$Average\ angular\ acceleration = \frac{\Delta\omega}{\Delta t}$$

If the object is increasing in speed (accelerating), then acceleration will be positive and if the object is decreasing in speed (decelerating), then acceleration will be negative. If the object is moving at constant velocity then acceleration will be equal to 0.

Solving for Angular Acceleration with Uniform ***Angular*** Acceleration *(α=constant angular acceleration; ω=angular velocity; θ_i = 0)*			
$\alpha = \frac{(\omega_f^2 - \omega_i^2)}{2\theta_f}$	$\alpha = \frac{\omega_f - \omega_i}{t}$	$\alpha = \frac{2(\theta_f - \omega_i t)}{t^2}$	$\alpha = \frac{2(\omega_f t - \theta_f)}{t^2}$

3.0 KINETIC FRICTION

Friction is the force that resists the movement of surfaces that slide upon one another. Most problems on the FE exam will assume that the surfaces are frictionless; if a friction factor or frictional force is not provided, it is safe to assume that the surfaces are frictionless. However, there should be a few problems that incorporate friction. For these problems it is important to understand two main concepts.

First, the object in question will have a normal force that is perpendicular to the contacted surface. Secondly, there will be a force that is tangent to the surface and opposite to the movement of the object. If the object is not moving, then the force is opposite the direction that the object is approaching movement. For example, if the object is on a slope, then the impeded motion is downward and the frictional force will be opposite this direction.

There are two types of friction: static friction and kinetic friction. When an object is at rest, there will be no motion until the applied force is greater than the maximum force due to static friction. Once the applied force is greater than this value, the object will move and the friction force that opposes the movement will change from static friction to kinetic friction, as shown below. Static friction was covered in more detail in *Section 7.0 Statics*.

$$Static\ Friction \rightarrow F_{max} = \mu_s * N$$

$$Kinetic\ Friction \rightarrow F_k = \mu_k * N$$

$$where\ \ \mu_s = coefficient\ of\ static\ friction\ [unitless];$$
$$\mu_k = coefficient\ of\ kinetic\ friction\ [unitless]; N = normal\ force\ [N\ or\ lbf]$$

There are three types of problems involving friction on the FE exam. These problems are based on a sliding block of mass “m,” under the following three scenarios: (1) A sliding block on a horizontal surface, (2) a sliding block on an angled surface and (3) a sliding block on an angled surface with an external force.

3.1 HORIZONTAL MOVING BLOCK

In this scenario a horizontal force, $F_{pull/push}$, is applied in the opposite direction of the frictional force, F_f. Because the external force is parallel to the ground, its entire magnitude is applied to the horizontal component. If, however, the external force is applied at an angle, it must first be broken up into its x and y components. Then, a balanced equations for the x and y forces can be made to solve for the unknowns.

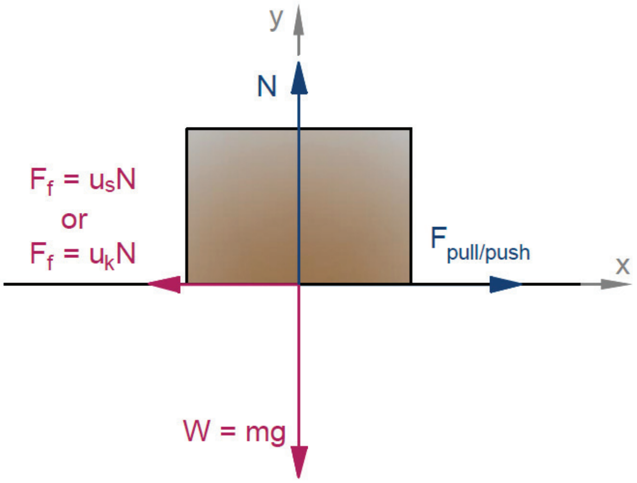


Figure 1: A horizontal moving block is subject to frictional forces in the opposite direction of movement.

The following table is a summary of the equations that balance both the x and y direction forces. Although this example is very simple, you should be able to distinguish between the x and y directions for the more complex FE problems.

X-Direction (F_x)	Y-Direction (F_y)
$F_f = F_{pull/push};$	$N = mg$
$Impending\ Motion:$ $\mu_{static} N = F_{pull/push};$	$Impending\ Motion:$ $N = mg$
$Moving\ F_x:$ $\mu_{kinetic} N = F_{pull/push};$	$Moving:$ $N = mg$

3.2 ANGLED SLIDING BLOCK WITHOUT EXTERNAL FORCE

In this scenario a block is placed on an angled surface. The block is subject to both gravitational and frictional forces. Because the block is angled, a portion of the gravitational force is applied in the x-direction, and the remaining portion is applied in the y-direction. Note that the x-axis and y-axis are always in relation to the object. In this case, the axes are rotated counterclockwise, where the x-axis is in the direction of the angled surface and the y-axis is perpendicular to this surface.

Tip: For these types of problems, begin by drawing all the forces on the object: gravity, friction, and normal. Then, draw the x-axis so that it is in-line with the surface and the y-axis so that it is in line with the normal component. Finally, rotate the entire object with the forces so that the x-y plane is upright. This will help you to visualize the x-components and y-components better.

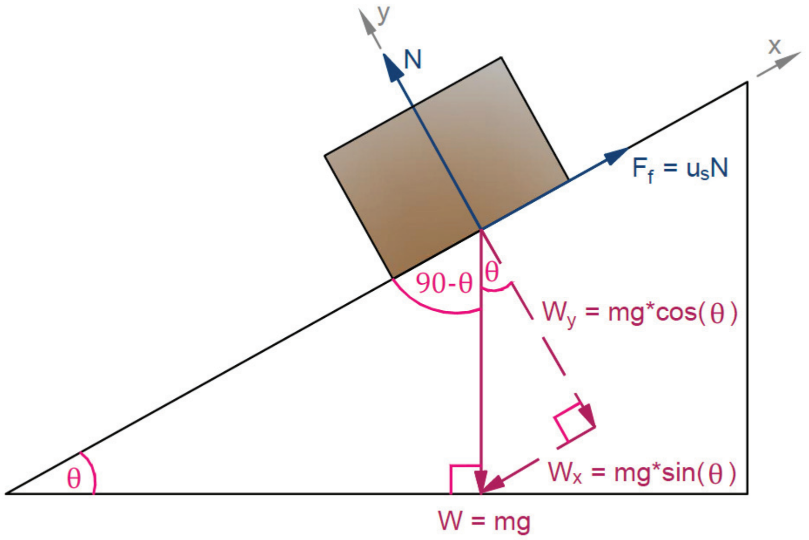


Figure 2: A block sliding down an incline is subject to frictional forces in the opposite direction of movement.

Similar to the first example, the forces are summed for both the x and y directions.

X-Direction (F_x)	Y-Direction (F_y)
$Impending\ Motion$: $\mu_{static}N = mg * sin(\theta);$	$Impending\ Motion$: $N = mg * cos(\theta);$
$Moving\ F_x$: $\mu_{kinetic}N = mg * sin(\theta);$	$Moving$: $N = mg * cos(\theta);$

3.3 ANGLED SLIDING BLOCK WITH EXTERNAL FORCE

This scenario is similar to the previous scenario. The block is subject to both gravitational and frictional forces. The main difference is that there is an additional external force applied to the block, shown as $F_{push/pull}$ in the next figure. This external force can either be directed up or down the ramp. The problem can become slightly more complex if the external force is not parallel with the surface, i.e. the x-axis. If this is the case then the external force must be broken into its x and y components, similar to how the weight is broken into its x and y components.

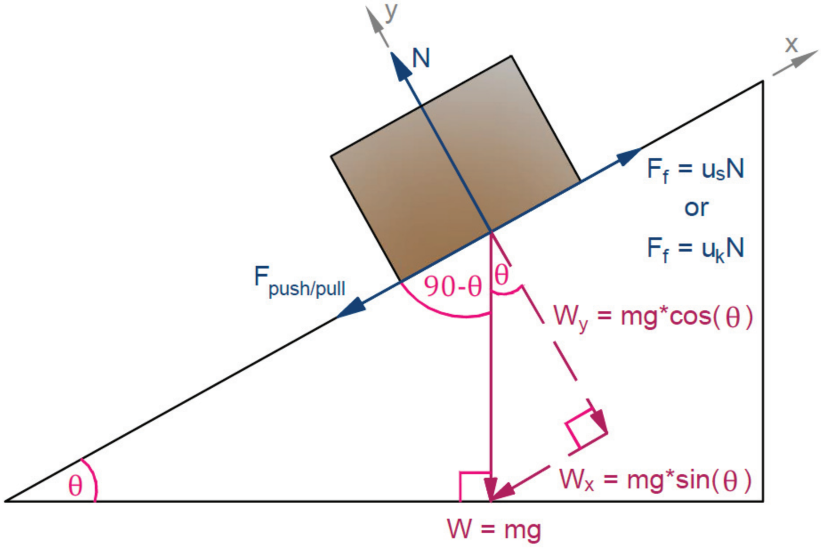
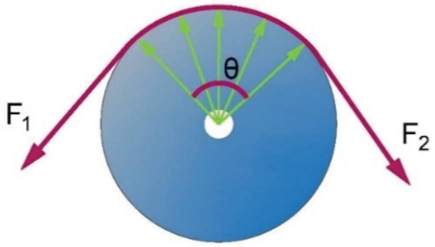


Figure 3: A sliding block with an external force acting upon the block

X-Direction (F_x)	Y-Direction (F_y)
$Impending\ Motion$: $\mu_{static}N = mg * sin(\theta) + F_{push\ or\ pull}$;	$Impending\ Motion$: $N = mg * cos(\theta)$;
$Moving\ F_x$: $\mu_{kinetic}N = mg * sin(\theta) + F_{push\ or\ pull}$;	$Moving$: $N = mg * cos(\theta)$;

3.4 BELTS AND PULLEYS FRICTION

Unlike the previous discussions of friction, the friction at belts and pulleys is dependent on the amount of contact with the rotating surface. At each point of contact, there will be a normal force from the center of the rotating part to the point of contact. In addition, there will be a frictional force at that point of contact, perpendicular to the normal force. A simple moment or force balance equation will not suffice for this situation. Instead, you must use the following formulas.



$$Pulley, Flat\ Belt\ Friction \rightarrow F_2 = F_1e^{\mu\theta}$$

$$\theta = Contact\ surface\ [radians];\ \mu = static\ or\ kinetic\ friction$$

Always make sure that your angle values are in radians, when using this equation.

If you encounter a V-belt situation, which is discussed in *Section 14.0 Mechanical Design & Analysis*, then you should use the following equation. The alpha term is the angular gap in radians between contact surfaces, since the belt is grooved. See Mechanical Design & Analysis for more discussion on this topic.

$$V - Belt\ Friction \rightarrow F_2 = F_1 e^{\mu\theta/\sin\left(\frac{\alpha}{2}\right)}$$

4.0 NEWTON'S SECOND LAW FOR PARTICLES

Newton's second law for particles states that the motion of particles or objects, for which net forces are not balanced, will be a function of the particles' or objects' mass and the net force acting upon the particles or objects. This law leads to the study of Kinetics.

$$Relative\ motion\ occurs \rightarrow \sum F = ma \neq 0$$

Kinetics is the study of motion. As shown above, if there is a non-zero resultant force acting upon a particle, then the particle will change in velocity, i.e. it will accelerate/decelerate. If there is a zero resultant force acting upon a particle, then the particle will remain at its same speed.

4.1 LINEAR MOMENTUM

The resultant force acting upon a particle over a period of time is equal to the particle's change in momentum. Momentum has the same vector direction as the velocity of the particle and it is equal to the particle's mass times its velocity.

$$Momentum = mv$$

$$SI \rightarrow m = kg; v = \frac{m}{s}$$

The units for mass for USCS is in slugs or lbf-s^2/ft. It is important that when you use the momentum and force equations in USCS that you watch the units. The units of mass you use in thermodynamics and fluids will be in pound-mass, while the units of mass you use in statics and dynamics will be in slugs. The units of force will be in lbf in statics and dynamics.

$$USCS \rightarrow m = slug = \left(\frac{lbf - s^2}{ft}\right); v = \frac{ft}{s}; a = \frac{ft}{s^2}; F = (lb\ or\ lbf)$$

For example, if you are given a particle mass of 200 lbm and wanted to know the force due to gravity, then you would have to first convert the mass to slugs.

$$m = \frac{200\ lbm}{32.2 \frac{ft}{s^2} * \frac{lbm}{lbf}} = 6.21\ slugs\ or \left(\frac{lbf - s^2}{ft}\right)$$

Now you can use the F=ma equation to find the force due to gravity.

$$F = ma = 6.21\left(\frac{lbf - s^2}{ft}\right) * 32.2\frac{ft}{s^2} = 200\ lbf$$

If you always show the units, when completing these problems or any other similar problems, then there should be no confusion.

The conservation of momentum equations state that the sum of the momentum of two objects will equal a constant, if there is no change in resultant force. The equation below shows two objects, with masses m_1 and m_2, moving at speeds v_1 and v_2, respectively. After a certain event, the mass and velocity of the two objects may change, but the sum of the momentums of both objects will equal the previous momentum.

$$Momentum\ Conserved \rightarrow \sum F = ma = 0$$

$$Conservation\ of\ Momentum \rightarrow m_1v_1 + m_2v_2 = m'_1v'_1 + m'_2v'_2$$

A very simple example of conservation of momentum is illustrated in the figure below. The forces before and after the collision of the two objects is equal to 1F to the right.

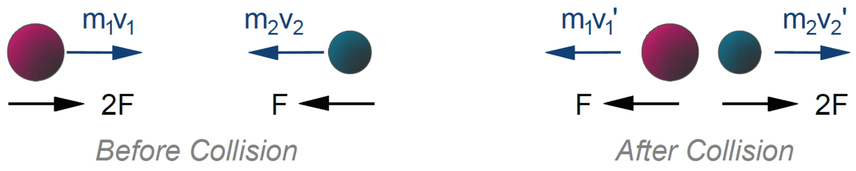


Figure 4: Object 1 with mass m_1 and velocity v_1 collides with object 2 with mass m_2 and velocity v_2. After collision, objects 1 and 2 have a new velocities v_1' and v_2', respectively. The resultant force and the total momentum are equal before and after the collision.

If there is a non-zero resultant force, then the momentum will change. This momentum change will be equal to the resultant force.

$$Momentum\ Not\ Conserved \rightarrow \sum F = ma \neq 0$$

The next equation shows that if there are two objects in the question, then the change in velocity multiplied by the mass of each object will equal the resultant force.

$$\sum F = m_1\dot{v}_1 + m_2\dot{v}_2$$

4.2 ANGULAR MOMENTUM

The conservation of angular momentum is the same as the previous conservation of momentum equation, except instead of mass, inertia is used and instead of velocity, angular velocity is used.

$$Momentum\ Conserved \rightarrow \sum F = ma = 0$$

$$Conservation\ of\ Angular\ Momentum \rightarrow I_1\omega_1 + I_2\omega_2 = I'_1\omega'_1 + I'_2\omega'_2$$

Similarly, a non-zero resultant force will result in a change in angular velocity of one or both objects.

$$\sum F = I_1\dot{\omega}_1 + I_2\dot{\omega}_2$$

4.3 MASS MOMENT OF INERTIA

Mass moment of inertia, generally referred to as just moment of inertia, characterizes an objects resistance to angular acceleration. This is distinguished between area moment of inertia, also known as the second moment of area, in the Statics section. The mass moment of inertia is described by the general equation below.

$$I = \int r^2 dm$$

Where $r = distance\ between\ the\ axis\ of\ rotation\ and\ the\ mass\ (m\ \ or\ \ ft)\ and\ dm = infinitesimally\ small\ sliver\ of\ mass$

For a point mass, M, rotating around an axis at distance, R, the mass moment of inertia is calculated as follows.

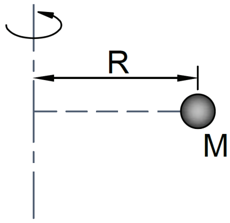


Figure 5: Moment of inertia of a particle with mass, M

$$I = \int_0^M R^2 dm = MR^2$$

$$M = mass\ (kg\ \ or\ \ slugs), R = distance\ (m\ \ or\ \ ft),$$
$$I = moment\ of\ inertia\ (kg \cdot m^2\ \ or\ \ slug \cdot ft^2)$$

For a rod with axis of rotation located at the end of the rod, the mass moment of inertia is as follows.

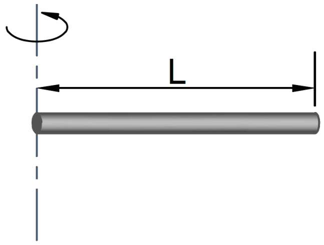


Figure 6: A solid cylinder, with one end on the y-axis, rotated about the y-axis has a moment of inertia as shown below.

$$I_y = \int_0^M x^2 dm = \frac{1}{3} ML^2;$$

For a rod with axis of rotation located at the center of the rod, the mass moment of inertia is as follows.

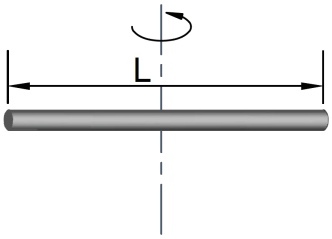


Figure 7: A solid cylinder, with its center at the y-axis, rotated about the y-axis has a moment of inertia as shown below.

$$I_y = \int_0^M x^2 dm = \frac{1}{12} ML^2;$$

Additional mass moment of inertias for various object shapes are provided in the *NCEES FE Reference Handbook*.

The radius of gyration is the distance from the axis of rotation to a mass point that has a mass equal to the mass of the object's body.

$$radius\ of\ gyration, R_G = \sqrt{I/M}$$

$$M = mass\ (kg\ \ or\ \ slugs), R_G = radius\ of\ gyration\ (m\ \ or\ \ ft),$$
$$I = moment\ of\ inertia\ (kg \cdot m^2\ \ or\ \ slug \cdot ft^2)$$

5.0 WORK-ENERGY OF PARTICLES

Another way to solve kinetic type problems on the FE exam is to use the conservation of energy and conservation of momentum equations. The basic concept is that energy is conserved and you can use these conservation equations to solve for unknowns. You should use Work-Energy equations for when the length of time is not a factor and you should use Impulse Momentum equations when length of time that a force acts is a factor. This may be a little confusing at first, but in Impulse Momentum equations the length of time that a force acts is important, where as in the Work-Energy equations it is not the time at which a force acts but the comparison between the forces, work, energy at one point in time compared to the forces, work and energy at another point in time.

5.1 CONSERVATION OF ENERGY

The conservation of energy equations state that the sum of all energy will equal a constant. Thus at time t =1 and at time t = 2, the sum of the total energy at both times will equal each other. On the FE exam, the most common types of energy will be kinetic energy and potential energy. The equations are shown below.

Kinetic Energy is the energy of a particle due to motion. Potential energy is the energy of a particle due to its relative height. There is also another potential energy for springs. When a spring is compressed there is potential energy due to the deformation of the spring from its normal length or free length.

5.1.1 SI Equations

The SI version is shown below.

$$Conservation\ of\ Energy \rightarrow KE_1 + PE_{1,height} + PE_{1,sping} = KE_2 + PE_{2,height} + PE_{2,spring}$$

$$\frac{1}{2}mv_1^2 + mgh_1 + \frac{1}{2}k\Delta x_1^2 = \frac{1}{2}mv_2^2 + mgh_2 + \frac{1}{2}k\Delta x_2^2$$

$$m = mass\ of\ particle\ (kg); v = velocity\ \left(\frac{m}{s}\right); g = 9.81\frac{m}{s^2}; h = relative\ height\ (m)$$

$$k = spring\ coefficient\ (N-m), \Delta x = spring\ displacement\ (m)$$

5.1.2 USCS Equations

The USCS version is shown below. The units for energy should be lbf-ft.

$$Conservation\ of\ Energy \rightarrow KE_1 + PE_{1,height} + PE_{1,sping} = KE_2 + PE_{2,height} + PE_{2,spring}$$

$$\frac{1}{2}(\frac{m}{g_c})v_1^2 + m\left(\frac{g}{g_c}\right)h_1 + \frac{1}{2}k\Delta x_1^2 = \frac{1}{2}(\frac{m}{g_c}s)v_2^2 + m\left(\frac{g}{g_c}\right)h_2 + \frac{1}{2}k\Delta x_2^2$$

$$m = mass\ of\ particle\ (lbm); v = velocity\ \left(\frac{ft}{s}\right); h = relative\ height\ (ft)$$

$$g = 32.2\frac{ft}{s^2}; k = spring\ coefficient\ (lb/ft), \Delta x = spring\ displacement\ (ft)$$

Be very careful with the USCS version by making sure that each component has the units (lbm-ft). Sometimes you may need to convert (lbf) to (lbm) with the gravitational constant (g_c term).

$$lbm * g = lbf * g_c \rightarrow$$

$$lbm * 32.2\frac{ft}{s^2} = lbf * 32.2\frac{lbm - ft}{lbf - s^2}$$

5.2 WORK VS. ENERGY

You also need to understand the difference between work and energy. Work is defined as the product of a force vector and the caused displacement. The work of a force acting upon a displacement is shown as the equation below.

$$W = F * \Delta d$$

$$F = N\ or\ lbf;\ \Delta d = m, ft. or\ in.;$$

$$W = N - m\ or\ Joule; W = lbf - ft\ or\ lbf - in$$

Work has the same units as energy, so it can also be added to the conservation of energy equations as shown below.

$$SI\ Units \rightarrow F_1 x_1 + \frac{1}{2}mv_1^2 + mgh_1 = F_2 x_2 + \frac{1}{2}m_2 v_2^2 + mgh_2$$

5.3 VECTOR EQUATIONS (X AND Y-DIRECTIONS)

You can also create balanced equations for the initial and final conditions for the various vectors too. On the FE exam you may need to break down the force vector into its x and y components.

$$x - direction \rightarrow F_{x,1} * x_1 + \frac{1}{2}mv_{x,1}^2 = F_{x,1} * x_x + \frac{1}{2}m_2 v_{x,2}^2$$

$$y - direction \rightarrow F_{y,1} * y_1 + \frac{1}{2}mv_{y,1}^2 + mgh_{y,1} = F_{y,1} * y_x + \frac{1}{2}m_2 v_{y,2}^2 + mgh_{y,2}$$

There is no potential energy due to relative height for the x-direction. The potential energy due to a spring could occur in either the x or y direction depending on the problem, for this reason the potential energy is not included in the equations above.

5.4 ROTATIONAL MOTION CONSERVATION OF ENERGY

Similar to the x and y-direction equations, you can also conserve energy in rotational motion. Work in rotational motion is equal to the torque multiplied by the angular distance. For the other equations, mass is replaced by moment of inertia and velocity is replaced by angular velocity.

$$T_{r,1} * \theta_1 + \frac{1}{2} I \omega_{r,1}^2 = T_{r,1} * \theta_x + \frac{1}{2} I_2 \omega_{r,2}^2$$

6.0 IMPULSE-MOMENTUM OF PARTICLES

Impulse-momentum equations can be used to solve problems where you are given force, mass, velocity and time. You should use these equations especially when there is an impact or when force is only applied for a certain period of time and then removed. Impulse is defined as the integral of force over a specific time period.

$$Impulse = \int_{t=t_1}^{t_2} F * dt$$

The units for impulse are shown below.

$$Impulse = F\Delta t = (N * s)\ or \left(\frac{kg * m}{s^2} x\ s\right)\ or\ (lbf * s)$$

If an object is subject to an impulse, then that impulse will change the momentum. An example of this effect is shown in the equation below. In this example, a particle has a mass and velocity as shown, then the particle is subject to an impulse with no change in the mass. This results in a change in velocity of the particle.

$$m_1 v_1 + Impulse = m_1 v_2$$

$$m_1 v_1 + F\Delta t = m_1 v_2$$

7.0 KINEMATICS OF RIGID BODIES

NCEES makes the distinction between rigid bodies and particles on its NCEES FE Mechanical outline, however the governing equations and concepts are the same for Kinematics. A rigid body is defined as a solid body that does not deform. This means that any force or energy acting upon a rigid body will be translated to motion as opposed to changing the internal energy or shape of the body.

The equations for kinematics of simple rigid bodies will be the same as those presented in Kinematics of Particles for linear and angular motion. The major difference will occur when you are doing problems on kinematics of mechanisms which is a specific and more complex type of rigid body. This is discussed in the next topics.

8.0 KINEMATICS OF MECHANISMS

A mechanism is a system of mechanical parts like members, frames, etc. A force acts upon the mechanism which causes motion throughout the mechanism. A simple example of a mechanism is a scissors. Force is applied to one side of a scissors, which causes motion on the sharp side of the scissors, which is used to cut materials.

The equations for kinematics of mechanisms will be the same equations as those for kinematics of particles and rigid bodies. However, mechanisms can become even more complex as you will need to determine the relationships between links, cams, levers, etc. These types of questions typically require a software program to complete.

The main topic that you should understand for Kinematics of Mechanisms is the two-bar mechanism, because this is the simplest mechanisms that can test your understanding of kinematics without requiring a software program. You may need to know the four-bar mechanism, but the four-bar requires multiple equations with multiple unknowns which will be difficult to test in a few minutes.

8.1 TWO BAR MECHANISM (SLIDER-CRANK)

The two bar mechanism or slider crank is used in mechanical engineering to translate rotational motion to linear motion. A bar is rotated about a point and the rotating bar is linked to a second bar, such that rotation is allowed. The other end of the second bar is constrained to linear motion. In the following figure, the 1st bar is labeled A-B and it is rotated about the point A. The 2nd bar is labeled B-C and point C is on rollers such that its motion is constrained to linear movement back and forth. For the two bar mechanism, you need to be able to calculate the displacement, velocity and acceleration.

8.1.1 Displacement

The displacement is fairly easy to model with the math skills learned in the geometry topic of *Section 1.0 Mathematics*.

8.1.2 Velocity

You need to be familiar with calculating the velocity at various points along the two bar mechanism. The following example will be used to develop your understanding of the slider-crank mechanism. In this example, bar A-B is 3 feet long, bar B-C is 5 feet long and bar A-B rotates at 1,000 RPM.

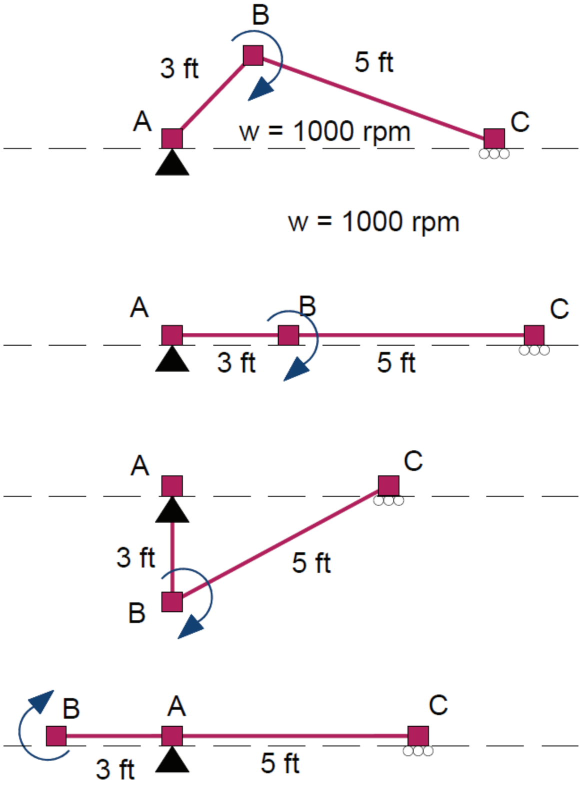

B
3 ft
5 ft
A
w = 1000 rpm
C
w = 1000 rpm
A
B
C
3 ft
5 ft
A
C
3 ft
5 ft
B
B
A
C
3 ft
5 ft

Figure 8: This figure shows the position of the bars at various rotations. Position A does not move, but position C moves linearly in the x-direction.

The first skill that you need for this problem is converting the rotational speed to a linear velocity. This is done by converting revolutions per minute to an angular speed in radians per second.

$$w_B = 1{,}000\ RPM = 1000\frac{rev}{min} * \left(\frac{1\ min}{60\ s}\right) * \left(\frac{2\pi\ rad}{1\ rev}\right) = 104.7\frac{rad}{s}$$

Then you can calculate the linear speed of point B at any of the situations shown below.

$$v_B = \left(104.7\frac{rad}{s}\right) * (3\ ft) = 314.2\frac{ft}{s}$$

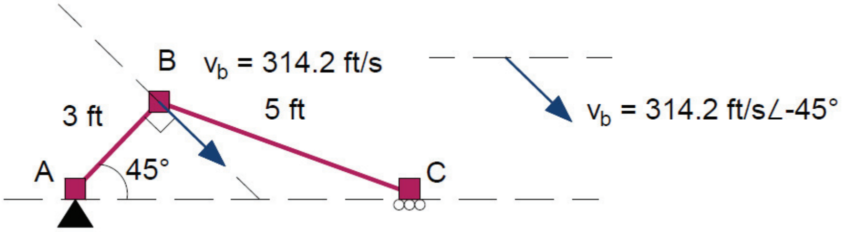

B
v_b = 314.2 ft/s
3 ft
5 ft
v_b = 314.2 ft/s∠-45°
A
45°
C

Figure 9: The velocity vector at B points to negative 45 degrees.

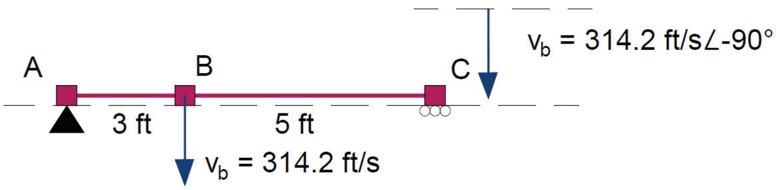

A
B
C
3 ft
5 ft
v_b = 314.2 ft/s
v_b = 314.2 ft/s∠-90°

Figure 10: At this point, the velocity vector at B points to negative 90 degrees.

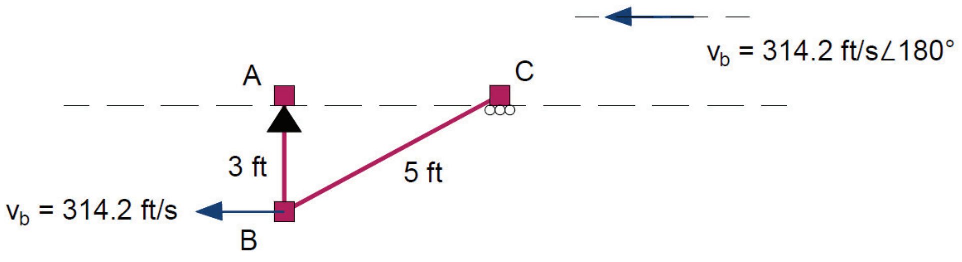

A
C
3 ft
5 ft
B
v_b = 314.2 ft/s
v_b = 314.2 ft/s∠180°

Figure 11: At this point, the velocity vector points to the left at an angle of 180 degrees.

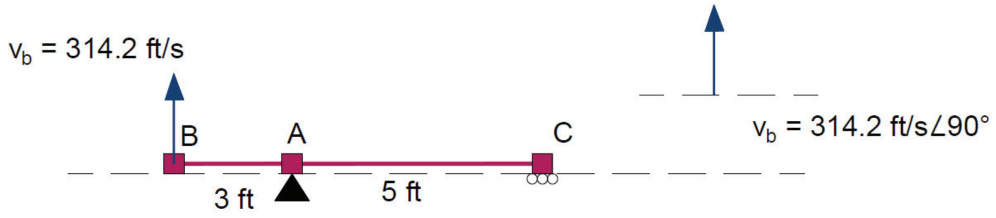

v_b = 314.2 ft/s
B
A
C
3 ft
5 ft
v_b = 314.2 ft/s∠90°

Figure 12: At this point, the velocity vector at point B points at an angle of 90 degrees.

After you calculate the velocity vector at various points, you can then calculate the velocity vector at point C. This is done by translating the linear velocity and the rotational velocity from point B. Translating the linear velocity simply means moving vector B to point C. Adding the rotational velocity from point B to point C is a little more difficult. First, you need the angle at point C as shown below.

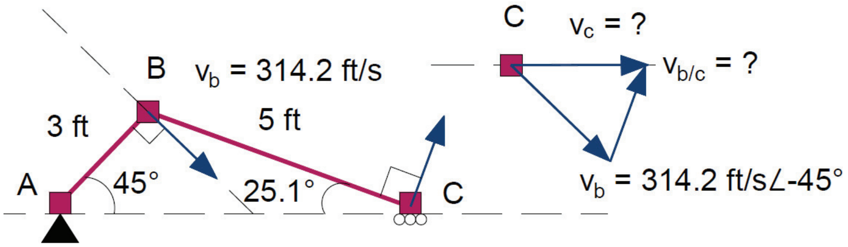

B
v_b = 314.2 ft/s
3 ft
5 ft
A
45°
25.1°
C
C
v_c = ?
$v_{b/c}$ = ?
v_b = 314.2 ft/s∠-45°

Figure 13: This figure shows graphically how to find vector c.

The Law of Sines is used on the triangle A-B-C, in order to find angle C as 25.1 degrees.

$$Law\ of\ Sines \rightarrow \frac{\sin(45°)}{5\ ft} = \frac{\sin(x°)}{3\ ft}$$

$$x = 25.1°$$

Vector B/C results from the rotational velocity, so it is shown as tangent to bar B-C. The angles of the vector b/c can be found through geometry. The angle of vector c is constrained to be only horizontal, so its angle is equal to 0 degrees.

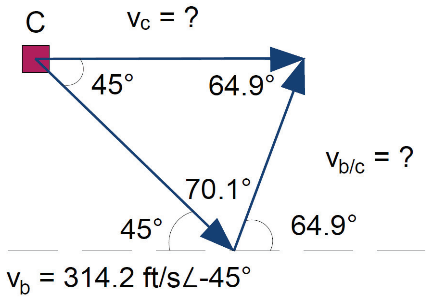


Figure 14: This figure focuses on point C and the addition of vectors b and b/c.

The Law of Sines must be used again to find the velocity vector magnitudes V_c and $V_{b/c}$.

$$Law\ of\ Sines \rightarrow \frac{\sin(45°)}{V_{b/c}} = \frac{\sin(70.1°)}{v_c} = \frac{\sin(64.9°)}{314.2}$$

$$V_{b/c} = 245\ \angle 64.9°; V_c = 326.2\ \angle 0°\ ft/s;$$

This same process can be repeated for all the other positions of bars A-B and B-C.

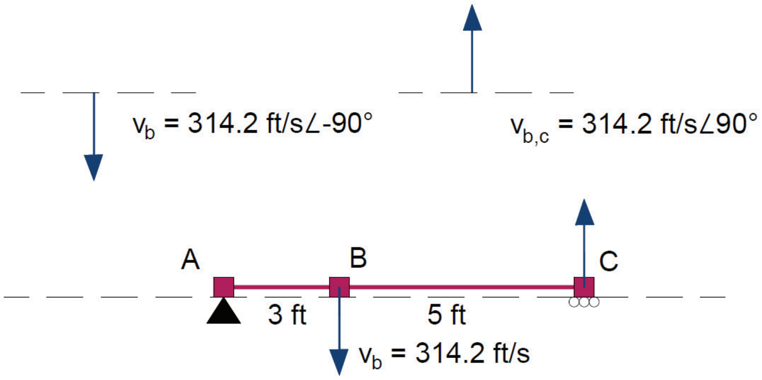
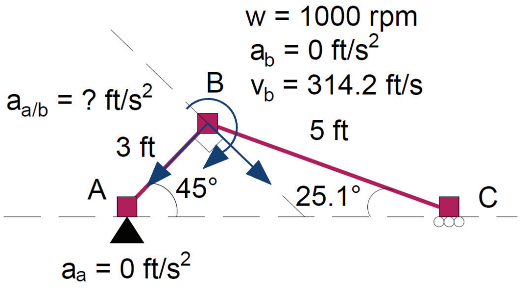


Figure 15: At this position, the vectors b and b, c cancel out and there is no linear velocity for vector c. The same is true when point B is located mirrored around point A from this current figure.

8.1.3 Acceleration

The equations that you need to translate the acceleration from one point of the mechanism to another point on the mechanism is shown below, using the example that was used in the velocity paragraphs.



$$a_b = a_a + a_{b/a;n} + a_{b/a;t}$$

$$a_a = accerlation\ of\ point\ a;\ a_{b/a;n} = accerlation\ of\ bar\ a - b\ pointing\ normal\ towards\ a$$

$$a_{b/a;t} = accerlation\ of\ bar\ a - b\ pointing\ tangent$$

The normal vector of acceleration can be thought of as that centripetal force that pulls point B inwards in this rotational motion. The tangent vector of acceleration is pointing the same direction as the velocity vector. Since the velocity magnitude does not change (only the direction changes), then the acceleration in the tangent direction is equal to zero. Also the acceleration of point A is also equal to zero.

$$a_b = \cancel{a_a} + a_{b/a;n} + \cancel{a_{b/a;t}}$$

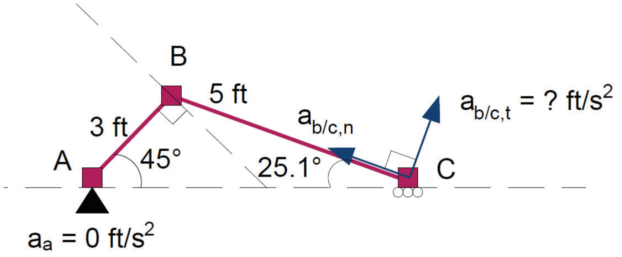
The only acceleration vector you need for point B is the normal acceleration. This acceleration is equal to the square of the angular velocity multiplied by the length of bar A-B.

$$a_b = a_{b/a;n} = l_{AB} * w_b^2$$

$$w_B = 104.7\frac{rad}{s};\ l_{AB} = 3\ ft$$

$$a_b = a_{b/a;n} = 3 * \left(104.7\frac{rad}{s}\right)^2 = 32{,}886\frac{ft}{s^2}$$

The same general acceleration equation can be used to find the normal acceleration at point C.



$$a_{b/_{c;n}} = l_{BC} * w_c^2$$

$$w_c = \frac{V_{b/c}}{l_{BC}} = {245\frac{ft}{s}}\Big/{5\,ft} = 49\frac{rad}{s}$$

$$v_c = 245\frac{ft}{s}(given\ from\ previous\ example)$$

$$a_{b/_{c;n}} = l_{BC} * w_c^2 = 5\,ft * \left(49\frac{rad}{s}\right)^2 = 12{,}005\frac{ft}{s^2}$$

Please remember that the acceleration values will change as the mechanism moves, similar to how the velocity values change.

9.0 NEWTON'S SECOND LAW FOR RIGID BODIES

Similar to the topic Newton's Second Law for Particles, this topic focuses on the F = ma equation and also Kinetics but for rigid bodies. The same equations are used, so please refer to Newton's Second Law for Particles.

10.0 WORK-ENERGY OF RIGID BODIES

Kinetics is the study of the relationship between the motion in kinematics and forces acting upon masses. The previous force diagrams assumed that the forces balanced, thus there was zero net force acting upon the objects. However, if there is a net force that is not balanced, then the object will experience acceleration directly proportional to the magnitude of this force and inversely proportional to the mass of the object. The equation below summarizes this statement.

$$Newton's\ 2nd\ Law \rightarrow \sum F_x = ma_x\,; \sum F_y = ma_y\,; \sum F_z = ma_z$$

Similar to the topic on Friction, this law applies to all directions, x, y and z. This law also applies to angular type problems as well.

$$\sum \tau = I\alpha$$

$$where\ I = moment\ of\ inertia;\ \alpha = angular\ moment$$

10.1 CONSERVATION OF ENERGY AND MOMENTUM

Another way to solve kinetic type problems on the FE exam is to use the conservation of energy and conservation of momentum equations. The basic concept is that energy and momentum is conserved and you can use these conservation equations to solve for unknowns.

The conservation of energy equations state that the sum of all energy will equal a constant. Thus at time t =1 and at time t = 2, the sum of the total energy at both times will equal each other. On the FE exam, the most common types of energy will be kinetic energy and potential energy. The equations are shown below.

$$Conservation\ of\ Energy \rightarrow KE_1 + PE_1 = KE_2 + PE_2$$

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}m'v'^2 + m'gh'$$

The conservation of momentum equations state that the sum of the momentum of two objects will equal a constant. The principle is the same as the conservation of momentum in particles. The equation below shows two objects, mass – m_1 and m_2 and these objects are moving at speeds v_1 and v_2. After a certain event, the mass and velocity of the two objects may change, but the sum of the momentums of both objects will equal the previous momentum.

$$Conservation\ of\ Momentum \rightarrow m_1v_1 + m_2v_2 = m'_1v'_1 + m'_2v'_2$$

The conservation of angular momentum is the same as the previous conservation of momentum equation, except instead of mass, inertia is used and instead of velocity, angular velocity is used.

$$Conservation\ of\ Angular\ Momentum \rightarrow I_1\omega_1 + I_2\omega_2 = I'_1\omega'_1 + I'_2\omega'_2$$

11.0 IMPULSE-MOMENTUM OF RIGID BODIES

Impulse and momentum of rigid bodies uses the same equations as impulse-momentum of particles. There are two more concepts that you should know for impulse-momentum of rigid bodies. The concepts are inelastic/elastic collisions and the coefficient of restitution.

11.1 INELASTIC VS. ELASTIC COLLISIONS

An elastic collision occurs when energy is conserved. This means that the ratio of the final to initial relative velocities between two objects is equal to 1. The final and initial relative velocity between two objects after a collision is the same as before the collision, assuming there is no change in mass.

11.2 COEFFICIENT OF RESTITUTION

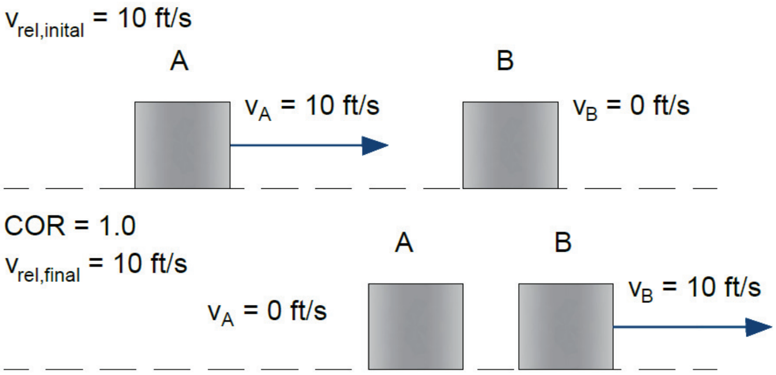
The coefficient of restitution is the ratio of the final to initial relative velocity between two objects after a collision occurs. This coefficient helps to characterize inelastic collisions. A perfectly elastic collision is characterized with a coefficient of restitution of 1.0. A perfectly inelastic collision is characterized with a coefficient of restitution of 0.0.

$$Coefficient\ of\ Restitution = \frac{Relative\ velocity\ after\ collision}{Relative\ velocity\ before\ collision}$$

The equation can also be represented in terms of kinetic energy.

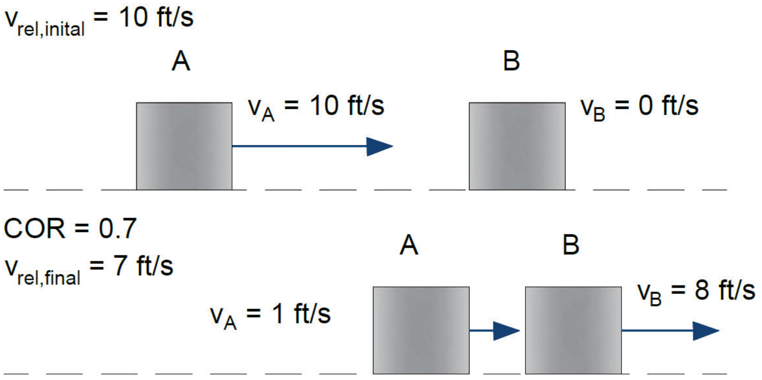
$$Coefficient\ of\ Restitution = \sqrt{\frac{\frac{1}{2}mv_{rel,final}^2}{\frac{1}{2}mv_{rel,initial}^2}}$$

The coefficient can also be shown graphically.



v_rel,inital = 10 ft/s
A
B
v_A = 10 ft/s
v_B = 0 ft/s
COR = 1.0
v_rel,final = 10 ft/s
A
B
v_A = 0 ft/s
v_B = 10 ft/s

Figure 16: In this figure the relative velocity between blocks A & B is 10 ft/s. After the collision, the relative velocity is still 10 ft/s.



v_rel,inital = 10 ft/s
A
B
v_A = 10 ft/s
v_B = 0 ft/s
COR = 0.7
v_rel,final = 7 ft/s
A
B
v_A = 1 ft/s
v_B = 8 ft/s

Figure 17: In this figure the relative velocity between blocks starts at 10 ft/s, but the relative velocity after the collision is 7 ft/s. This results in a COR of 0.7.

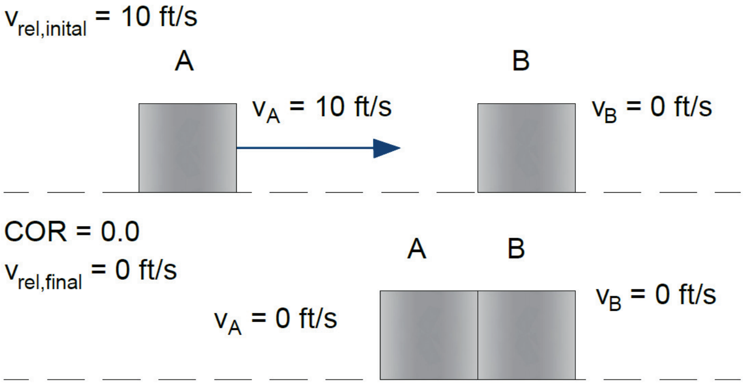


Figure 18: In this figure the relative velocity between blocks starts at 10 ft/s, but the relative velocity after the collision is 0 ft/s. This results in a COR of 0.0.

12.0 FREE AND FORCED VIBRATIONS

The topic of Vibration Analysis encompasses motion in machine design objects that are characterized by periodic, sinusoidal functions. Each object is characterized by inertia, stiffness or elasticity, damping and imparted energy or work. The sinusoidal functions that govern the movement of these objects take into account the characteristics of the machine design object. Free vibration refers to an object's vibration based on its internal characteristics. Forced vibration occurs when there is an external force also applied to the object.

12.1 SINUSOIDAL FUNCTIONS

Before these characteristics are discussed, you should first understand some basics of sinusoidal functions. First, these sinusoidal functions occur in periods or cycles. The number of cycles or periods completed in one second is defined as the frequency. Frequency is given in units of hertz (seconds^{-1}). A frequency of 60 hertz corresponds to a period of 0.0167 seconds, which means that one cycle is completed in 0.0167 seconds.

$$f = \frac{1}{T} \; or \; f * T = 1$$

$$f = frequency\ (Hz); T = period\ (sec)$$

Another basic term that is closely related to frequency is angular frequency. Angular frequency is the rotational frequency and is given in terms of radians per second. For example, if one rotation is completed in one second, then the angular frequency is 2π. Angular frequency is related to physical frequency through the following equation. Angular frequency is used for rotational machine design objects like shafts.

$$angular\ frequency\ \ \omega\ (radians/sec) = 2\pi f$$

Another defining feature of a sinusoidal function is the amplitude. The amplitude can be represented as either a root mean square (RMS) value or a maximum amount. Most often the amplitude will be represented as the maximum amount, however the RMS conversion is provided for completeness. The RMS value is used to find the average or effective value of a constantly varying value. The term RMS is found by taking the sum of "n" number of points along the waveform, squaring each value, summing all "n" values and then taking the square root of the sum.

The fundamental equation for calculating RMS is shown below. The term "y_n" is used to represent the y-value that corresponds to an x-value of "n".

$$RMS = \frac{\sqrt{y_1^2 + y_2^2 + y_3^2 \ldots + y_n^2}}{n}$$

In a basic sinusoidal waveform, the RMS value is determined for an infinite number of "n" points. This value is shown as 0.707 times the peak value of the sinusoidal graph.

$$RMS_{sine\ wave} = \frac{y_{max}}{\sqrt{2}}$$

The following figure shows a typical sinusoidal graph. The RMS equation applies to this sinusoidal function because there is no damping, which indicates that the maximum amplitude of the function does not change from one period to the next. In practice friction and damping will cause free vibrations to have a reduction in amplitude from one period to the next. In forced vibrations, there may be a consistent sinusoidal function and the second RMS equation shall apply. However, the first RMS equation will apply to both free and forced vibrations.

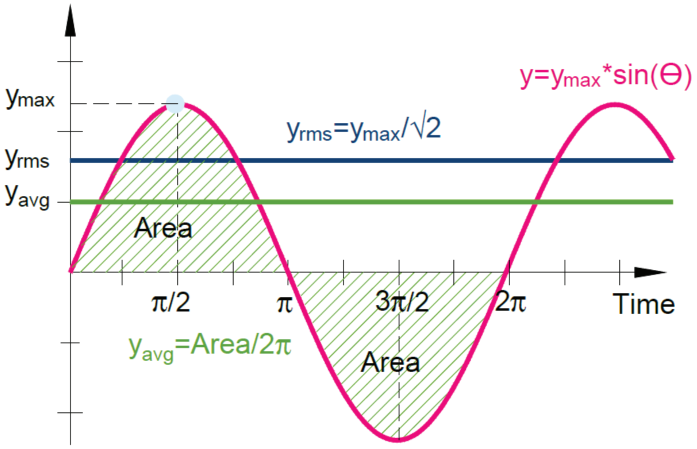


Figure 19: A typical sinusoidal function has a period of 2π and its RMS value is 0.707 times its maximum amplitude.

The basic sinusoidal function has the following equation in radians (not degrees) format.

$$y = \mathrm{A} * \cos(2\pi f t) + \mathrm{B} * \sin(2\pi f t)$$

$$f(hertz); t(seconds);$$

$$A = y(0) \rightarrow initial\ displacement\ at\ time = 0$$

$$B = \frac{\dot{y}(0)}{\omega_n} \rightarrow initial\ speed\ at\ time = 0;\ \omega_n\ is\ the\ natural\ frequency$$

This equation format will be applied to the position equation of the following spring-mass system. Understanding how to derive this equation be discussed in the next section. The main concept here is the characteristics of the sinusoidal function: amplitude, frequency, and period.

12.2 SPRING-MASS

The simplest type of vibration is a spring with mass moving along a single direction, thus having a single degree of freedom. A spring is used as the basis to analyze more complex machine design objects. A spring is described as a system with a spring constant or stiffness (k) and a mass (m). The stiffness is used to characterize the amount of force required to cause a certain amount of deflection in the spring. This is shown in the equation below.

$$F_{spring} = k * x$$

$$SI\ Units \rightarrow F[N];\ k = stiffness\ [N/m]; x = deflection[m]$$

$$US\ Units \rightarrow F[lbf];\ k = stiffness\ [lbf/in]; x = deflection[in]$$

Because this is a dynamics problem, and not a statics problem, the object connected to the spring will oscillate and the value of x in the equation above will vary with time. This deflection, x, is thus a function and not a constant variable, which can be demonstrated by the following figure. The force imparted by the mass and spring will result in a movement of the mass, up and down. The oscillation continues as the potential energy due to the compression or elongation of the spring balances with the kinetic energy of the moving mass. The following figure shows how the oscillation of the spring can be modeled as a sinusoidal, periodic function. Without friction or damping, the spring would oscillate forever and thus will have a continuous sinusoidal function.

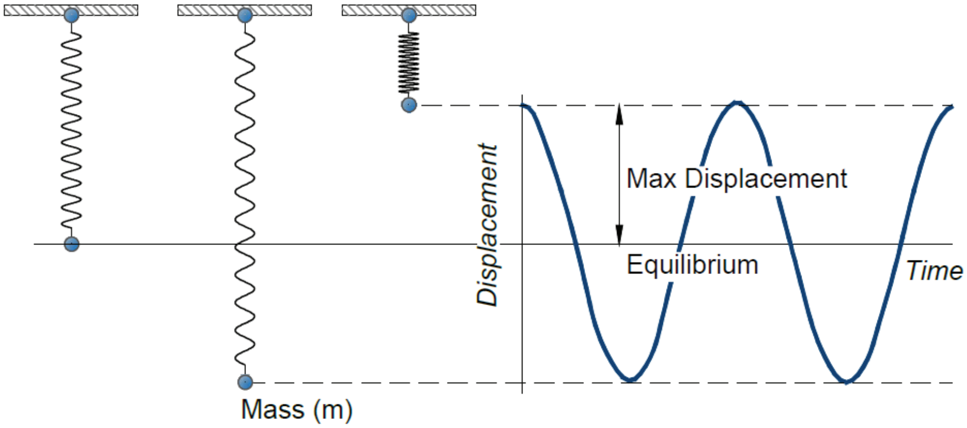


Figure 20: Vibrations can be modeled as a spring. This figure assumes an un-dampened spring that will oscillate until an external force is applied.

12.2.1 Energy

The total energy in an un-dampened spring remains constant based on the following equations. An un-dampened spring means that there are no frictional forces or damping forces.

$$Total\ Energy = PE_{spring} + KE_{mass};\ KE_{mass} = \frac{1}{2}mv^2; PE_{spring} = \frac{1}{2}kx^2$$

The total energy is equal to the potential energy of the spring at max displacement. At this point the kinetic energy of the mass is equal to zero because the velocity is equal to zero.

$$Total\ Energy = PE_{spring}\ @\ Max\ Displacement; KE_{mass} = 0$$

The total energy is equal to the kinetic energy of the mass at equilibrium. At this point the potential energy of the spring is equal to zero, because the displacement is equal to zero.

$$Total\ Energy = KE_{mass}\ @\ Equilibrium; PE_{spring} = 0$$

12.2.2 Force

By using a free body diagram, the total force in the spring-mass system can be equated to characterize the movement of the system, with and without an external force.

Free Vibration:

The following single degree of freedom, free vibration, spring-mass system has an initial deflection and moves freely without an external force. A free body diagram shows that the movement, *F=ma*, is produced by the force due to the resistance in the spring, F_k.

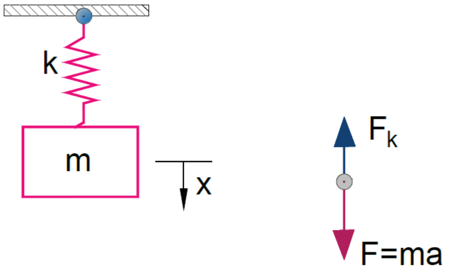


Figure 21: Left – Free vibration on a mass-spring system; Right – Free body diagram.

$$\Sigma F = m\ddot{x} = -kx$$

$$\Rightarrow\ m\ddot{x} + kx = 0$$

Side note: At rest, the force from the weight of the object will counteract the deflection of the spring when it is at equilibrium. Therefore, these two forces cancel out and are ignored; they do not *affect the displacement equation for x(t). Additionally,* $\ddot{x}$ *is the notation for acceleration, which is also the second derivative of displacement,* $a = d^2x/dt^2 = \ddot{x}$*.*

This above spring-mass equation is a second order, linear, *homogeneous* differential equation with constant coefficients. From the Mathematics chapter, recall that the solution for this type of equation falls under one of three scenarios. Since there is no first derivative coefficient, the solution will fall under scenario 1, underdamped. In fact, this is the special case where there is no damping; the system is "undamped." The solution for this common configuration is given as follows, where *x(t)* is the displacement.

$$x(t) = x_0 \cos(\omega_n \mathrm{t}) + \frac{v_0}{\omega_n} \sin(\omega_n \mathrm{t})\,;$$

$$where\ \ \omega_n = \sqrt{k/m}\ (natural\ frequency\ in\ radians), x_0 = initial\ position, v_0 = initial\ velocity$$

The natural frequency, ω_n, in the above equation will be discussed in the next section. The equation above can be simplified using trigonometric identities.

$$x(t) = C\sin(\omega_n \mathrm{t} + \varphi)$$

$$where\ the\ amplitude\ C = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}\ and\ the\ phase\ angle\ \varphi = \tan^{-1}\left(\frac{x_0}{v_0/\omega_n}\right)$$

Forced Vibration:

When there is an external force with its own frequency acting on the system, the vibration is "forced." Sum the forces of the free body diagram to produce the forced vibration equation.

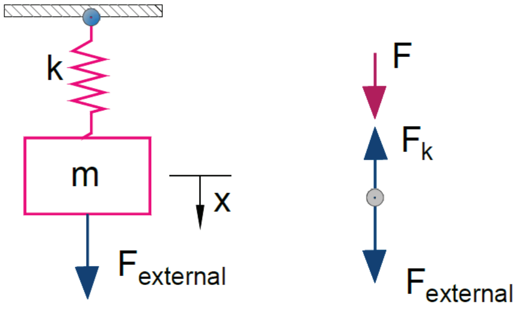


Figure 22: Left - Forced vibration on a mass-spring system; Right – Free body diagram.

$$\Sigma F = m\ddot{x} = -kx + F_{external}$$

$$\Longrightarrow\ \ m\ddot{x} + kx = F_{external}$$

This is a linear, *non-homogeneous* differential equation with constant coefficients. From the Mathematics chapter, we know that the solution has the following format.

$$x(t) = x_{homogeneous} + x_{particular}$$

The homogeneous solution, $x_{homogeneous}$, is solved separately, with the methods previously discussed. The particular solution, $x_{particular}$, is solved based on the format of the external force function. This process was also discussed in the Mathematics chapter. Summing the two solutions together will provide the full displacement function equation

Assume the force applied is given by the following equation, where F_0 is the amplitude and ω is the angular frequency of the applied force.

$$F_{external} = F_0 \sin(\omega t)$$

The total displacement equation with the particular solution for this common, single degree of freedom, forced vibration spring-mass system is solved below.

$$x(t) = x_{homogeneous} + x_{particular} = x_0 \cos(\omega_n \mathrm{t}) + \frac{v_0}{\omega_n} \sin(\omega_n \mathrm{t}) + X_0 \sin(\omega t)$$

$$where\ the\ amplitude\ X_0 = \frac{F_0}{k\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}}$$

For other spring-mass configurations, where there are multiple spring, varying orientations, etc, produce a free body diagram to equate all forces in the scenario. One useful rule to remember is that springs in parallel will have an equivalent spring constant equal to the sum of individual constants and springs in series will have an equivalent spring constant equal to the inverse of the sum of inverses.

$$k_{equiv,parallel} = k_1 + k_2 + k_3 + \cdots$$

$$k_{equiv,series} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots}$$

12.3 NATURAL FREQUENCY

The period and natural frequency of the periodic, sinusoidal function will depend on the mass and the spring stiffness. The formula that governs this equation is shown below. The term natural frequency is used because this frequency is only dependent on the properties of the mass and stiffness. This does not depend on forced or disturbing frequencies due to external forces acting upon the spring system. It is critical to determine the natural frequency of the mechanical system, because if a forced or disturbing frequency is close to the natural frequency then resonance will occur and vibrations will be magnified.

$$SI\ Units \rightarrow f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ (Hz)$$

$$English\ Units \rightarrow f_n = \frac{1}{2\pi}\sqrt{\frac{kg}{m}}\ (Hz)$$

Although springs are used as the basis for vibration analysis, all machine design items can be modeled similarly. Each machine design situation has a corresponding, k-value or stiffness based on its geometry, modulus of elasticity and/or shear modulus. A few examples of the stiffness values for the following situations are shown below.

Situation	*k-value equation*
Helical Coil Spring	$k = \frac{GD^4}{64Nr^4}$
Longitudinal Bar	$k = \frac{AE}{L}$
Simply Supported Beam at Mid-span	$k_{beam} = \frac{48EI}{L^3}$
Stiffness of a Cantilever Beam at its End	$k_{beam} = \frac{3EI}{L^3}$
Torsional Stiffness of Shaft	$k_{t,shaft} = \frac{JG}{L}$
Equivalent Stiffness of n Springs in Parallel	$k_{springs\,\|\|} = k_{s1} + k_{s2} + k_{sn};$
Equivalent Stiffness of n Springs in Series	$k_{springs,series} = \frac{1}{k_{s1} + k_{s2} + k_{sn}}$

Once you have found the correct k-value, then you can use the k-value to find the corresponding natural frequency with the previous equations. Another way frequency can be expressed is in radians format. In this format the 2π term is removed.

$$\omega_n = \sqrt{\frac{k}{m}}\ or\ \sqrt{\frac{kg_c}{m}}\left(\frac{radians}{second}\right);$$

The units can be found to be units as shown in the dimensional analysis below. The units for radians are dimensionless, since the definition of radians is the circular arc divided by the radius of the arc. This results in meters divided by meters for US units.

$$\omega_n = \sqrt{\frac{k}{m}} \rightarrow \sqrt{\frac{N/m}{kg}} \rightarrow \sqrt{\frac{\frac{\cancel{kg} * \cancel{m}}{s^2}/\cancel{m}}{\cancel{kg}}} \rightarrow \sqrt{\frac{1}{s^2}} \rightarrow \frac{1}{s}; (next\ multiply\ by\ \frac{circular\ arc(m)}{radius(m)})$$

The following shows the dimensional analysis for US units.

$$\omega_n = \sqrt{\frac{kg_c}{m}} \rightarrow \sqrt{\frac{\left(\frac{\cancel{lbf}}{\cancel{ft}}\right) * \left(\frac{\frac{\cancel{lbm * ft}}{\cancel{s^2}}}{\cancel{1\ lbf}}\right)}{\cancel{lbm}}} \rightarrow \sqrt{\frac{1}{s^2}} \rightarrow \frac{1}{s};$$

12.4 SPRING-MASS-DAMPER

When a damper is added to the spring-mass system, the amplitude of the displacement will change with time. An example is the friction caused by the spring that will cause the oscillation to stop. The figure below shows a simple spring-mass-damper configuration.

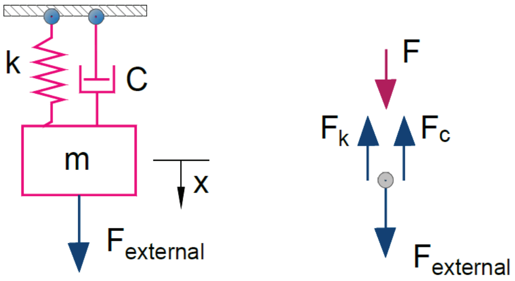


Figure 23: Left - Force on a spring-mass-damper system; Right – Free body diagram.

The force due to a damper is given by the equation below.

$$F_{damper} = c * \dot{x}$$

$$SI\ Units \rightarrow F[N];\ c = damping\ coefficient\ [N \cdot s/m]; \dot{x} = velocity[m/s]$$

$$US\ Units \rightarrow F[lbf];\ c = damping\ coefficient\ [lbf \cdot s/in]; \dot{x} = velocity[in/s]$$

The displacement equation is adjusted to add the effects of the damper, based on the free body diagram above. Below is the generic equation for a simple spring-mass-damper system. For other spring-mass-damper configurations, use a free body diagram to produce a specific force equation.

$$\Sigma F = m\ddot{x} = -kx - c\dot{x} + F_{external}$$

$$\Rightarrow \quad F_{external} = m\ddot{x} + c\dot{x} + kx$$

This is another second order, linear ordinary differential equation, this time with a first derivative coefficient.

12.4.1 Free Vibration

When $F_{external}$ is equal to zero, the equation is *homogeneous* and can be solved with the methods explained in the mathematics section.

$$m\ddot{x} + c\dot{x} + kx = 0 \;\; \Rightarrow \;\; x(t) = x_{homogeneous}$$

The three scenarios for the homogeneous solution are underdamped, overdamped, or critically damped. A fourth is added to the summary below: undamped, which was previously discussed in the mass-spring scenario without the damper.

Before discussing the scenarios, the terms critical damping coefficient and damping ratio are introduced. These characteristics are used to identify the system properties.

Critical Damping Coefficient: The damping coefficient of a critically damped system, which brings the oscillation to rest the fastest.

$$c_{critical} = 2\sqrt{km} = 2m\omega_n$$

Damping Ratio: The damping ratio has dimensionless units. It describes how slowly or quickly a system is returned to its original conditions after a temporary change to the original conditions. The damping ratio is defined as the ratio of the actual damping to the critical damping.

$$Damping\ Ratio \rightarrow \zeta = \frac{c_{actual}}{c_{critical}} = \frac{c_{actual}}{2\sqrt{km}}$$

The damping ratio is another way to determine which homogeneous solution scenario the displacement function x(t) will follow. It simplifies the format and the process for the spring-mass-damper type of problems.

Undamped $\rightarrow \zeta = 0$**:** If there is no damping then the damping ratio will equal 0. This situation was previously discussed in the spring-mass section and is called undamped. There will be no decay in the sinusoid and the amplitude of the sinusoid will remain the same as time progresses.

Underdamped$\rightarrow 0 < \zeta < 1$**:** If the ratio of the actual damping and critical damping is in-between 0 and 1, then the system will be underdamped. The sinusoid will decay slowly and with oscillations. See the mathematics section for the solution to an underdamped scenario.

Critically Damped$\rightarrow \zeta = 1$**:** This classification occurs when the actual damping equals the critical damping value. The sinusoid will exponentially return back to equilibrium. See the mathematics section for the solution to an underdamped scenario.

Overdamped$\rightarrow \zeta > 1$**:** The final classification occurs when the damping ratio is greater than 1. The system will decay exponentially back to equilibrium but not as fast as the critically damped classification. See the mathematics section for the solution to an underdamped scenario.

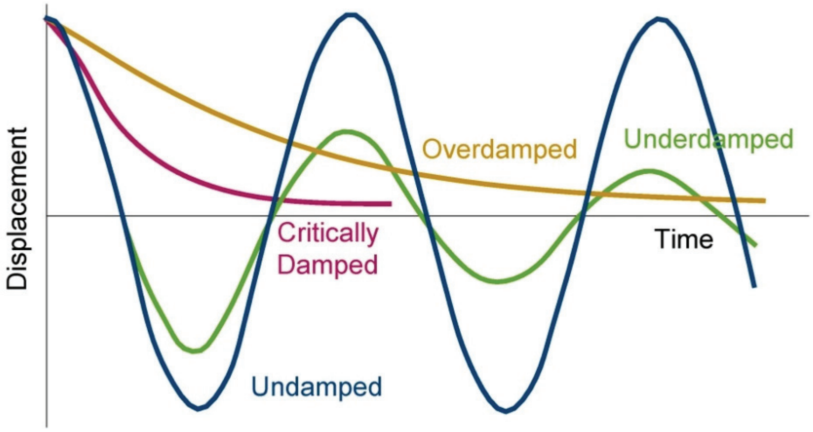


Figure 24: There are four classifications of damped systems that depend on the damping ratio. This graph shows the different effects of the different damping classifications.

12.4.1 Forced Vibration

When an external force is applied, the displacement equation becomes *non-homogeneous* and has the following solution, similar to the spring-mass system.

$$F_{external} = m\ddot{x} + c\dot{x} + kx \ \Rightarrow$$

$$x(t) = x_{homogeneous} + x_{particular}$$

However, with the damper added to the system, the homogeneous part of the solution, $x_{homogeneous}$, will diminish. This is known as a transient response, which means the function will only have amplitude early on. As time increases, the transient response (i.e. homogenous solution) will eventually equal zero. When this happens, the total displacement function will equal the particular solution and they system is said to be at steady state. Often times, we are more interested in the steady state solution.

$$x(t) \ = x_{particular}, for\ time\ t \gg 0 \ \ [steady\ state]$$

A common format for the external force is presented below, where F_0 is the amplitude of the force and ω is the forced angular frequency.

$$F_{external} = F_0 \sin(\omega t)$$

If given an external force with the above arrangement, the displacement function at steady state will equal to the following. The particular solution for this case is presented in the *NCEES FE Reference Handbook*. Observe how this solution compares to the forced spring-mass solution, without damper.

$$x(t) \ = x_{particular} = X_0 \sin(\omega t + \varphi)$$

$$where\ the\ amplitude\ X_0 = \frac{F_0}{k\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}};phase\ shift\ \varphi = \tan^{-1}\left(\frac{-\frac{2\zeta\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

The amplitude of the displacement equation, X_0, can also be found graphically based on the damping ratio and the frequency ratio. Locate the intersection of the two ratios on the graph to find the amplitude ratio, then solve for X_0. The same process can be done to find the phase angle. As you can see from the graph, when the forced frequency ω is close to the natural frequency, the amplitude explodes. This is the point of resonance. Damping helps to reduce this extreme amplification. The greater the damping ratio, the lower the resulting displacement amplitude, and thus the greater the effectiveness of the damping. The displacement amplitude can also be reduced by shifting the frequency ratio, away from resonance.

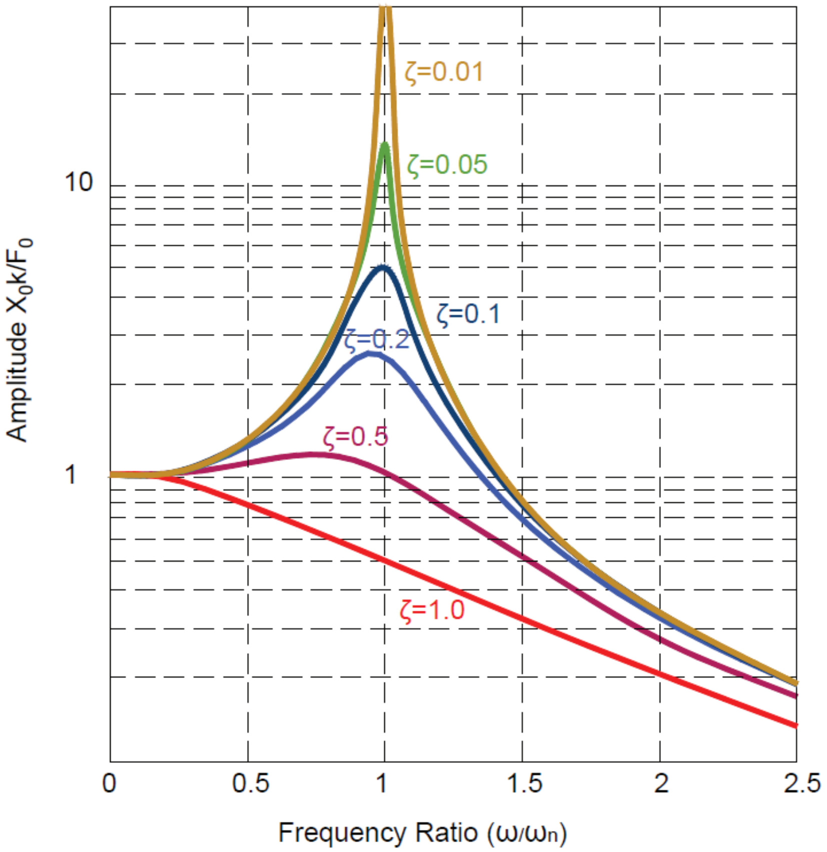


Figure 25: This graph shows the amplitude ratio as a function of various frequency and damping ratios.

12.5 TORSIONAL VIBRATION

Similar vibration principles apply when the displacement is angular and the object is under torsion. One common application is torsional vibration in rotating shafts. To illustrate this concept, assume there is a rod of length, *L*, with negligible mass. The rod is held in place at one end and connected to a large mass with moment of inertia, *I*, at the other end, as shown in the figure below. In this case, the "spring" is now the rod, which has a stiffness characterized by a torsional spring coefficient (stiffness), k_t.

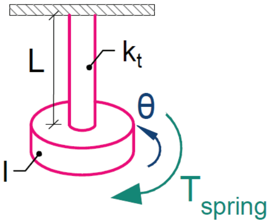


Figure 26: Torsional vibration of a rod with length, L, and torsional stiffness, k_t, connected to a mass, with moment of inertia, I.

When the rod is twisted, by an angle, *θ*, the stiffness of the rod resists the angular deflection and wants to twist back, like a spring. This produces a torque in the rod. The large mass at the bottom of the rod will propagate the twisting motion back and forth, by continuously translating the movement between kinetic and potential energy. The motion will continue to oscillate if there is no friction or damping, similar to the spring scenario.

The angular displacement in the rod can be modeled as a sinusoidal function over time with amplitude, A, oscillating around the equilibrium. In this discussion, the torsional vibration is free without external torque or damping.

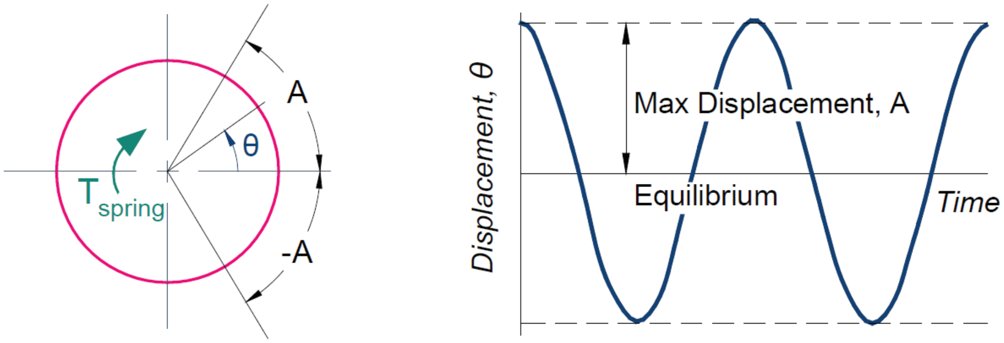


Figure 27: Angular displacement, θ, of the system due to torsional vibration.

The angular displacement equation can be found with a similar process to the spring-mass system, except the equations are now in turns of angular movement. The variables are first defined as below.

Torque: Similar to how the force due to the spring was defined by deflection times the spring stiffness, the torque due to the rod is the angular deflection times the torsional stiffness, k_t.

$$Torque, T\ [N \cdot m\ or\ lbf \cdot ft] = k_t\theta$$

Torsional Stiffness: Recall from Mechanics of Materials topic that angular deflection of a shaft due to torque has the following relationship: $\theta = TL/JG$. With these two equations, the torsional stiffness, k_t, of the rod can be solved below.

$$k_t = \frac{GJ}{L}$$

$$where\ k_t = torsional\ stiffness\ \left[\frac{N \cdot m}{rad}\ or\ \frac{lbf \cdot in}{rad}\right];$$

$$G = rod's\ shear\ modulus\ of\ elasticity\ [Pa\ or\ psi];$$

$$J = rod's\ polar\ moment\ of\ inertia\ [m^4\ or\ in^4]; and\ L = length\ of\ rod\ [m\ or\ in].$$

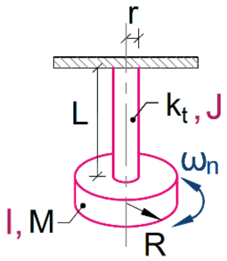
Natural Frequency: The natural frequency of the system can be solved based on the torsional stiffness and the mass moment of inertia of the large mass that propagates the rotation.

$$\omega_n\left[\frac{rad}{s}\right] = \sqrt{\frac{k_t}{I}}$$

$$where\ I = mass\ moment\ of\ inertia\ of\ the\ large\ mass\ [kg \cdot m^2\ \ or\ \ lbf \cdot ft \cdot s^2]$$

This equation can be simplified for a rod, with a radius, r, and polar moment of inertia, $J = \pi r^4/\,2$ connected to a disk with a mass M, radius R, and mass moment of inertia, $I = MR^2/\,2$. Tables for various polar and mass moments of inertia can be found in the *NCEES FE Reference Handbook.*

$$\omega_{n,rod\ w/\ disc} = \sqrt{\frac{k_t}{I}} = \sqrt{\frac{GJ}{IL}} = \sqrt{\frac{G\pi r^4}{MR^2L}}$$



Finally, using the free body diagram, the angular displacement equation for a free vibration system is found below.

$$\Sigma T = I\ddot{\theta} = -k_t\theta\ \Rightarrow$$

$$\ddot{\theta} + \frac{k_t}{I}\theta = 0$$

The solution to this second order, linear homogeneous differential equation is similar to the mass-spring solution.

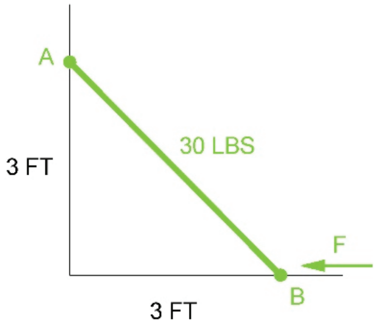
$$\theta(t) = \theta_0 \cos(\omega_n \mathrm{t}) + \frac{\dot{\theta}_0}{\omega_n} \sin(\omega_n \mathrm{t})$$

$$where\ \theta_0 = initial\ angular\ displacement\ [rad];\ \dot{\theta}_0 = intial\ angular\ velocity\ [rad/s]$$

13.0 PRACTICE EXAM PROBLEMS

13.1 PRACTICE PROBLEM 1 - FRICTION

A beam is leaning against a frictionless wall. The dimensions are shown on the below figure. The beam weighs 30 lbs and the surface at point B is assumed to have a static friction value of 0.4. What is the force, F, required to keep the beam from moving?



The answer is most nearly,

(a) 3 LBS

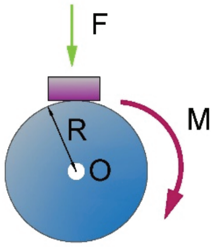
(b) 14 LBS

(c) 30 LBS

(d) 33 LBS

13.2 PRACTICE PROBLEM 2 - FRICTION

A braking system applies a force of 1,000 N (F) to a rotating part that has a moment of 100 N-m. The rotating part is pinned at O and is allowed to rotate about O. The coefficient of kinetic friction is 0.4. What is the reaction force in the X-direction at point O? The radius of the rotating part is R = 250 mm.



The answer is most nearly,

(a) 100 N

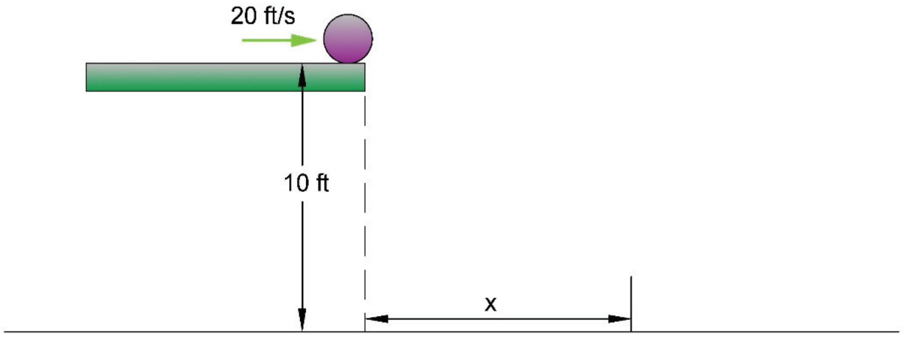
(b) 200 N

(c) 300 N

(d) 400 N

13.3 PRACTICE PROBLEM 3 - KINEMATICS

An object has a horizontal speed of 20 ft/s. It leaves a surface that is 10 feet above the ground. At what distance “x” will the object hit the ground?



(a) 10 ft

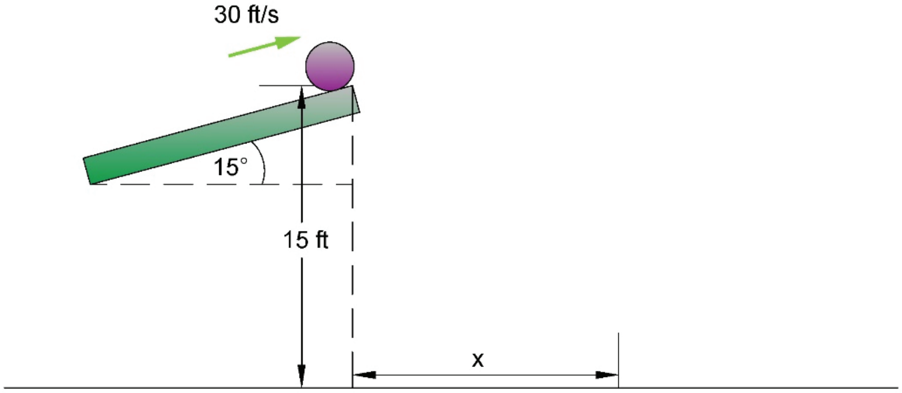
(b) 12 ft

(c) 14 ft

(d) 16 ft

13.4 PRACTICE PROBLEM 4 - KINEMATICS

An object is traveling at a velocity of 30 ft/s and leaves a surface at an angle of 15 degrees. The surface is 15 feet above the ground. At what distance “x” will the object hit the ground?



(a) 12 ft

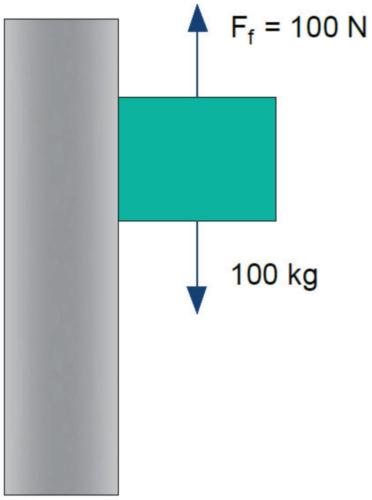
(b) 19 ft

(c) 36 ft

(d) 41 ft

13.5 PRACTICE PROBLEM 5 – WORK ENERGY

What is the velocity of the block after it has dropped 5 meters? Assume the block starts at a velocity of 1 m/s. There is an opposing frictional force that has been calculated to be as shown in the figure below.



(a) 1 m/s

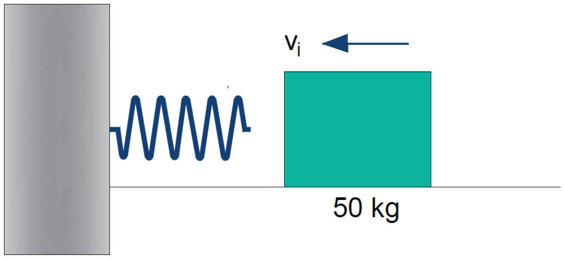
(b) 3.3 m/s

(c) 9.4 m/s

(d) 14 m/s

13.6 PRACTICE PROBLEM 6 – WORK ENERGY

At what distance will the spring compress such that it brings the block to rest in the figure below. Assume the spring is not compressed and k = 1,000 N/m. The block starts at an initial velocity of 5 m/s. Assume a frictionless surface.



(a) 1 m

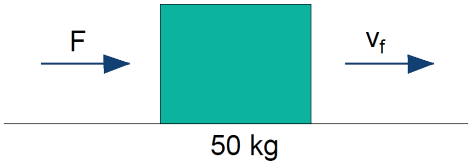
(b) 3 m

(c) 5 m

(d) 7 m

13.7 PRACTICE PROBLEM 7 – IMPULSE MOMENTUM

An average force is impacted upon a block for 0.01 seconds. The average force is equal to 20 N. The mass of the block is 50 kg and the force acts in the directions shown in the figure below. What is the resultant velocity of the block, assume a frictionless surface?



(a) 0.004 mm/s

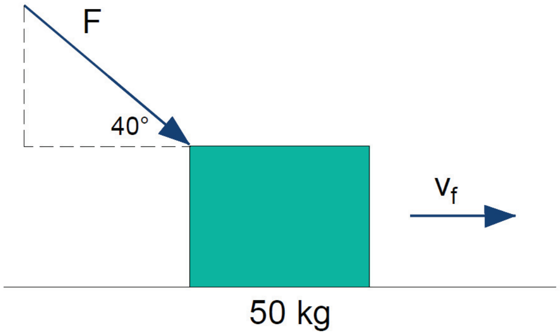
(b) 4 mm/s

(c) 11 mm/s

(d) 13 mm/s

13.8 PRACTICE PROBLEM 8 – IMPULSE MOMENTUM

A force acts upon a block as shown in the figure below. The average force is 100 N and the impact time is 0.1 seconds. The force acts in the direction shown. What is the horizontal velocity of the block after impact? Assume a frictionless surface.



(a) 0.15 m/s

(b) 0.99 m/s

(c) 1.52 m/s

(d) 2.91 m/s

13.9 PRACTICE PROBLEM 9 – VIBRATION

A ceiling hangar supports a mass of 50 lbs. The hangar consists of a spring with a spring constant of 50 lbf/in. If the mass undergoes free vibration, then how long will it take for one period to be completed?

The answer is most nearly,

(a) 0.3

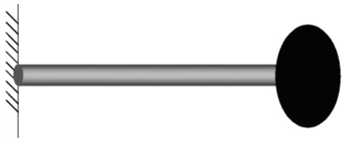
(b) 0.9

(c) 1.7

(d) 3.1

13.10 PRACTICE PROBLEM 10 - VIBRATION

A 5 lb. thin disk is located on one end of a 36” long steel shaft, while the other end is fixed, as shown below. The shaft has a 3” diameter and the disk has a 10” diameter. Assume that the shaft has negligible mass and the shear modulus for steel is 11.3x10^6 psi. What is the torsional natural frequency of the shaft?



The answer is most nearly,

(a) 700 rad/sec

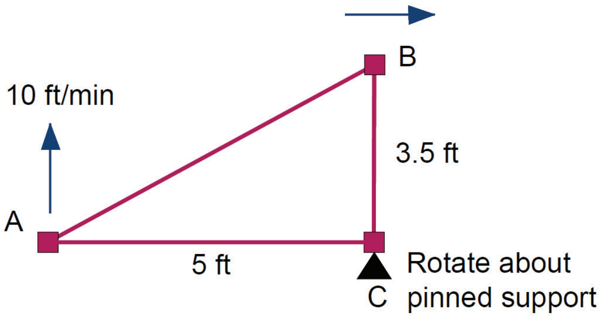
(b) 2,000 rad/sec

(c) 4,000 rad/sec

(d) 5,100 rad/sec

13.11 PRACTICE PROBLEM 11 – KINEMATICS OF MECHANISMS

What is the horizontal velocity of point B, given the information in the figure below?



(a) 3 ft/min

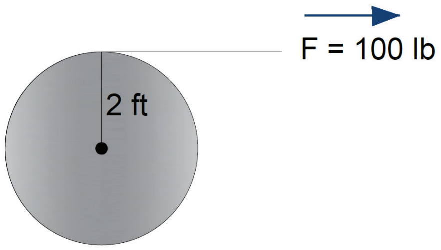
(b) 5 ft/min

(c) 7 ft/min

(d) 10 ft/min

13.12 PRACTICE PROBLEM 12 – WORK-ENERGY

An average force of 100 lbs is used to pull a cable that is wrapped around a spool as shown in the figure below. Assume the initial velocity at the end of the cable where the force is applied is 0 ft/s. The moment of inertia of the spool and cable is 10 lb-ft-s^2. Assume the cable and spool surfaces are frictionless. After the cable has been pulled for 5 feet of horizontal length, what will be the velocity at the end of the cable where the pulling force is applied?



(a) 5 ft/s

(b) 10 ft/s

(c) 15 ft/s

(d) 20 ft/s

13.13 PRACTICE PROBLEM 13 – KINEMATICS

A motor shaft is rotating at 1750rpm. Power is cut to the motor and it takes 110 seconds for the shaft to come to rest. Assuming uniform acceleration, how many rotations does the shaft make before coming to rest?

(a) 1604 rotations

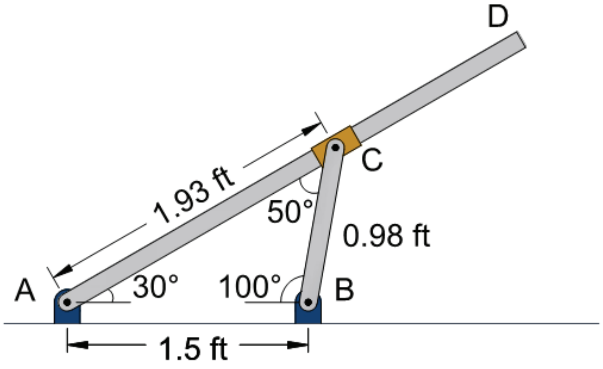
(b) 4812 rotations

(c) 9600 rotations

(d) 10,080 rotations

13.14 PRACTICE PROBLEM 14 – KINEMATICS OF MECHANISMS

In the figure below, rod BC rotates at a constant speed of 15 rad/s in a clockwise direction. Point C slides along rod AD and rotates with rod BC. What is the velocity of the slider at point C, relative to rod AD?



(a) 11.3 ft/s

(b) 14.7 ft/s

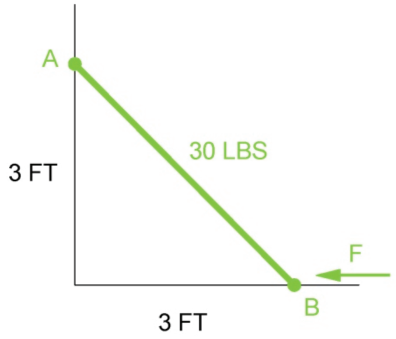
(c) 16.9 ft/s

(d) 19.2 ft/s

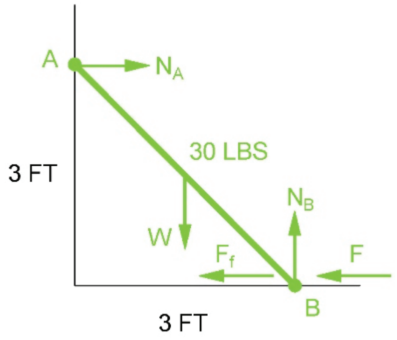
14.0 SOLUTIONS

14.1 SOLUTION 1 - FRICTION

A beam is leaning against a frictionless wall. The dimensions are shown on the below figure. The beam weighs 30 lbs and the surface at point B is assumed to have a static friction value of 0.4. What is the force, F, required to keep the beam from moving?



The first step is to draw the free body diagram. The normal forces will be perpendicular to the surfaces at the contact points. The frictional force will be opposite the direction of impending movement and the weight of the beam will act at the center of gravity.



There are three unknowns, the two normal forces and the force F. The frictional force is a function of the normal force B. The three equations are the balanced equation in the X-direction, Y-direction and the moment about point B.

$$X\ Direction \rightarrow F_f + F = N_A$$

$$Y\ Direction \rightarrow W = 30lbs = N_B$$

$$Moment\ B \rightarrow 0 = -W * 1.5ft + N_A * 3$$

Substitute into moment equation.

$$0 = -30lbs * 1.5ft + N_A * 3$$

$$N_A = 15\ lbs$$

Next, substitute N_A and N_B into the X-direction equation.

$$0.4N_B + F = N_A$$

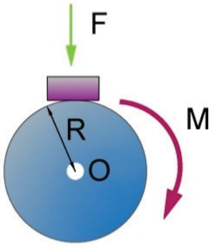
$$0.4 * 30 + F = 15$$

$$F = 3\ LBS$$

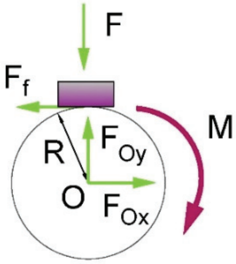
The answer is most nearly, **(a) 3 LBS.**

14.2 SOLUTION 2 - FRICTION

A braking system applies a force of 1,000 N (F) to a rotating part that has a moment of 100 N-m. The rotating part is pinned at O and is allowed to rotate about O. The coefficient of kinetic friction is 0.4. What is the reaction force in the X-direction at point O? The radius of the rotating part is R = 250 mm. Assume the weights of the components are negligible.



First draw a free body diagram. The brake force F is applied downward and the frictional force is opposing the rotating motion. The reactionary forces at O will be opposite the brake force F in the y-direction and opposite the frictional forces in the x-direction.



You should use the moment at O to solve for the reactionary force in the x-direction, which will simply be equal to the frictional force.

$$Moment\ O \rightarrow 0 = -M + F_f * 0.25m$$

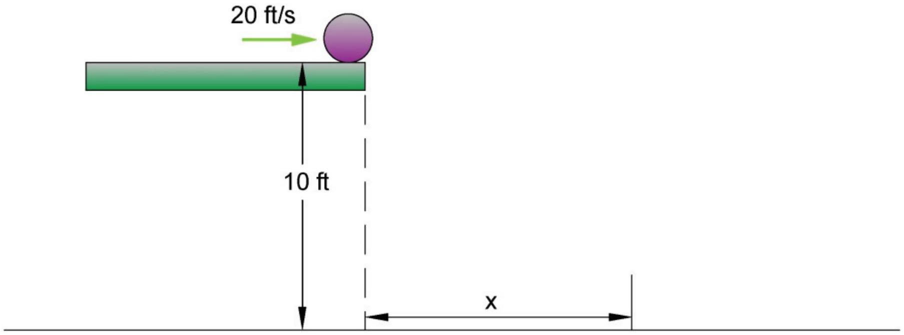
$$Moment\ O \rightarrow 0 = -100\ N * m + F_f * 0.25m$$

$$F_f = F_{ox} = 400\ N$$

The correct answer is most nearly, **(d) 400 N.**

14.3 SOLUTION 3 - KINEMATICS

An object has a horizontal speed of 20 ft/s. It leaves a surface that is 10 feet above the ground. At what distance "x" will the object hit the ground?



The first step is to determine the amount of time the object will be in the air. Since there is no vertical velocity, the starting vertical velocity will be equal to zero. The distance traveled will be 10 feet and the acceleration will be equal to gravity.

$$y_f = y_i + v_y t + \frac{1}{2}at^2$$

$$0\,ft = 10\,ft + 0 - \frac{1}{2}32.2\frac{ft}{s^2}t^2$$

$$t = 0.79\,seconds$$

Next use the same equation but for the x direction. There is no acceleration in the x-direction.

$$x_f = x_i + v_x t + \frac{1}{2}at^2$$

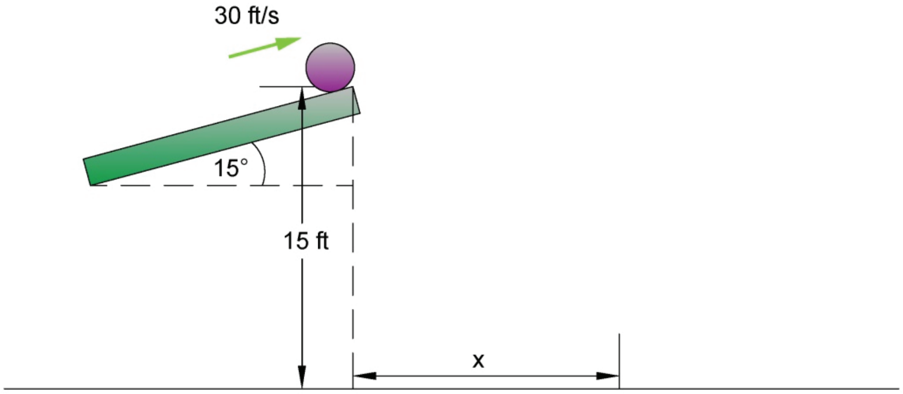
$$x_f = 0 + \frac{20ft}{s} * 0.79s + 0$$

$$x_f = 15.8\,ft$$

The correct answer is most nearly, **(d) 16 ft**.

14.4 SOLUTION 4 - KINEMATICS

An object is traveling at a velocity of 30 ft/s and leaves a surface at an angle of 15 degrees. The surface is 15 feet above the ground. At what distance "x" will the object hit the ground?



The first step is to determine the x and y velocity.

$$v_x = 30\frac{ft}{s} * (\cos(15)); v_y = 30\frac{ft}{s} * (\sin(15));$$

$$v_x = 28.98\ ft/s; v_y = 7.77\frac{ft}{s};$$

The next step is to determine the amount of time the object will be in the air.

$$y_f = y_i + v_y t + \frac{1}{2}at^2$$

$$0\ ft = 15\ ft + 7.77t - \frac{1}{2}32.2\frac{ft}{s^2}t^2$$

$$t = 1.24\ seconds$$

Next use the same equation but for the x direction. There is no acceleration in the x-direction.

$$x_f = x_i + v_x t + \frac{1}{2}at^2$$

$$x_f = 0 + 28.98\frac{ft}{s} * 1.24s + 0$$

$$x_f = 35.9\ ft$$

The correct answer is most nearly, **(c) 36 ft**.

14.5 SOLUTION 5 – WORK ENERGY

In this problem, you can use the energy balance equation with potential energy, work and kinetic energy.

Initially, you can assume that the work is equal to zero because the net force has not acted over a distance yet. If you assume the block is 5 meters above the 0 meter point, then the potential energy will be "mg" multiplied by "5 meters". The initial velocity is 1 m/s, so there will be some kinetic energy.

$$Initial\ Condition \rightarrow W = 0; PE = mgh; KE = \frac{1}{2}mv_i^2$$

$$Initial\ Condition \rightarrow (100\ kg) * \left(9.81\frac{m}{s^2}\right) * (5\ m) + \frac{1}{2}(100\ kg) * (1^2) = 4{,}955\ J$$

At the final condition, at point 0 meters, there will have been 5 meters of work done by the net force and the potential energy will be equal to zero. The kinetic energy will account for the loss of potential energy.

$$Final\ Condition \rightarrow 100N(5) + \frac{1}{2}(100\ kg) * (v_f^2)$$

Next equate the initial and final conditions.

$$Initial\ Condition = Final\ Condition$$

$$4{,}955\ J = 100(5) + \frac{1}{2}(100\ kg) * (v_f^2)$$

$$v_f = 9.43\ m/s$$

Another method is to solve for the net force, which is equal to the frictional force minus the gravitational force. In this problem, you can assume that upward forces are positive and downward forces are negative.

$$Net\ Force \rightarrow 100\ kg * \left(9.81\frac{m}{s^2}\right) - 100\ N -= 881\ N \rightarrow a = 8.8\frac{m}{s};$$

$$(v_F)^2 = (v_i)^2 + 2ax; v_f = 9.43\frac{m}{s^2};$$

The correct answer is most nearly (C) 9.4 m/s.

14.6 SOLUTION 6 – WORK ENERGY

In this problem you must conduct an energy balance between the initial and final conditions.

$$Initial\ Condition = Final\ Condition$$

$$Work_{initial} + KE_{initial} + PE_{initial} = Work_{final} + KE_{final} + PE_{final}$$

$$\cancel{F} * d + \frac{1}{2}mv_i^2 + \frac{1}{2}k\cancel{\Delta x^2} = \cancel{F} * d + \frac{1}{2}m\cancel{v_f^2} + \frac{1}{2}k\Delta x^2$$

$$0+\frac{1}{2}50*\left(5\frac{m}{s}\right)^2+0=0+0+(\frac{1}{2}*1{,}000*x^2)$$

$$x=1.1\ m$$

The correct answer is most nearly (A) 1 m.

14.7 SOLUTION 7 – IMPULSE MOMENTUM

For this problem, you need to use the impulse momentum equation.

$$F*\Delta t=mv_f$$

$$20\ N*0.01\ s=50\ kg*v_f$$

$$20\ kg\frac{m}{s^2}*0.01\ s=50\ kg*v_f$$

$$v_f=0.004\ m/s$$

The correct answer is most nearly (B) 4 mm/s.

14.8 SOLUTION 8 – IMPULSE MOMENTUM

For this problem, you need to use the impulse momentum equation.

$$F*\Delta t=mv_f$$

$$100\ N*\cos(40°)*0.1\ s=50\ kg*v_f$$

$$6.669\ N-s=50\ kg*v_f$$

$$v_f=0.15\ m/s$$

The correct answer is most nearly (A) 0.15 m/s.

14.9 SOLUTION 9 – VIBRATION

In order to complete this problem, you should use the frequency equation shown below.

$$f_n=\frac{1}{2\pi}\sqrt{\frac{kg_c}{m}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{50\,\frac{lbf}{in} * \frac{12\ in}{1\ ft} * 32.2\,\frac{lbm - ft}{lbf - s^2}}{50\ lbm}}$$

$$f_n = 3.13\ cycles/sec$$

$$T = \frac{1}{3.13} = 0.32\ seconds\ per\ cycle$$

The correct answer is most nearly, **(a) 0.3**.

14.10 Solution 10 – Vibration

The torsional natural frequency is found by the following equation.

$$\omega_n = \sqrt{\frac{k_t}{I}}, where\ k_t = \frac{GJ}{L}$$

Find the torsional stiffness, k_t.

The shear modulus, G, is given as $11.3x10^6$ psi and the length of the shaft, L, is 36in. The torsion occurs at the shaft. Therefore, the polar area moment of inertia for the shaft, J, is found.

$$J = \frac{\pi}{32}d^4 = \frac{\pi}{32}(3\ in)^4 = 7.9\ 5in^4$$

Solve for k.

$$k_t = \frac{GJ}{L} = \frac{\left(11.3 * 10^6\,\frac{lbf}{in^2}\right)(7.95in^4)}{36\ in} = 2.5 * 10^6 in * lbf$$

Since the shaft mass is negligible, the polar mass moment of inertia, J, is found from the disc.

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(5\ lbm) * \left(\frac{10\ in}{2}\right)^2 = 62.5\ lbm * in^2$$

Finally, solve for the natural frequency

$$\omega_n = \sqrt{\frac{k_t}{I/g_c}} = \sqrt{\frac{(2.5 * 10^6\ in * lbf) * \left(\frac{32.2\,\frac{lbm * ft}{s^2}}{1\ lbf}\right) * \left(\frac{12\ in}{ft}\right)}{62.5\ lbm * in^2}} = 3930\,\frac{rad}{s}$$

The answer is most nearly, **(d) 4,000 rad/sec.**

14.11 SOLUTION 11 – KINEMATICS OF MECHANISMS

In order to solve this problem you must first convert the linear velocity of A to an angular velocity.

$$w_a = 10\frac{ft}{min} \div 5ft = 2\frac{rad}{min}$$

Since the mechanism is linked together, then point B will also have the same angular velocity, which you can use to convert to a horizontal linear velocity.

$$w_b = 2\frac{rad}{min} = v_b \div 3.5ft$$

$$v_b = 7\frac{ft}{min}$$

The correct answer is most nearly (C) 7 ft/min.

14.12 SOLUTION 12 – WORK-ENERGY

In order to solve this problem you must do a conservation of energy equation for the initial and final conditions. Initially the energy is equal to zero.

$$Initial\ Energy + Work = Final\ Energy$$

The force will conduct positive work that will cause a rotational energy in the spool/cable. You can use the below equation to solve for rotational velocity.

$$0 + 100\ lb * (5\ feet) = \frac{1}{2}I * \omega^2$$

$$500\ lb - ft = \frac{1}{2} * 10\ lb - ft - s^2 * \omega^2$$

$$\omega = 10\frac{rad}{s}$$

The next step is to convert the angular velocity to linear velocity.

$$v = \omega * r = 10\frac{rad}{s} * 2\ ft = 20\frac{ft}{s}$$

The correct answer is most nearly (D) 20 ft/s.

14.13 SOLUTION 13 - KINEMATICS

Refer to the *NCEES FE Reference Handbook* and use the rigid body motion equation around a fixed axis with constant angular acceleration. First solve for the angular acceleration from the initial rotational speed to the time of rest using the following equation.

$$\omega = \omega_0 + \alpha_0 t$$

The shaft starts at 1750 rpm, then decelerates to 0 rpm. Convert the rotational velocity to radians per second.

$$\omega_0 = 1750\frac{rotation}{min} * 2\pi\frac{rad}{rotation} * \frac{1min}{60sec} = 183.3\frac{rad}{s}$$

Solve for acceleration

$$0 = 183.3\frac{rad}{s} + \alpha_0 * 110sec$$

$$\alpha_0 = -1.67\frac{rad}{s^2}$$

Finally, solve for the number of rotations.

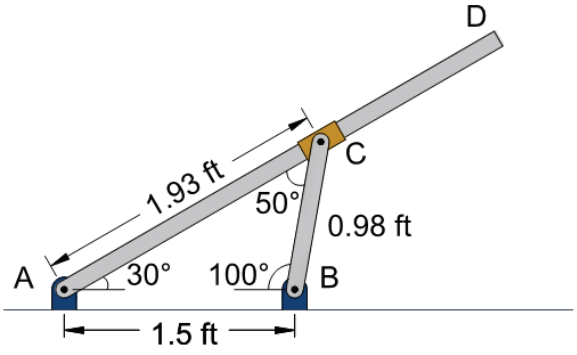
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2$$

$$\theta = 0 + 183.3\frac{rad}{s} * 110s + \frac{1}{2} * \left(-1.67\frac{rad}{s^2}\right) * (110s)^2 = 10{,}079\,rad$$

$$\theta = 10{,}079\,rad * \frac{1\,rot}{2\pi\,rad} = 1604\,rotations$$

The answer is (A) 1604 rotations.

14.14 SOLUTION 14 – KINEMATICS OF MECHANISMS



Since point C slides along rod AD, the linear velocity at point C will be perpendicular to rod AD. Therefore the linear velocity at point C is the angular velocity of rod AD times the distance AC.

$$V_c = AC * \omega_{AD}$$

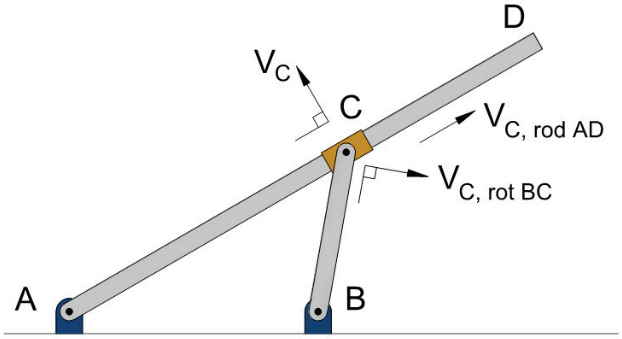
The linear velocity at point C can also be written as the linear velocity obtained from the rotational velocity from rod BC plus the linear velocity of point C, relative to rod AD

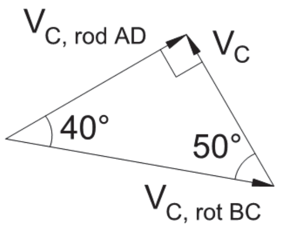
$$V_c = V_{C,rot\ BC} + V_{C,rod\ AD}$$

We can solve for $V_{C,\text{ rot BC}}$

$$V_{C,rot\ BC} = BC * \omega_{BC} = 0.98\ ft * 15\frac{rad}{s} = 14.69\ ft/s$$

The vector diagram for the velocities are shown below.





Use the vector diagram to solve for $V_{C,\text{ rod AD}}$

$$V_{C,rod\ AD} = V_{C,rot\ BC} * \cos(40°) = 14.7\ ft/s * \cos(40°) = 11.3 ft/s$$

The answer is (A) 11.3 ft/s.

8 – Mechanics of Materials

MECHANICAL FE TEXTBOOK

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Section 8.0 – Mechanics of Materials

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1.0 INTRODUCTION

Mechanics of Materials accounts for approximately 9 to 14 questions on the Mechanical FE exam. Mechanics of Materials questions basically will cover calculating the stress or strain due to different loadings, like axial loads, bending loads, torsional loads and shear loads. In addition, you will have to calculate the displacement, shear force and moments for beams and the buckling forces and stresses in columns.

The equations shown in the *NCEES FE Reference Handbook* on Mechanics of Materials is comprehensive and all of these equations are fairly simple and easy to use. This means that the equations do not have any complex math. After going through this section, you should be familiar with the different types of loadings and all the equations within the handbook and should be able to quickly recognize when to use each equation for each problem.

Section 9.0 Mechanics of Materials (9 to 14 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
8A	Section 2.0	Shear and Moment Diagrams
8B	Section 3.0 & 4.0	Stress Transformations & Mohr's Circle
8C	Section 5.0	Stress and Strain caused by Axial Loads
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8E	Section 7.0	Stress and Strain caused by Torsion
8F	Section 8.0	Stress and Strain caused by Shear
8G	Section 9.0	Stress and Strain caused by Temperature Change
8H	Section 10.0	Combined loading
8I	Section 11.0	Deformations
8J	Section 12.0	Column Buckling
8K	Section 13.0	Statically Indeterminate Systems
	Section 14.0	Practice Exam Problems

2.0 SHEAR AND MOMENT DIAGRAMS

Shear and moment diagrams are used to graphically show how forces, displacement and moments change depending on the location within a component. The components that you will most likely need to know are beams, since the shear and moment diagrams included in the *NCEES FE Reference Handbook* are only for beams.

However, before you use the diagrams you should understand the concepts behind the beam diagrams and the resulting equations. Once you have a good grasp on the concepts and how to use the equations, then you should be able to solve these types of problems.

There are three main types of beam diagrams, (1) Free Body Diagram, (2) Shear Diagram and (3) Moment Diagram. The first step (1) is to determine the forces acting upon the beam in order to construct the beam diagram. Beams can be loaded with a load at a point or a distributed load along the entire length of the beam. The loads are primarily downward and in order to create equilibrium there will be reaction forces upward at the supports. Equilibrium equations for both forces and moments about each of the supports are used to find the reaction forces. The forces must equal zero, since the beam is restricted from movement. Also the sum of the moments at each of the supports must be zero, since the beam is restricted from twisting. Once you have found all loads and reaction forces, then you can construct the free body diagram.

The next step (2) is to construct the shear diagram. The shear diagram describes the internal forces within the beam at any point of the beam. This diagram is created by splitting up each segment separated by a point force or reaction force. Then you must cover all external forces to the right of each point and sum up all external forces and this sum is the shear force at that point.

The final step (3) is to construct the moment diagram. The moment diagram is comprised of the moment at any location along the length of the beam. The moment diagram is constructed in a similar fashion as the shear diagram.

In practice, shear diagrams and moment diagrams are not often re-constructed, since the diagrams have already been documented. Please see the *NCEES FE Reference Handbook* for completed beam diagrams with equations. Beam diagrams will most likely be used on the FE exam in order to find the maximum stress in the beam or the maximum deflection in the beam. This can be done by using the given beam diagrams and using the corresponding equations. However, there may be conceptual type problems which will require you to have an understanding of how these diagrams are constructed. The next few pages will show a few diagrams and walk you through some points on the diagram.

In practice, beam supports can be classified based on how much the beam is restricted by the support.

- Loose support (simply supported): A loose support, which would be the same as the simply support diagram, is used to support the weight vertically. The beam can be moved in all other directions. Including along the length of the beam (longitudinal) and perpendicular to the direction of the pipe (transverse). The beam is only restricted from moving vertically (up and down).

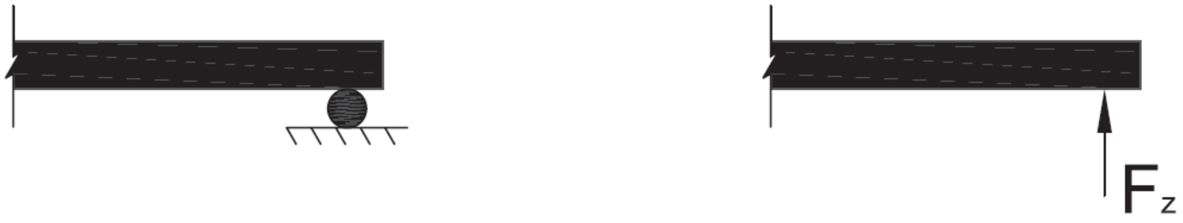


Figure 1: This support shows a loose support but it is more often called a roller support. There only is a reaction force in the z-direction.

- Loose support with longitudinal guide: This type of support includes the loose support but also has a restriction on the transverse. The beam will not be allowed to move perpendicular to the direction of the beam. This would include putting a clamp around a beam to stop the side to side movement.

- Loose support with transverse guide: This type of support includes the loose support but also has a restriction on the longitudinal. This would stop the beam from expanding and contracting in the longitudinal direction.

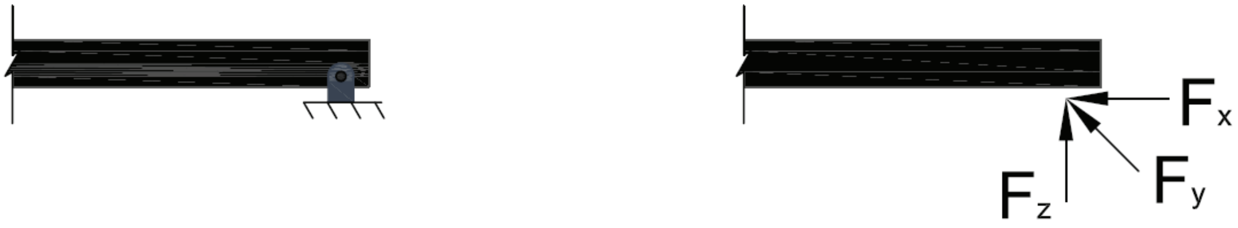


Figure 2: A pinned or hinged support will have reactionary forces in the z & x directions. However, it will not have a reactionary moment.

- Loose support with anchor: This type of support restricts movement in both the longitudinal and transverse direction.

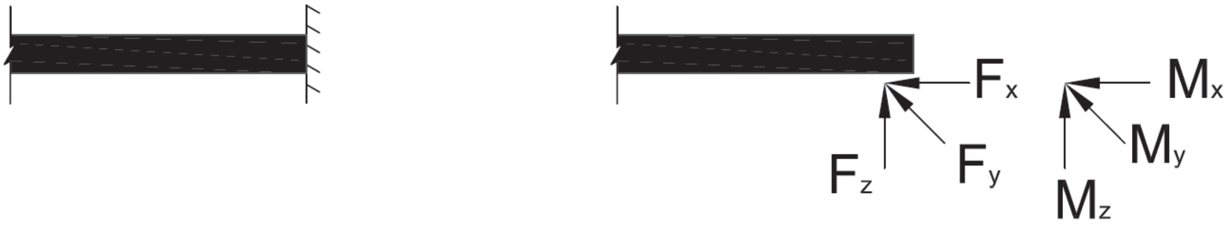


Figure 3: This support shows a fixed support. There are reaction forces in all directions and reactionary moments.

2.2 SIMPLE BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

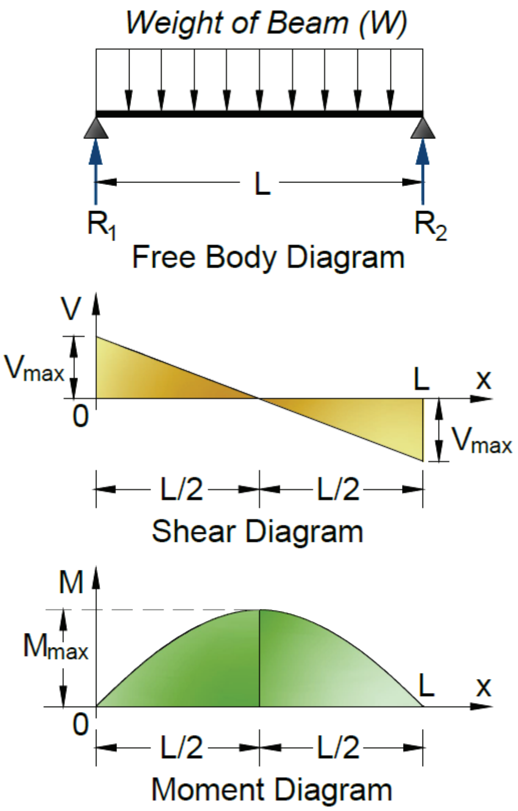


Figure 4: Simple beam with uniformed load diagrams

This diagram is typically used when a beam is supported at both ends for only the weight of the beam. The downward force is equal to the weight of the beam. The weight is evenly distributed evenly the entire length of the beam, which is why the vectors are of equal size.

$$Total\ Weight\ of\ Beam = W\ (lbf) = w\ (lbf/ft) * L\ (ft)$$

$$Total\ Weight\ of\ Beam = W\ (N) = w\ (N/m) * L\ (m)$$

There will be upward reaction forces at the support that will counteract the weight of the beam. The force at each support will be equal to one-half the entire weight.

$$R_1 = R_2 = \frac{W}{2}\ (N\ \ or\ \ lbf)$$

The next figure shows the shear force diagram. The shear force acting at any point "x" on the beam is governed by the below equation. The shear force is at its maximum at the supports. You can see that when 0 and L are inserted into the equation for x, the value will be of magnitude equal to ½ the weight.

$$V_x = w * \left(\frac{L}{2} - x\right) \quad [N\ \ or\ \ lbf]$$

$$V(0) = w * \left(\frac{L}{2} - x\right) = W\frac{1}{2} and\ V(L) = -W\frac{1}{2} \quad [N \ \ or \ \ lbf]$$

The bending moment acting at any point "x" on the beam is governed by the following equation. The moment is at its maximum at the center. The moment will be equal to zero when "x" is equal to "L" or "0".

$$M_x = \frac{Wx}{2}(L - x) \qquad [N \cdot m \quad or \quad lbf \cdot in]$$

$$Maximum\ Moment \rightarrow M_{L/2} = \frac{WL^2}{8} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

The deflection at any point "x" on the beam is governed by the following equation. The deflection is at its maximum at the center.

$$\Delta_x = \frac{Wx}{24 * E * I} * (L^3 - 2 * L * x^2 + x^3) \quad [m \ \ or \ \ in]$$

$$where\ E = modulus\ of\ elasticity\ (Pa\ \ or\ \ psi) and\ L = \ length\ (m\ \ or\ \ in)$$

$$\Delta_{center} = \frac{5WL^4}{384 * E * I} [m\ \ or\ \ in]$$

The moment can then be used to find the maximum stress in the beam. The maximum stress in the beam will help to influence the beam dimensions and material choices. The maximum stress will be discussed in the bending topic.

2.3 SIMPLE BEAM WITH A CONCENTRATED LOAD

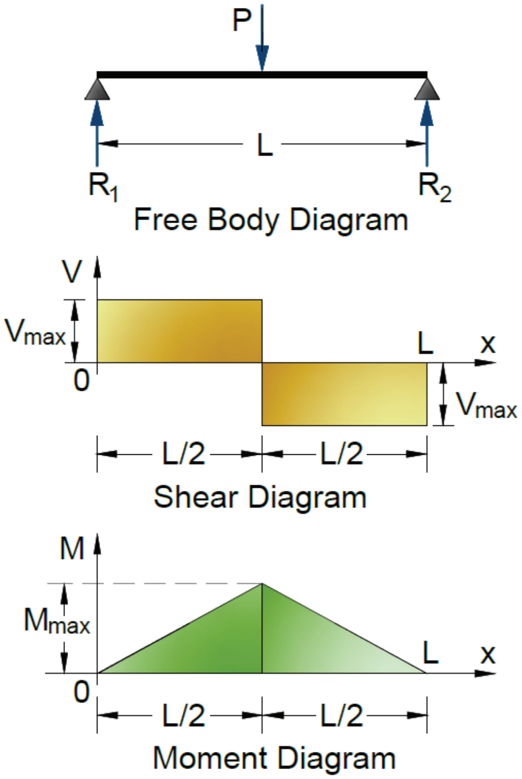


Figure 5: Simple beam with concentrated load diagrams.

This diagram is typically used when a beam is supported at both ends and it is used to support a concentrated load. The downward force is equal to load. There will be upward reaction forces at the support that will counteract the concentrated load. The force at each support will be equal to one-half the entire weight.

$$R_1 = R_2 = \frac{P}{2} \quad [N \;\; or \;\; lbf]$$

The next figure shows the shear force diagram. The shear force acting at any point "x" on the beam is either +P/2 or –P/2. The shear force is at its maximum at the supports. The slope of the line in this diagram is straight, because unlike the distributed load, the only forces acting upon the beam are the reaction forces and the concentrated load. The previous diagram had external forces acting on the beam throughout the entire length, which changed the shear force at each point along the beam.

The bending moment acting at any point "x" on the beam is governed by the following equation. The moment is at its maximum at the center. The moment will be equal to zero when "x" is equal to "L" or "0".

$$M_{x,upward\ sloping\ portion} = \frac{Px}{2} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

$$M_{x,downward\ sloping\ portion} = -\frac{P(x-L)}{2} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

$$Maximum\ Moment \rightarrow M_{L/2} = \frac{PL}{4} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

The deflection at any point “x” on the beam is governed by the following equation. The deflection is at its maximum at the center.

$$\Delta_{x,upward\ sloping} = \frac{Px}{48 * E * I} * (3L^2 - 4x^2) \ [m \ or \ in]$$

$$\Delta_{x,downard\ sloping} = -\frac{P(x-L)}{48 * E * I} * (3L^2 - 4(x-L)^2) \ [m \ or \ in]$$

$$where\ E = modulus\ of\ elasticity\ (Pa \ or \ psi)\ and\ L = \ length\ (m \quad or \quad in)$$

$$\Delta_{center} = \frac{PL^3}{48 * E * I} \quad [m \ or \ in]$$

The moment can then be used to find the maximum stress in the beam. The maximum stress in the beam will help to influence the beam dimensions and material choices. The maximum stress will be discussed in the bending topic.

2.4 FIXED END BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

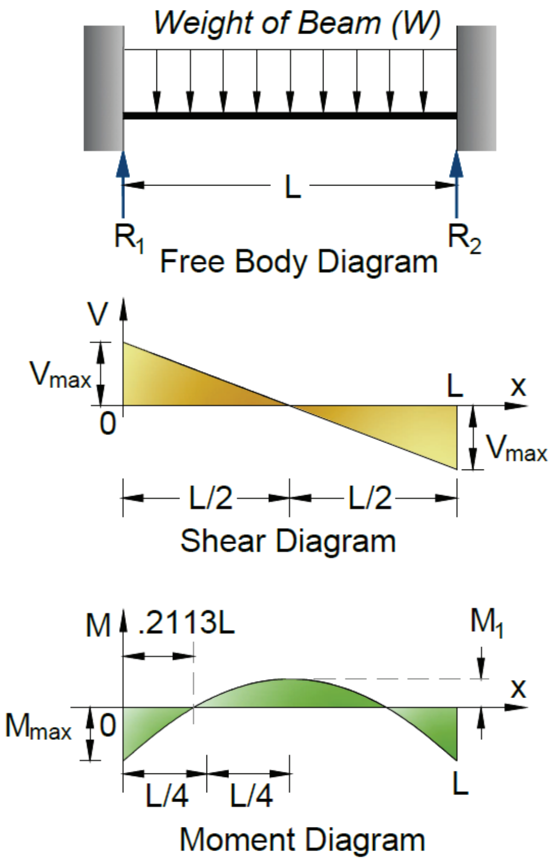

Weight of Beam (W)
L
R1
R2
Free Body Diagram
V
Vmax
0
L
x
Vmax
L/2
L/2
Shear Diagram
M
.2113L
M1
x
Mmax
0
L/4
L/4
L
Moment Diagram

Figure 6: Fixed end beam with uniformly distributed load.

This diagram is different from the first diagram because the ends are fixed. This means the ends are fixed in all planes and are not allowed to rotate, thus the moment at the ends will no longer be zero. The downward force is equal to load. There will be upward reaction forces at the support that will counteract the concentrated load. The force at each support will be equal to one-half the entire weight.

$$R_1 = R_2 = \frac{W}{2} \quad [N \ \ or \ \ lbf]$$

The next figure shows the shear force diagram, which will be the same as the first diagram.

The bending moment acting at any point "x" on the beam is governed by the following equation. The moment is at its maximum at the ends. The moment will be equal to zero when "x" is equal to "0.2113L".

$$M_x = \frac{W}{12}\left(6Lx - L^2 - 6x^2\right) \quad [N \cdot m \quad or \quad lbf \cdot in]$$

$$Maximum\ Moment \rightarrow M_{0\ or\ L} = \frac{WL^2}{12} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

$$Moment\ at\ Center \rightarrow M_{L/2} = \frac{WL^2}{24} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

The deflection at any point "x" on the beam is governed by the following equation. The deflection is at its maximum at the center.

$$\Delta_x = \frac{Wx^2}{24 * E * I} * (L - x)^2 \quad [m\ \ or\ \ in]$$

$$where\ E = modulus\ of\ elasticity\ (Pa\ \ or\ \ psi)\ and\ L = \ length\ (m\ \ or\ \ in)$$

$$\Delta_{center} = \frac{WL^4}{384 * E * I} \quad [m\ \ or\ \ in]$$

The moment can then be used to find the maximum stress in the beam. The maximum stress in the beam will help to influence the beam dimensions and material choices. The maximum stress will be discussed in the bending topic.

2.5 FIXED END BEAM WITH A CONCENTRATED LOAD

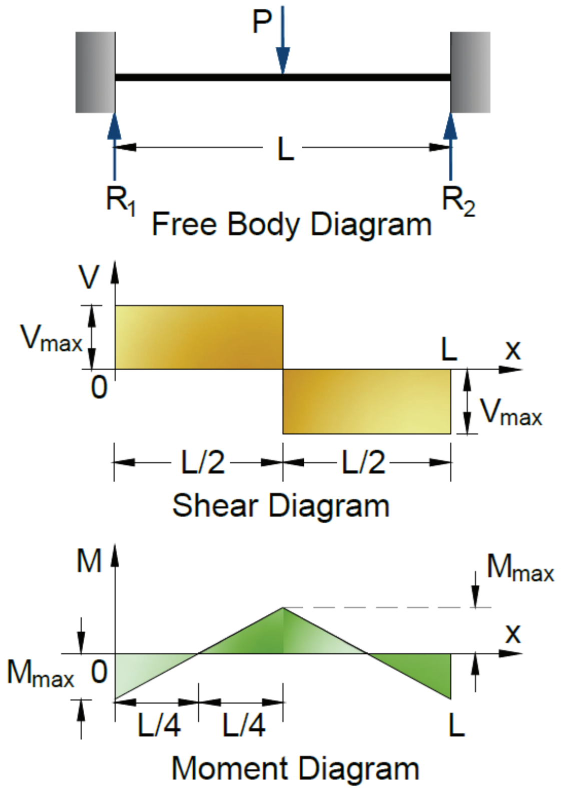


Figure 7: Fixed end beam with concentrated load diagrams.

This diagram is different from the second diagram because the ends are now fixed. This will cause the moment at the ends to not equal zero, since the ends are no longer allowed to rotate. The downward force is still equal to load and the reaction force at each support will be equal to one-half the entire load.

$$R_1 = R_2 = \frac{P}{2} \quad [N \ or \ lbf]$$

The shear force diagram will remain the same.

The bending moment acting at any point "x" on the beam is governed by the following equation. The moment is at its maximum at the center and at the ends. The moment will be equal to zero when "x" is equal to "L/4" or "3L/4".

$$M_{x,upward\ sloping} = \frac{P}{8}(4x - L) \quad [N \cdot m \quad or \quad lbf \cdot in]$$

$$Maximum\ Moment \rightarrow M_{0,\ \frac{L}{2},\ L} = \frac{PL}{8} \quad [N \cdot m \quad or \quad lbf \cdot in]$$

The deflection at any point "x" on the beam is governed by the following equation. The deflection is at its maximum at the center.

$$\Delta_{x,upward\ sloping} = \frac{Px^2}{48 * E * I} * (3L - 4x) \quad [m \ \ or \ \ in]$$

$$where\ E = modulus\ of\ elasticity\ (Pa \ \ or \ \ psi)\ and\ L = \ length\ (m \ \ or \ \ in)$$

$$\Delta_{center} = \frac{PL^3}{192 * E * I} \quad [m \ \ or \ \ in]$$

The moment can then be used to find the maximum stress in the beam. The maximum stress in the beam will help to influence the beam dimensions and material choices. The maximum stress will be discussed in the bending topic.

More beam diagrams can be found in the *NCEES FE Reference Handbook*.

3.0 STRESS TYPES & TRANSFORMATIONS

Stress is described as the internal force acting upon a specific cross sectional area within an object. The equation below shows that stress is equal to the force divided by the area. The units of stress are in "psi" or "kips" for imperial units and Megapascal for SI units.

$$\sigma = \frac{F}{A}$$

$$Stress\ (Pa) = \sigma = \frac{Load\ (N)}{Area\ (m^2)}\ \text{[SI Units]}$$

$$Stress\ (psi) = \sigma = \frac{Load\ (lbf)}{Area\ (in^2)}\ \text{[English Units]}$$

You may encounter both sets of units on the FE exam, so you should be familiar with both and should commit these conversions to memory.

SI Units	*Imperial Units*	*Conversion*
Megapascal(MPa) *1 Pascal(Pa) = 1 N/m^2* *Kilo (10^3), Mega (10^6), Giga (10^9)*	*lb/in^2(psi)*	*1 MPa = 145 psi*
	Kip/in^2(ksi) [kip = 1,000 lb]	*7 MPa = 1 ksi*

The strength of a material with respect to stress is the maximum stress that a material can withstand before failing. For brittle materials, failure occurs at ultimate stress.

$$Brittle \rightarrow \sigma_{ultimate};$$

For ductile materials, the maximum stress for a design *could be* the yield stress, because an object will no longer perform to design if it passes the yield strength and becomes permanently deformed.

$$Ductile \rightarrow \sigma_{yield};$$

On the FE exam, you should be careful to make sure you use the correct maximum stress for the correct type of material and situation. The following two graphs illustrate the difference between the yield stress and ultimate stress for brittle and ductile materials.

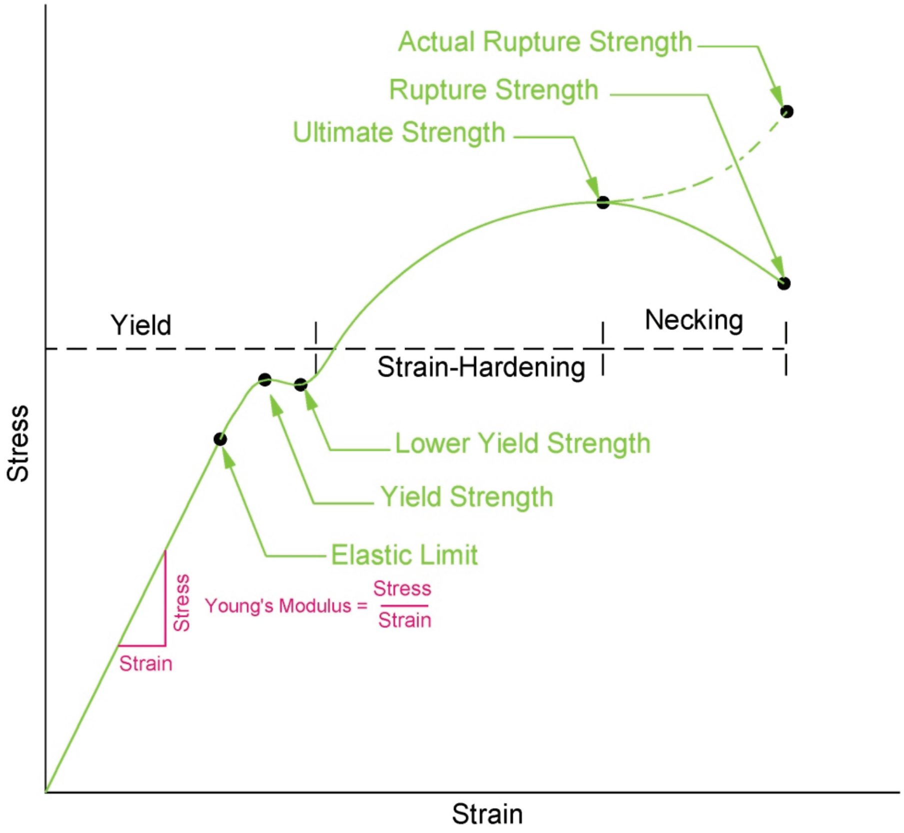

Actual Rupture Strength
Rupture Strength
Ultimate Strength
Yield
Necking
Strain-Hardening
Lower Yield Strength
Yield Strength
Elastic Limit
Stress
Young's Modulus = Stress / Strain
Strain
Stress
Strain

Figure 8: This figure shows the stress and strain relationship for ductile materials. The yield strength is typically used as the maximum design stress for ductile materials.

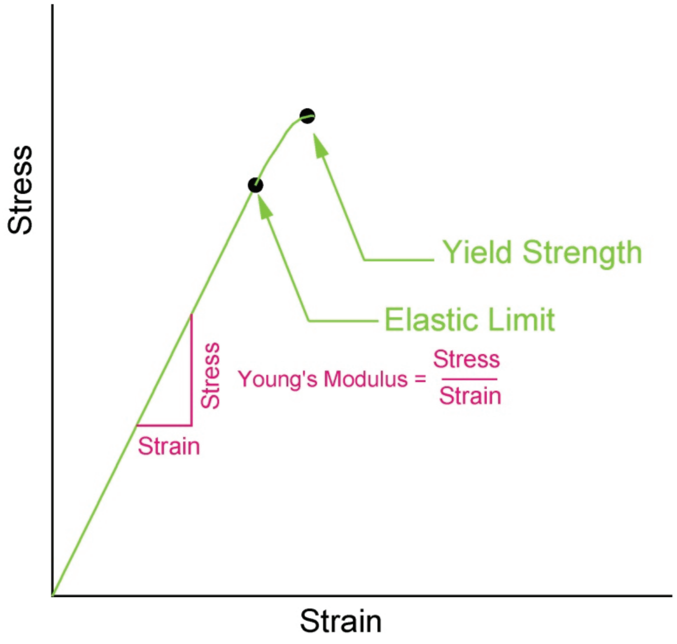

Yield Strength
Elastic Limit
Stress
Young's Modulus = Stress / Strain
Strain
Stress
Strain

Figure 9: This figure shows the stress-strain relationship for brittle materials. The yield strength is equal to the ultimate strength and is typically used for maximum design stress.

There are four types of stress discussed in this section, (1) Axial, (2) Bending, (3) Torsion and (4) Shear.

3.1 AXIAL

Axial stress is due to any stress caused by a force acting along the axis of an object. Axial stress is also known as tension or compression stresses. The force divided by the incident cross sectional area is equal to the axial stress and the stress can either be tensile or compressive. The definitions of tensile and compressive are shown in the figures below.

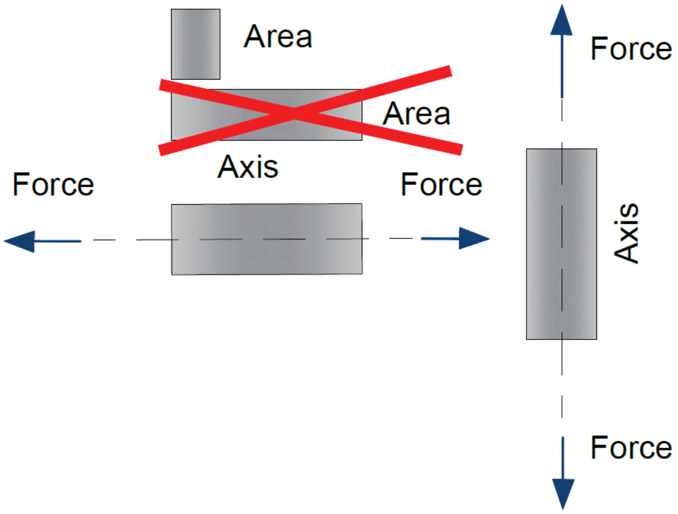


Figure 10: Forces that pull apart an object cause a tensile stress. The axial force acts along the axis, so the area used in the calculations must be the cross sectional area is perpendicular to the force vector.

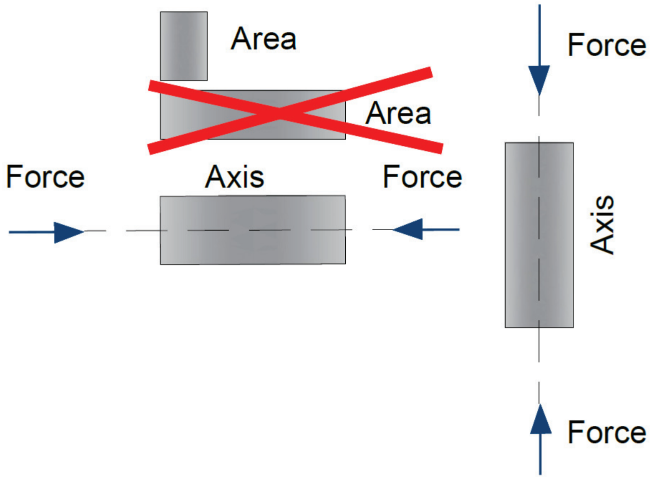


Figure 11: Forces that push together an object cause a compressive stress. The axial force acts along the axis, so the area used in the calculations must be the cross sectional area that is perpendicular to the force vector.

The equation for axial stress is shown below. Remember that the area used for this equation is the area that is perpendicular to the force vector.

$$\sigma_{axial} = \frac{Force}{Axial\ Area}; \sigma_{axial} = \frac{lbf}{in^2}; F = lb; A = in^2$$

$$\sigma_{axial} = \frac{Force}{Axial\ Area}; \sigma_{axial} = Pa\ or\ \frac{N}{m^2}; F = N; A = m^2$$

3.2 SHEAR

Shear stresses are the opposite of axial stresses. These stresses are caused by unaligned forces acting in parallel to a cross sectional area. The equation for shear stress is the same as axial stress, except the area is the area parallel to the force vector.

$$\tau_{shear} = \frac{Force}{Shear\ Area}; \tau_{shear} = \frac{lbf}{in^2}; F = lbf; A = in^2$$

$$\tau_{shear} = \frac{Force}{Shear\ Area}; \tau_{shear} = Pa\ or\ \frac{N}{m^2}; F = N; A = m^2$$

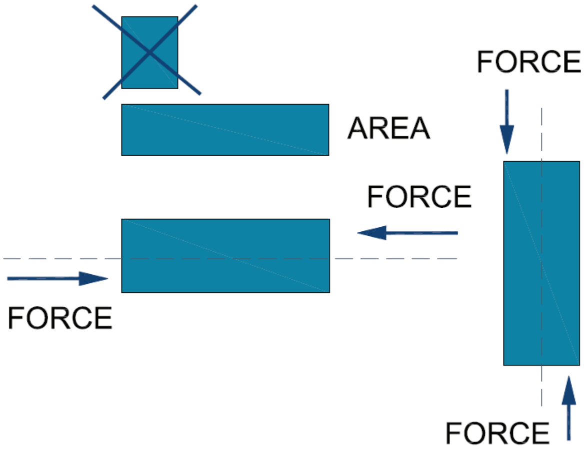


Figure 12: The shear stress is found by dividing the force by the area that is in parallel to the force vector.

3.3 BENDING

Bending stresses are discussed more in the beam topic. Bending stress can also be called a flexure stress. Bending stresses put the top half of an object in compression and the bottom half in tension. So the stress varies depending on the location in the object. In the previous topics on shear and axial stresses, the stress is consistent throughout the entire object's cross section. The equation for bending stress is shown below. Bending stress is a function of the moment, distance from the neutral axis and the moment of inertia.

$$\sigma_{bending} = \frac{M * y}{I}$$

$$M = bending\ moment\ (N \cdot m)\ \ or\ \ (lbf \cdot in)\,;$$

$$y = distance\ from\ neutral\ axis\ (in)\ or\ (m);$$

$$I = moment\ of\ inertia\ (m^4)\ \ or\ \ (in^4); \sigma_{bending} = \left(Pa\ or\ \frac{N}{m^2}\right)\ or\ \left(\frac{lbf}{in^2}\right)$$

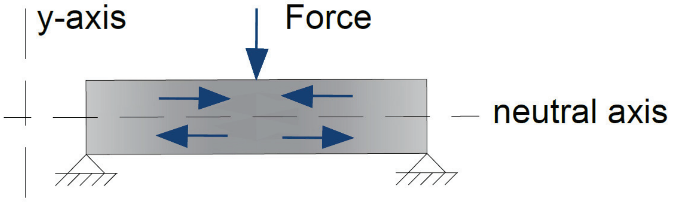
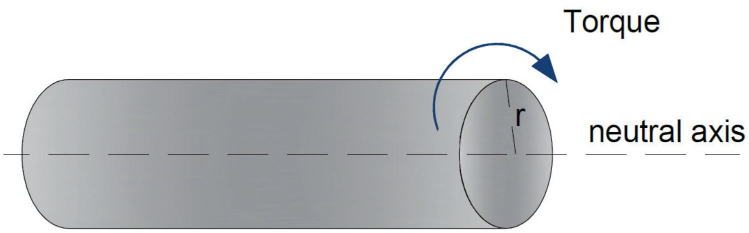


Figure 13: The force above causes a bending stress that varies as you move away from the neutral axis.

3.4 TORSION

Torsional stresses occur when a twisting moment is applied. Similar to how the bending force causes a bending moment, a torsional force can causes a torsional (twisting) moment. This twisting moment is simply called torque. The equation for torsional stress is shown below. Torsional stress is a function of the torque, distance from the neutral axis and the polar moment of inertia.



$$\tau_{torsion} = \frac{T * r}{J}$$

$$T = torque\ (lb - in)\ or\ (N - m);$$

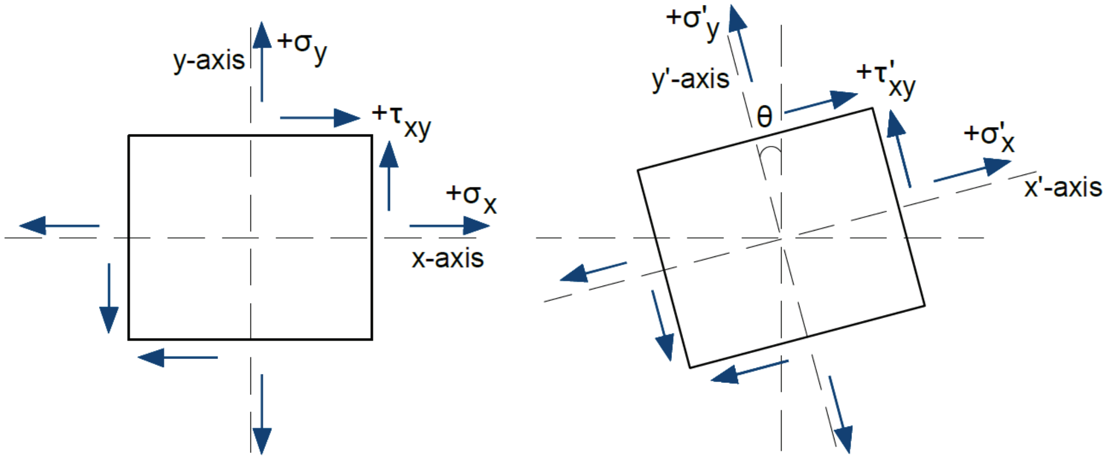
$$r = radial\ distance\ from\ neutral\ axis\ (in)\ or\ (m);$$

$$J = polar\ moment\ of\ inertia\ (m^4)\ \ or\ \ (in^4); \sigma_{bending} = \left(Pa\ or\ \frac{N}{m^2}\right)\ \ or\ \ \left(\frac{lbf}{in^2}\right)$$

Stress transformation equations are used to transform the stresses that occur in one set of axes (x, y & z) to another set of axes (x', y' and z'). This technique is used when you need to find the stress acting upon a cross sectional area that isn't on the original set of axes.

3.5 TRANSFORMATION EQUATIONS

The stresses in one axes can be transformed to another set of axes with these sets of equations.



$$\sigma_{x,new} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\sigma_{y,new} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

$$\tau_{xy_{new}} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

3.6 PRINCIPAL STRESSES

Principal stresses are used to combine shear stresses with axial stresses in multiple directions.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

4.0 MOHR'S CIRCLE

4.1 MOHR THEORY - DUCTILE

Mohr's theory is another theory used to assess when a material will fail. This theory is used when a material does not have similar compressive and tensile yield strengths. Mohr's theory can be best understood through the use of Mohr's Circle. This circle is a representation of 2-d stresses in the tensile/compressive direction and in shear. The circle is used to analyze complex stresses to determine the principle stresses that are applied to a component. The principle stresses can then be compared to the tensile, compressive and shear strengths.

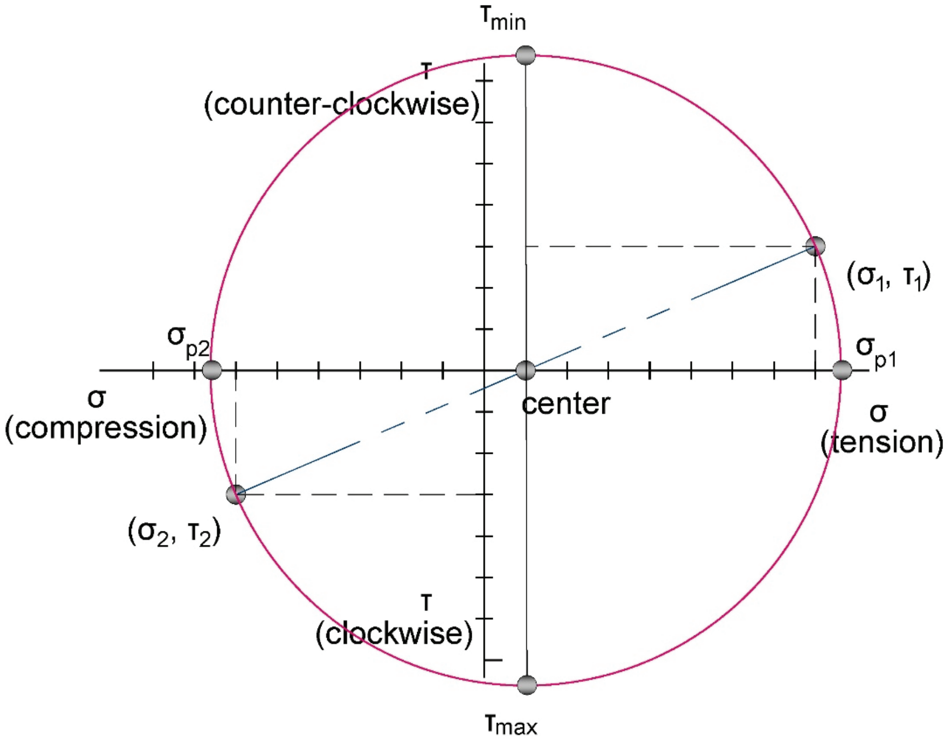

τ_{min}
τ
(counter-clockwise)
(σ_1, τ_1)
σ_{p2}
σ_{p1}
σ
(compression)
center
σ
(tension)
(σ_2, τ_2)
τ
(clockwise)
τ_{max}

Figure 14: Mohr's circle is used to find the primary stresses in tension, compression and shear.

The complex stresses of (σ_1, τ_1) and (σ_2, τ_2) are typically given in a FE problem. Then you must use Mohr's circle to find the principal forces that act in tension, compression and shear. These principal forces are shown as the extreme points on the circle.

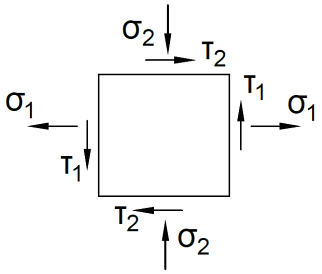

σ_2
τ_2
σ_1
τ_1
τ_1
σ_1
τ_2
σ_2

Figure 15: The given stresses that act upon an element within a component are shown in this figure. Notice how the clockwise and counter-clockwise shears correspond to the points shown on the previous figure. Also notice how the tension and compression stresses are shown in this figure correspond to the correct location in the previous figure.

Mohr's circle is created by first determining the center point of the circle from the complex stresses given in the problem. The center is found through this equation.

$$Center = \left(\frac{\sigma_1 + \sigma_2}{2}, \frac{\tau_1 + \tau_2}{2}\right)$$

Next you must find the radius of the circle. The radius is found through the use of the Pythagorean Theorem. First you take the difference in the x-directions, to find the x-length of the triangle. Then do the same for the y-direction. Finally, square root the sum of the squares to find the hypotenuse of the triangle, which is equal to the diameter of the circle. The radius is equal to ½ the diameter.

$$D = \sqrt{(\sigma_1 - \sigma_2)^2 + (\tau_1 - \tau_2)^2}; r = \frac{D}{2}$$

Finally, the principal stresses can be found by adding or subtracting the radius to the center values.

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + r;\ \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - r$$

$$\tau_{min} = \frac{\tau_1 + \tau_2}{2} + r; \tau_{max} = \frac{\tau_1 + \tau_2}{2} - r$$

4.2 MOHR THEORY - BRITTLE

Mohr's theory does not cover brittle materials. There is a modification to Mohr's theory for brittle materials, but this is outside the scope of this technical study guide and is most likely too complex to be tested on the FE exam.

5.0 STRESS AND STRAIN CAUSED BY AXIAL LOADS

Axial loads will cause tension and compression. On the FE exam you may encounter problems where an object is under a force due to tension or compression. A tension force is a force that attempts to pull a material apart along its axis. A compression force is a force that squeezes a material together along its axis. The forces due to tension and compression will cause a stress within the material that is equal to the force divided by the cross sectional area that is perpendicular to the force.

5.1 STRAIN

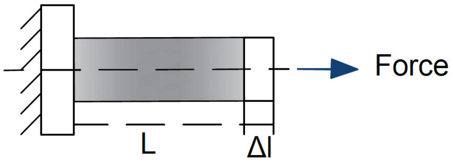
Strain type problems use either the Young's Modulus or the Stress-Strain graph to determine the amount of elongation or compression of the object. Deformation will occur based on the stress and the modulus of elasticity of the material. The modulus of elasticity is also known as

Young's Modulus and this will be discussed in more detail in *Section 11.0 Material Properties & Processing*. This equation is called Hooke's Law.

$$\sigma = E\varepsilon$$

$$\sigma = axial\ stress\ (psi\ or\ Pa);\ E = modulus\ of\ elasticity\ (psi);\ \varepsilon = strain\ (unitless)$$

Strain is defined as the change in length divided by the original length.



$$\varepsilon = \frac{\Delta l}{L}$$

$$\varepsilon = axial\ strain\ (unitless);\ \Delta l = change\ in\ length\ (in\ or\ m);\ L = original\ length\ (in\ or\ m)$$

6.0 STRESS AND STRAIN CAUSED BY BENDING LOADS

Bending occurs when slender objects are used to support loads perpendicular to their length. The most common type of problem that you will encounter on the FE exam is the bending of beams. Bending of beams was first introduced earlier in this section under shear and moment diagrams. This section will tie together the material properties and the beam diagrams to determine the strength of a beam during bending.

Once you use the force and moment diagrams from the previous section to determine the maximum forces and moments in the beam, then you can determine the stress in the beam through the following equation. The maximum stress in the beam is equal to the maximum moment divided by the section modulus of the beam. The section modulus will be defined later in this section. The equation to find the stress in a beam is shown below.

$$\sigma_{beam} = \frac{M}{S};$$

$$M = moment\ (lb - in); S = section\ modulus\ (in^3);\ \sigma_{beam} = stress\ in\ beam (psi)$$

6.1 SECTION MODULUS

The section modulus is a function of the geometry of the cross section of a beam. The section modulus is defined as the moment of inertia divided by the perpendicular distance from the neutral axis of the beam to the surface of the beam.

$$S = \frac{I}{c}$$

$$S = section\ modulus\ (m^3 \quad or \quad in^3); I = moment\ of\ inertia\ (m^4 \ \ or \ \ in^4);$$

$$c = perpendicular\ distance\ from\ neutral\ axis\ to\ surface\ of\ beam\ (m \ \ or \ \ in)$$

The section modulus is a measure of the strength of the beam's geometry. A larger moment of inertia or a smaller distance between the center of gravity and the surface will result in a stronger beam than a beam of equal material properties but a smaller moment of inertia or a larger distance "c". This is reiterated with the next figures.

The cross section of a beam will determine its moment of inertia. Thus a larger cross section will result in a larger section modulus, which will result in lower stresses in the beam.

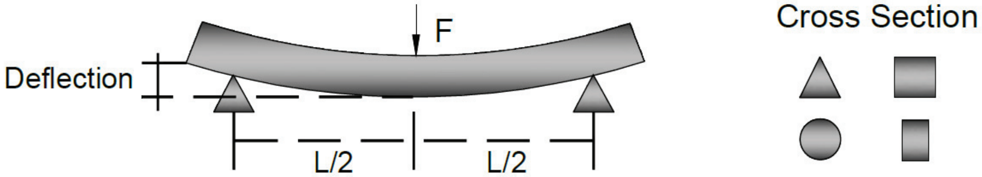


Figure 16: The strength of a material during bending is dependent on the length of the object, the material properties and the cross section of the object.

The following figure shows that the stresses due to tension and compression in a beam during bending vary based on the location relative to the center of the cross section of the beam. The top and bottom of the beam will experience the highest levels of stress.

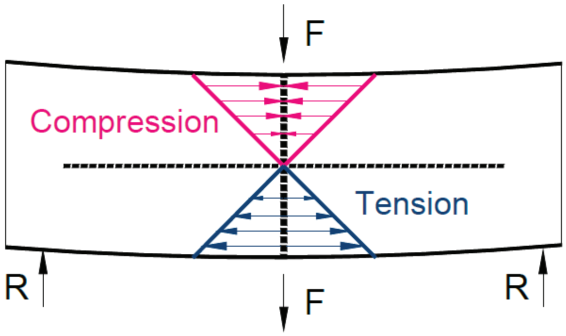


Figure 17: Cross section of a beam in bending. The maximum stress will occur at the surface of the beam. The distance from the center of the beam to the surface uses the variable, "c".

The final maximum beam stress equation will be the maximum moment multiplied by the maximum distance, "c" divided by the second moment of area (moment of inertia).

$$\sigma_{beam\ max} = \frac{M_{max} * c}{I};$$

6.2 DEFLECTION

Beams can be designed with the focus of limiting the amount of deflection in the beam. The strength of the material must be much stronger than normal to limit deflection to a certain amount as opposed to designing a beam to withstand an external loading. There will most likely be a deflection type problem on the FE exam, because beam deflection regularly occurs in machine design.

In order to calculate the amount of deflection, you must first find the moment, second moment of area and the material's modulus of elasticity. There is a method that allows you to find the deflection for a beam under any condition. This method involves developing an equation that governs the load or moment as a function of the distance along the length of the beam. Then this function is integrated twice and the constants are solved based on the boundary conditions of the beam and loading. This process is involved and does not meet the 2-3 minute requirement for the FE exam. Thus you should only be familiar with the equations that govern deflection for the most common situations, which are shown in the beam diagrams in the *NCEES FE Reference Handbook*.

7.0 STRESS AND STRAIN CAUSED BY TORSION

Torsional loading is the act of twisting an object, typically a rod or shaft in the Machine Design field. When the shaft is twisted, a torque is applied that causes the shaft to deform as shown in the figure below. The solid line is the new twisted axis and the dotted line shows the original axis. The angular difference between these two lines is called the angle of twist.

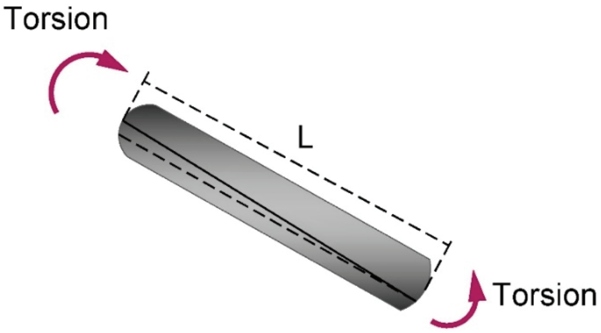


Figure 18: Torsion occurs when an object is twisted about its longitudinal axis. The dotted line is the original longitudinal axis and the solid line is the new axis after torsion.

Similar to beams during bending, the shear stress experienced in the shaft will vary based on the location from the center of the longitudinal axis of the shaft. The maximum shear stress will be at the surface of the shaft.

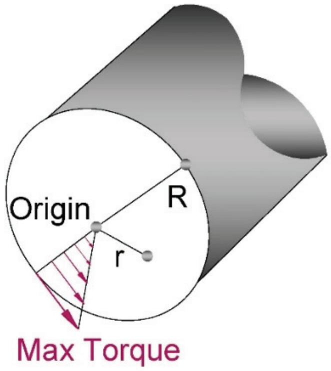


Figure 19: The maximum torque on a shaft occurs at the surface. The center of the shaft experiences zero torque.

The shear stress at any point from the origin to the radius R can be found with the ratio of the location to the total radius of the shaft. This equation shows you that the shear stress will be at a maximum at the surface of the rod or shaft.

$$\tau_r = \frac{r}{R}\tau_{max} \; or \; \frac{Tr}{J}$$

$\tau_r = shear\ stress\ at\ location\ r;\ \ R = radius\ of\ shaft;\ \tau_{max} = maximum\ shear\ stress,$

The maximum shear stress in a rod or shaft can be round with the following equation.

$$\tau_{max} = \frac{Tc}{J}$$

$\tau_{max} = maximum\ shear\ stress\ (Pa\ \ or\ \ psi), T = internal\ torque\ (N \cdot m\ \ \ or\ \ \ lbf \cdot in);$
$c = radius\ of\ shaft(m\ \ or\ \ in);\ \ \ J = polar\ moment\ of\ inertia\ (m^4\ \ or\ \ \ in^4)$

Another form of the above equation that solves for the internal torque is to use the shear modulus and the angle of twist. The internal torque is equal to the angle of twist in radians multiplied by the polar moment of inertia and shear modulus, divided by the length.

$$T = \frac{JG}{L}\theta$$

$G = shear\ modulus\ (Pa\ \ or\ \ psi); L = length\ of\ shaft\ (m\ \ or\ \ in);\ \varphi = angle\ of\ twist\ (radians)$

$T = internal\ torque\ (N \cdot m\ \ \ or\ \ \ lbf \cdot in);\ J = polar\ moment\ of\ inertia\ (m^4\ \ or\ \ \ in^4);$

The shear strain created by a torque is equal to the radius of the shaft, multiplied by the radius of the shaft and divided by the length. The following diagram illustrates the difference between the shear strain and the angle of twist.

$$\theta_1 = \frac{R}{L}\theta$$

$$\theta = angle\ of\ twist\ (radians);\ \theta_1 = shear\ strain\ (radians)$$

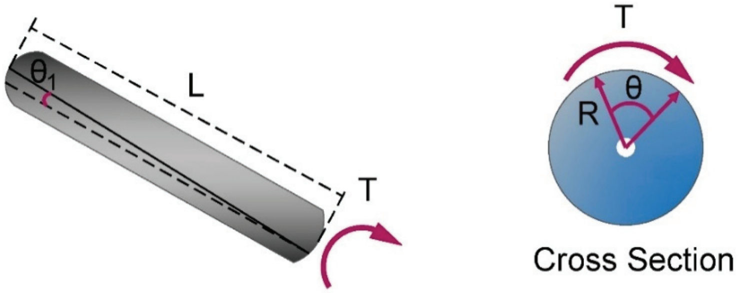


Figure 20: The torque on a shaft is related to the angle of twist with the above equation and the torsional strain.

On the FE exam, you may encounter different types of shafts in torsion. The main types of problems ask you to find the maximum shear stress due to torsion and then compare that value to the shear strength of the material. The biggest difficulty is using the correct polar moment of inertia value for the various shaft arrangements. The following paragraphs will highlight the most common types of arrangements and their respect polar moment of inertia equations. Polar moment of inertia can also be referred to as the polar second moment of area. This value is a function of the cross section of the shaft. You should find the table of second moments of area in the *NCEES FE Reference Handbook*.

7.1 SOLID CIRCULAR SHAFTS

When a pure torque acts upon a shaft, shear stresses will be set up which act in directions perpendicular to the radius at all points.

The polar moment of inertia of a solid circular shaft can be found with the following equation.

$$J = \frac{\pi}{2} R^4$$

7.2 TUBULAR CIRCULAR SHAFTS

If a shaft is hollow on the inside, then you will need the outer and inner radius to find the polar moment of inertia. Sometimes the exam will try to trick you and provide the outer diameter and the thickness. Just be careful and make sure you use the radius values.

$$J = \frac{\pi}{2}(R_{outer}^4 - R_{inner}^4)$$

7.3 SHAFTS OF VARYING DIAMETER

If the shaft has a varying diameter, but the same center axis, then you can add the polar moments of each individual shaft together.

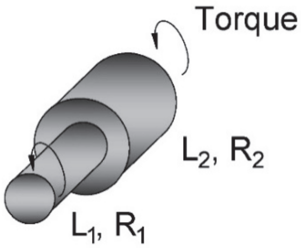


Figure 21: Connected shafts with varying diameters.

$$J_{total} = \frac{\pi}{2}R_1^4 + \frac{\pi}{2}R_2^4$$

7.4 PARALLEL AXIS THEOREM

The polar second moment of area is still the second moment of area. The polar term just refers to the fact that these equations focus on circular cross sections. Similar to the second moment of area section, the parallel axis theorem also applies. When a shaft is off-center and the axis needs to be adjusted to account for this difference, then you can use the parallel axis theorem. This theorem states that the second moment of area about an axis can be adjusted to another axis by adding the cross sectional area multiplied by the distance between the two axes squared.

$$J_{new} = J_{old} + A * d^2$$

$$d = perpendicular\ distance\ between\ the\ old\ axis\ and\ the\ new\ axis$$

7.5 TORSION FAILURE

The failure of a shaft during torsion will depend on whether the material is ductile versus brittle. If the material is ductile, then the material will most likely fail due to a maximum shear stress. If the material is brittle then the material will most likely fail due to a maximum tensile stress. This concept is a little difficult to grasp, but you can try to imagine a brittle material twisting and at the same time the material is twisting, the material is elongating and creating tension in a direction that is 45 degrees from the longitudinal direction.

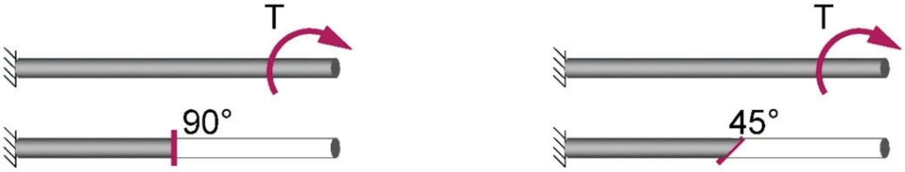


Figure 22: Ductile materials (left) will flex in torsion, until the maximum shear stress is met and the material will fail at a near 90 degree angle. Brittle materials (right) will not flex in torsion and will fail in tension at an angle near 45 degrees to the longitudinal direction.

When you are completing a torsion problem and you are finding the shear stress due to torsion, you can compare this shear stress value to either the shear or tensile strength of the material based on the type of material. On the FE exam and in practice, you will most likely not encounter a ductile material loaded in torsion. Ductile materials like copper and aluminum are rarely loaded in torsion. It is most common that all the various types of steel will be loaded in torsion, such that you can use the shear strength when comparing the design shear stress.

8.0 STRESS AND STRAIN CAUSED BY SHEAR

8.1 SHEAR STRESS

Shear stress is the stress that is in parallel to the area, as opposed to stress in tension or compression which acts perpendicular to the area. The figure below shows a force that acts in parallel to the area. The shear stress is defined as the shear force divided by the area.

$$\tau = \frac{V}{A}$$

$$\tau = shear\ stress\ (psi\ or\ Pa); V = shear\ force\ (lbs\ or\ N); A = area\ (in^2\ or\ m^2)$$

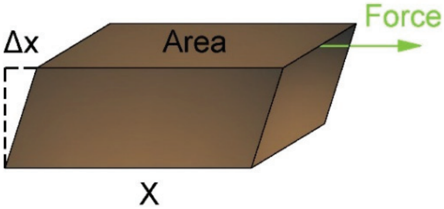


Figure 23: Shear stress acts in parallel to the cross sectional area of the object. Stress in tension or compression acts perpendicular to the cross sectional area.

8.2 SHEAR STRAIN

Shear strain is similar to the axial strain previously discussed, except that shear strain is a function of the change in angle of the object as opposed to the change in length/width of the object. Shear strain will be a function of the material's ability to resist shear stresses, which is shown as the material's modulus of rigidity or shear modulus.

The shear modulus is given the variable, "G". It describes the material's ability to resist shearing.

$$Modulus\ of\ Rigidity \rightarrow G\ (psi)$$

$$\tau = G\gamma$$

$$\tau = shear\ stress\ (psi\ or\ Pa); G = shear\ modulus\ (psi\ or\ Pa)$$

$$\gamma = shear\ strain(dimensionless)$$

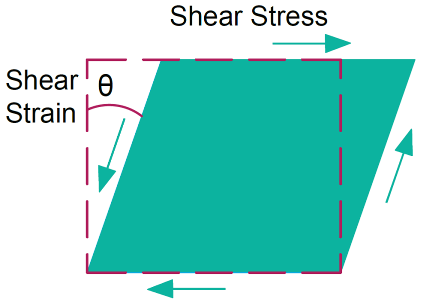


Figure 24: Shear strain is shown as the angle change.

8.3 POISSON'S RATIO

When a component elongates due to an axial stress, the component also contracts in the axis perpendicular to the elongation. The direction of the stress in tension will cause a positive longitudinal (axial) strain and a negative contraction in the lateral direction called lateral strain. The ratio of this elongation and contraction is called Poisson's ratio.

$$v = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}}$$

$$\varepsilon_{lateral} = \gamma = shear\ strain$$

Poisson's ratio can also be used to show the relationship between the modulus of elasticity in tension/compression and the modulus of elasticity in the shear direction. The equation that governs this relationship is shown below.

$$E\ = 2 * (1 + v) * G$$

9.0 STRESS AND STRAIN CAUSED BY TEMPERATURE CHANGE

As discussed further in *Section 9.0 Material Properties & Processing*, an increase in temperature can cause an object to expand and a decrease in temperature can cause an object to reduce in size. This change in dimensions is called thermal strain. As previously discussed thermal strain can be calculated with the equation below, which determines the change in length of an unrestrained object.

$$\Delta L = \alpha \Delta T L$$

$$\alpha = coefficient\ of\ thermal\ expansion\ (\frac{strain}{°\text{F}\ or\ °\text{C}\ or\ °K})$$

If this object is restrained, then the expansion/contraction will not result in a change in dimensions but will result in a change in the internal stresses of the object. An increase in temperature will cause internal stresses to rise in a restrained object and a decrease in temperature will cause internal stresses to drop. These internal stresses due to temperature changes are called thermal stress. Thermal stress is governed by the equation below.

$$\sigma_{thermal\ stress} = \alpha \Delta T E$$

$$\alpha = coefficient\ of\ thermal\ expansion\ (\frac{strain}{°\text{F}\ or\ °\text{C}\ or\ °K})$$

$$E = Young's\ Modulus\ (\frac{stress}{strain})$$

10.0 COMBINED LOADING

A combined loading problem occurs when you have multiple stresses acting upon an object. You could have a bending and axial load or a torsional and axial load or any other combination of loads. A combined loading problem will most likely not be on the exam, because the calculations will take much longer than 2-3 minutes. However, you should be familiar with the process, because they may ask you to complete part of a combined loading problem. In order to solve these types of problems, you must calculate the stress and strain due to each load. You also have to calculate the direction of the stress and strain due to each load at the point in question, then you can add-subtract the vectors for stress and strain due to each load. An example will be shown below to illustrate this process.

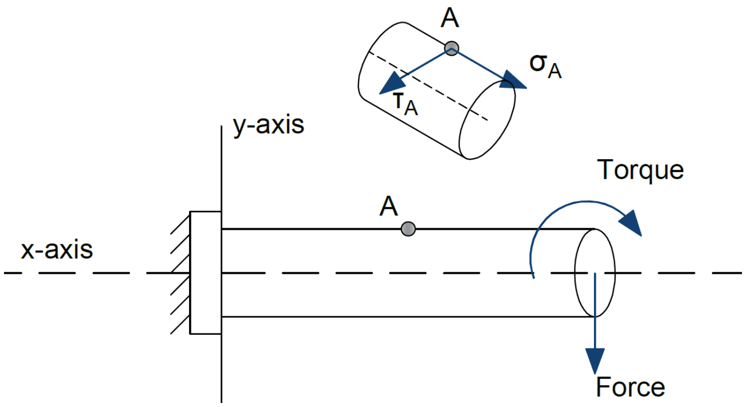


Figure 25: The stresses acting at point A are the bending stress due to the force and the torsional stress.

These vectors can then be added through the vector formulas that are shown in the mathematics section.

11.0 DEFORMATIONS

Deformation is the change in shape of an object. The change in shape is characterized by the strain values that were discussed in the previous topics. Deformations were shown in bending, torsion, compression, tension and combined loadings.

Two main types of deformation were also briefly introduced in Topic 3.0 Stress Types, (1) Elastic and (2) Plastic deformations.

Plastic deformations are also called permanent deformations. This means that the deformation has permanently changed the shape of the component. Elastic deformations can return back to their original shape. There is a certain point on the stress-strain graph called the elastic limit that indicates when a deformation will switch from elastic to plastic.

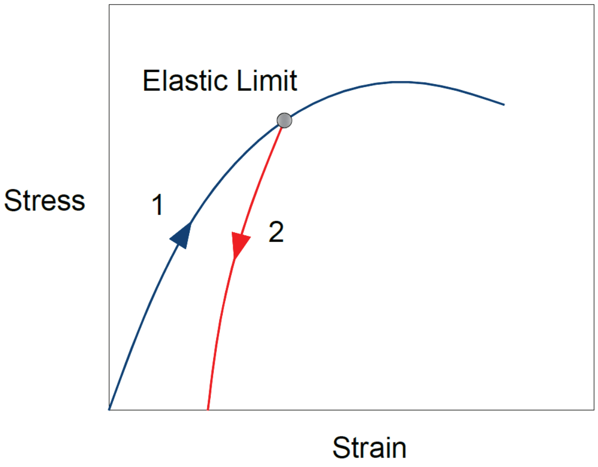


Figure 26: As stress in a component is increased in path 1, the component will begin to undergo strain, also known as deformation. Once the stress is increased to its elastic limit,

permanent deformation will have taken place. This means that when the stress is reduced as shown in path 2, there will be permanent strain, even when the stress is completely removed. When stress is equal to zero, there is still a strain value.

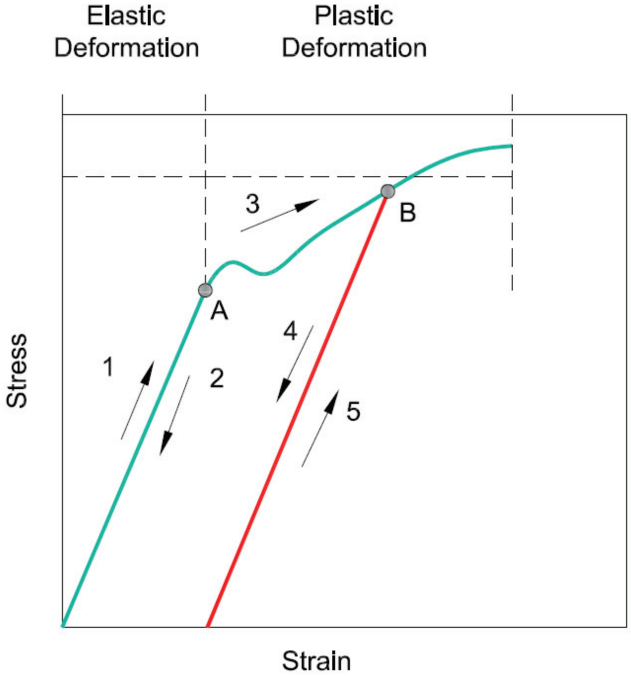


Figure 27: This figure shows a material that undergoes various levels of stress. In step 1, the stress goes to point A and the stress is relieved as it travels back to its original form (0 strain) in step 2. In step 3, the stress travels to point B, which is past the elastic limit. This means that plastic deformation has occurred. When the stress is relieved, the strain-stress follows path 4. This shows that even when all stress is relieved, there is permanent strain. When the stress is increased in path 5, the stress-strain graph follows the new path.

The *NCEES FE Reference Handbook* uses another way to show this elastic limit, called the elastic strain energy. The strain energy is the energy that characterizes the deformation of a component. The strain energy is calculated similar to work, it is the force acting upon a component, multiplied by the displacement. The displacement is the amount of deformation or strain that occurs due to the force.

$$Strain\ Energy\ (U) = \left(\frac{1}{2}\right) * F * \Delta d$$

$$\Delta d = displacement\ ;\ F = force\ (N\ or\ lb)$$

However, the component will be more complex and the force will also be more difficult to characterize on the actual exam. So you must use specific equations to calculate strain energy for the various loadings. If the strain energy due to the load is greater than the elastic strain energy, then the load will cause permanent deformation.

11.1 AXIAL LOAD

In an axial load, there will be an external force that causes the component to either stretch or compress. The displacement will be a function of the stress and the modulus of elasticity of the component.

$$Strain\ Energy\ (U) = \frac{F_{axial}^{2} * L_{initial}}{2AE}$$

$$E = modulus\ of\ elasticity\ (psi\ or\ Pa); A = cross\ sectional\ area\ (in^2\ or\ m^2)$$

$$F_{axial} = axial\ force\ (N\ or\ lbf); L = initial\ length\ (m\ \ or\ \ ft)$$

$$U = strain\ energy\ \ (N \cdot m = Joule\ \ or\ \ lbf \cdot ft)$$

This equation may look a little different, but it can be broken down to look just like the general strain energy equation that was previously shown. First, start with Hooke's law which shows the axial strain as a function of axial stress and Young's Modulus.

$$\sigma = E\varepsilon$$

But you know that the axial stress is equal to the force divided by the cross sectional area and the strain is equal to the deformation divided by the original length.

$$\sigma = \frac{F}{A}\ and\ \varepsilon = \frac{\Delta d}{L}$$

Then you can plug in the above equations back into Hooke's law.

$$\frac{F}{A} = E * \frac{\Delta d}{L}$$

$$\Delta d = \frac{FL}{AE}$$

Now plug in the deformation amount back into the original strain energy equation.

$$Strain\ Energy\ (U) = \left(\frac{1}{2}\right) * F * \frac{FL}{AE}\ (This\ is\ for\ a\ simple, elastic, bar\ under\ axial\ load\ only.\ )$$

11.2 SHEAR LOAD

In a shear load, there will be an external torque that causes the component to twist and deform. The displacement will be a function of the shear stress and the modulus of elasticity of the component.

$$Strain\ Energy\ (U) = \frac{T_{shear}^{2} * L_{initial}}{2GJ}$$

$$G = shear\ modulus\ (Pa\ \ or\ \ psi); A = cross\ sectional\ area\ (m^2\ \ or\ \ in^2)$$

$$F_{axial} = axial\ force\ (N\ or\ lbf); L = initial\ length\ (m\ \ or\ \ ft)$$

$$U = strain\ energy\ (N \cdot m = Joule\ \ or\ \ \ lbf \cdot ft)$$

The same conversion of this equation back into the original strain energy equation can also be done.

$$\tau_{shear\ stress} = G\gamma$$

But you know that the axial stress is equal to the force divided by the cross sectional area and the strain is equal to the deformation divided by the original length.

$$\tau_{shear\ stress} = \frac{T}{J}\ and\ \gamma = \frac{\Delta\theta}{L}$$

Then you can plug in the above equations back into Hooke's law.

$$\frac{T}{J} = G * \frac{\Delta\theta}{L}$$

$$\Delta\theta = \frac{TL}{GJ}$$

Now plug in the deformation amount back into the original strain energy equation.

$$Strain\ Energy\ (U) = \left(\frac{1}{2}\right) * T * \frac{TL}{GJ}$$

12.0 COLUMNS

12.1 BUCKLING

Buckling occurs when an object is subject to compression and as the compressive load increases, there is a quick bowing movement outwards, perpendicular to the direction of the loading. This situation becomes very unstable and the support will most likely be either deformed or unable to carry the force. A material's stress during buckling will most likely be less than the stress required to fracture or crush the same material in compression, which is typically the ultimate tensile stress or yield stress.

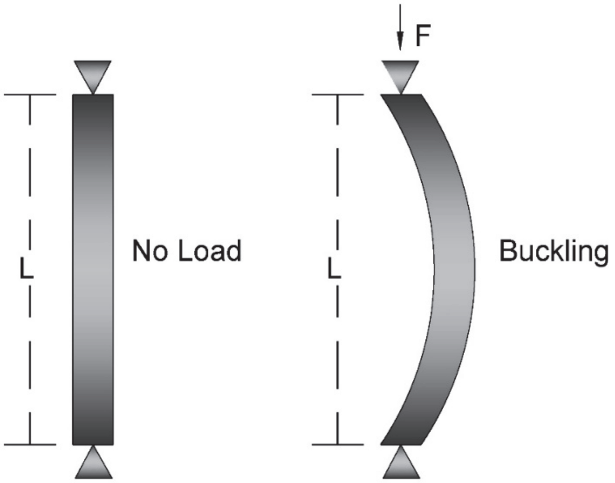


Figure 28: Buckling occurs when the length of a material is typically an order of magnitude greater than the width or diameter of the material.

Buckling occurs when a long, slender object is under compression. Mathematically, you must first confirm that an object is slender through the following steps.

Step 1: Find the effective length

The first step is to find the effective length of the object. The effective length will depend on the length of the object and the K-constant which depends on how the object is situated. If the object is fixed then the effective length will be smaller as compared to when one side of the object is fixed and the other is free. This will cause the effective length to increase. The following table provides the various situations and their corresponding K-constants.

$$Effective\ Length \rightarrow L_e = KL$$

K-Value	*Description*	*K-Value*	*Description*
1.0	*Pinned at both ends*	*0.5*	*Fixed at both ends*
0.7	*Fixed at one end, pinned at other*	*2.0*	*Free at one end, pinned at other end*

Step 2: Find radius of gyration

The radius of gyration compares the moment of inertia and the cross sectional area. The larger cross sectional area will cause the radius of gyration to increase, which means the object is less slender. You will need to compute the radius of gyration for both the x and y axis and use the minimum radius of gyration for the next step.

$$Radius\ of\ Gyration \rightarrow r_{min} = \sqrt{\frac{I}{A}}$$

Step 3: Find the slenderness ratio

Once you have the minimum radius of gyration and the effective length, then you can find the slenderness ratio with the following equation.

$$Slenderness\ Ratio \rightarrow R = \frac{L_e}{r_{min}};$$

Step 4: Find column constant

The column constant takes into account the modulus of elasticity and the yield stress of the material. A material with a larger yield stress will mean the column is much stronger and thus the column can be longer before it will be considered slender and therefore subject to buckling.

$$Column\ Constant \rightarrow C = \sqrt{\frac{2\pi^2 E}{\sigma_{yield}}}$$

$$E = Young's\ Modulus$$

Step 5: Compare slenderness ratio and column constant

Finally, compare the slenderness ratio and column constant. If the slenderness ratio is greater than the column constant then the object is considered to be slender and long. Thus the column will be subject to buckling

$R > C$	$R < C$
Column is considered slender	*Column is not slender*
Column is subject to buckling	*Column might not be subject to buckling before yield stress.*
Use Euler formula to determine critical buckling load	*Use Johnson formula to determine critical buckling load*

12.2 CRITICAL BUCKLING LOAD

The force at which a skinny column will fail under compression (buckling) is directly related to the material's Young's Modulus and moment of inertia. The force is inversely related to the square of the length and a constant which describes the degrees of freedom of the column. The yield stress of the material does not affect the critical buckling load in this formula. This formula is called the Euler Formula.

$$F_{Euler} = \frac{\pi^2 EI}{(KL)^2}$$

$$E = Young's\ Modulus; I = moment\ of\ inertia; L = free\ length$$

If the object is not considered long and slender, then the object's critical buckling load will be a function of the yield stress of the material. The equation used to solve for the critical buckling load is found through the below Johnson formula.

$$F_{Johnson} = A\sigma_{\text{yield}} * (1 - \frac{\sigma_{\text{yield}} * (K^2 L^2)}{4\pi^2 E r^2}$$

$$E = Young's\ Modulus; \sigma_{\text{yield}} = yield\ stress; L = free\ length$$

13.0 STATICALLY INDETERMINATE SYSTEMS

A statically indeterminate system is a system that when at rest has too many unknowns and not enough equilibrium equations to find the answers. The equilibrium equations include the force and momentum equations shown below.

$$Equilibrium\ Equations \rightarrow \sum F_x = 0; \sum F_y = 0; \sum F_z = 0; \sum M = 0;$$

The common example for statically determinate systems are the beam problems. You have a beam that is subject to a force and you must find the reactionary forces and moments. But if there are too many unknowns and not enough constraints, then the system will be statically indeterminate. The supports are typically where you have your constraints.

Support	**Unknown Reaction Forces**	**Unknown Reaction Moment**	**Movement Allowed**	**Movement Restricted**
Simple, roller	F_y	0	Horizontal & rotation	Upward or downward
Fixed	F_x & F_y	M	None	Horizontal, vertical & rotation
Pinned	F_x & F_y	0	Rotation	Horizontal & vertical

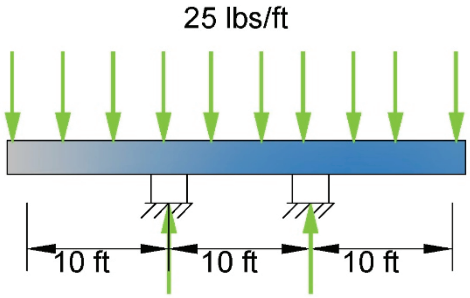
Constraints are created when you allow for movement. Unknown reactionary forces and moments are created when you make more restrictions as shown with the fixed support. The more unknown reactionary forces you have and the less degrees of freedom, the more likely you are going to have a statically indeterminate system.

Statically indeterminate systems are not unsolvable, these systems are just not able to be solved with Moment equilibrium and force equilibrium equations. You must start to use internal stresses, displacement and other techniques that were presented in this section and in the Mechanical Design Analysis section to see how your mechanical systems response to forces, moments or other external actions.

14.0 PRACTICE EXAM PROBLEMS

14.1 PRACTICE PROBLEM 1 – BEAM

A wood beam is situated as shown in the figure below. The material has strength of 900 psi. The beam shall be designed to have a safety factor of 1.0. What should be the dimension of the height of the beam? Assume the height of the beam is 2 times the width of the beam.



(a) 0.89 in

(b) 2.03 in

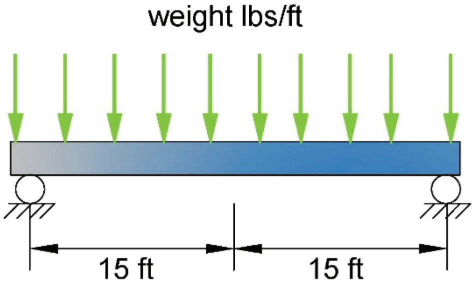
(c) 2.55 in

(d) 5.84 in

14.2 PRACTICE PROBLEM 2 – BEAM

A W 8 x 10 beam has an allowable stress of 50 ksi and dimensions as shown below. What is the maximum weight per foot that the beam can support?

$$I_x = 30.8\ in^4, h = 7.89in$$



(a) 120 lb/ft

(b) 210 lb/ft

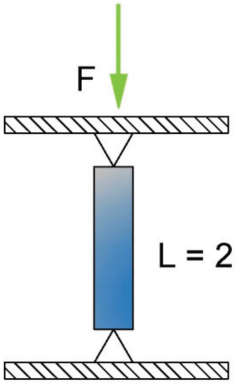
(c) 289 lb/ft

(d) 420 lb/ft

14.3 PRACTICE PROBLEM 3 – BUCKLING

A W 6 X 9 steel column with yield strength of 40 ksi and a modulus of elasticity of 29,000 ksi will be used to support an unknown load. What is the critical buckling load of this column? Assume the column is slender.

$$A = 2.68\ in^2; I_{xx} = 16.4\ in^4; I_{yy} = 2.2\ in^4; S = 5.56\ in^3$$



(a) 9 kips

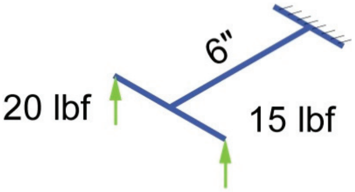
(b) 19 kips

(c) 26 kips

(d) 52 kips

14.4 PRACTICE PROBLEM 4 – TORSION

A handle is torqued to close a valve against the point shown in the figure below. What is the maximum shear stress developed at the outer wall of the solid stem? The length of the stem is 6" and the handle has a diameter of 4". The stem has a radius of 2". The forces are applied at the end of the handle at equal distances from the center.



(a) 0.8 psi

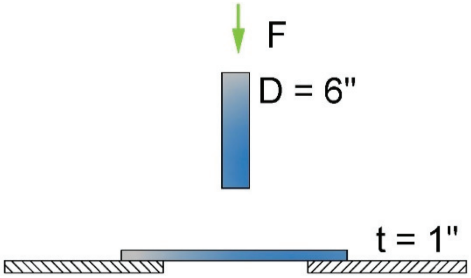
(b) 3.2 psi

(c) 5.0 psi

(d) 10.1 psi

14.5 PRACTICE PROBLEM 5 – SHEAR STRESS

A hole must be made in a metal plate (shear strength 30 ksi). A rod with a diameter of 6” will be used to puncture the metal plate. The metal plate has a thickness of 1”. What is the minimum force required to puncture the metal plate?



(a) 30 kips

(b) 100 kips

(c) 340 kips

(d) 570 kips

14.6 PRACTICE PROBLEM 6 – STRESS-STRAIN

A material is tested at various stresses and the strain is measured as shown in the table below. What is most nearly, the modulus of elasticity of the material?

Stress (psi)	Strain (in./in.)
0	0
15	.0004
30	.0008
38	.0012
42	.0016
44	.0020
45	.0024
45.5	.0028
45.8	.0032
46	.0036
46.1	.0040

(a) 10,000 psi

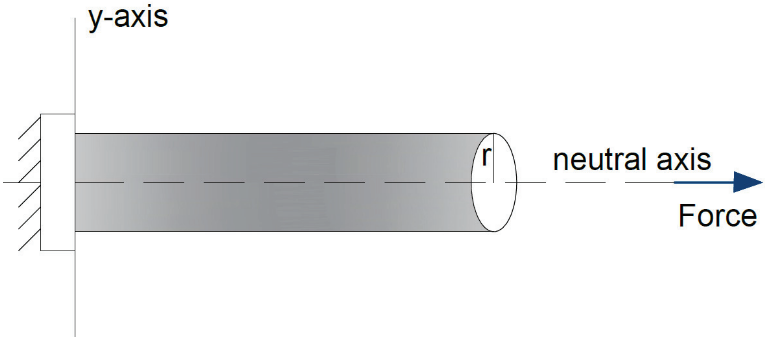
(b) 20,000 psi

(c) 30,000 psi

(d) 38,000 psi

14.7 PRACTICE PROBLEM 7 – STRAIN ENERGY

A rod is subject to a force of 10 kN. The rod has a diameter of 6 inches and has a modulus of elasticity of 100 GPa and a shear modulus of 50 GPa. The length of the rod is 12 feet. What is the strain energy?



(a) 0.1 J

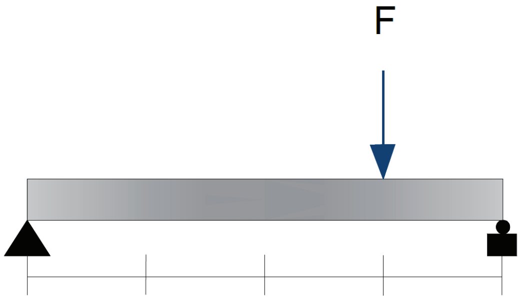
(b) 10 J

(c) 856 J

(d) 2,009 J

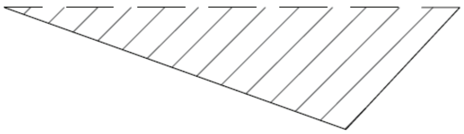
14.8 PRACTICE PROBLEM 8 – SHEAR AND MOMENT DIAGRAMS

Which of the following shear diagrams best fits the beam loading shown below?

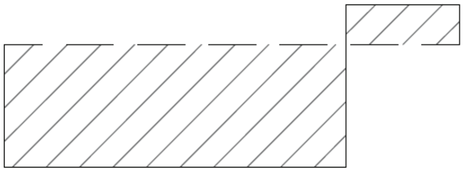


Possible Shear Diagrams:

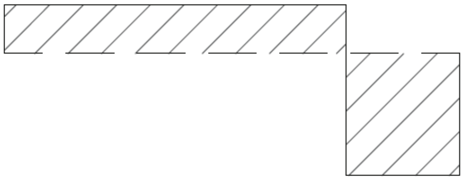
A



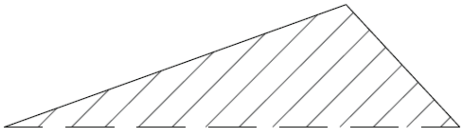
B



C

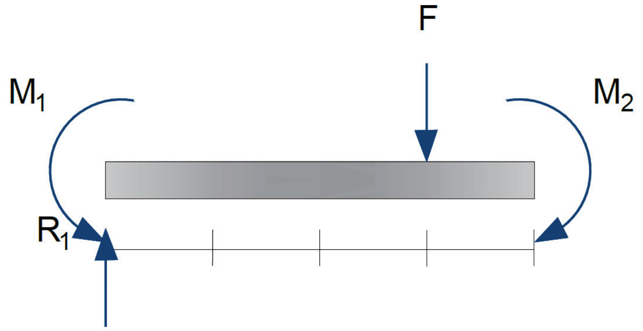


D

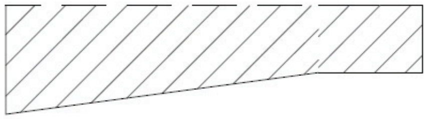


14.9 Practice Problem 9 – Shear and Moment Diagram

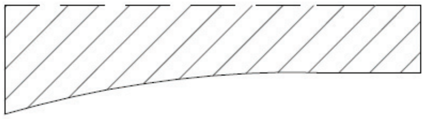
Which of the following moment diagrams best fits the free body diagram shown below? Assume M1 and M2 are -20 kN-m and -5 kN-m respectively. The forces are non-zero.



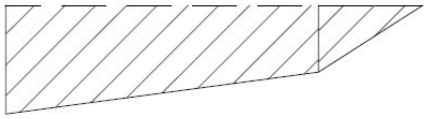
A



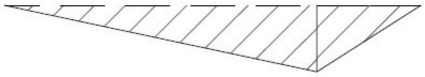
C



B



D



14.10 Practice Problem 10 – Stress Transformations

A compressive stress of 10 psi and a shear stress of 5 psi acts upon a material. What are the principal stresses?

(a) -12 psi & 2 psi

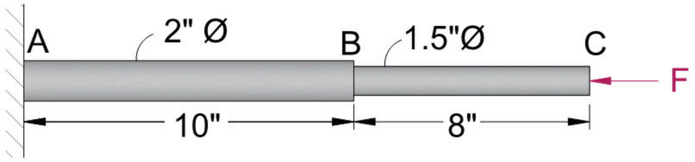
(b) -20 psi & 10 psi

(c) -30 psi & 30 psi

(d) -31 psi & 1 psi

14.11 PRACTICE PROBLEM 11 – STRESS FROM AXIAL LOAD

A force, F, of 45 kips is applied to a two segment steel rod as shown below. What is the stress experienced at segment BC?


A
2" Ø
B
1.5"Ø
C
F
10"
8"

(a) 6 kpsi

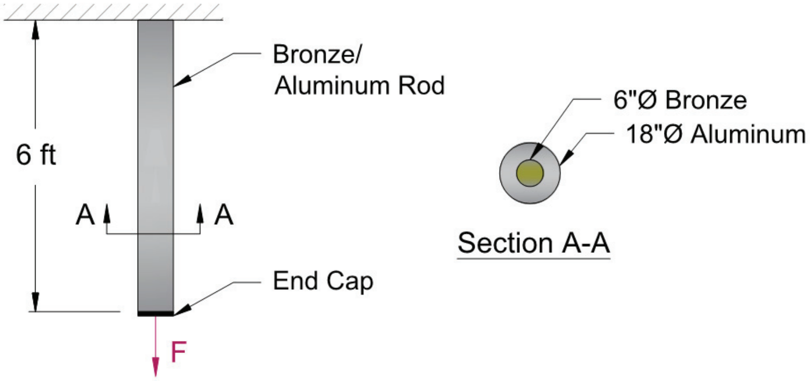
(b) 14 kpsi

(c) 18 kpsi

(d) 25 kpsi

14.12 PRACTICE PROBLEM 12 - STRESS FROM AXIAL LOAD

A 6ft long rod made up of bronze and aluminum as shown below has an external force, F, of 60 kips applied to the end of the rod. The end cap ensures that the rod bronze and aluminum components move together. What is the force acting on the bronze rod? Use 10Mpsi for the modulus of elasticity of Aluminum and 14.5Mpsi for the modulus of elasticity for bronze.


Bronze/
Aluminum Rod
6 ft
A
A
End Cap
F
6"Ø Bronze
18"Ø Aluminum
Section A-A

(a) 4.8 kips

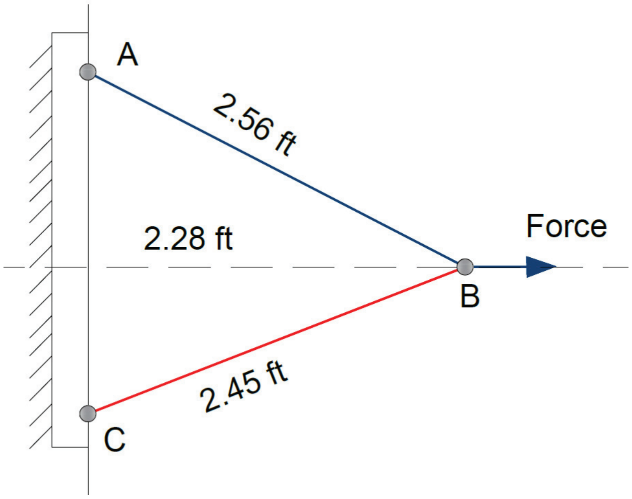
(b) 8.3 kips

(c) 9.2 kips

(d) 18.1 kips

14.13 PRACTICE PROBLEM 13 – AXIAL STRAIN

Rods AB and CB are secured to a wall. The rods are connected at point B and a force is applied as shown. The free lengths of rods AB and CB are shown. Point B is initially at a distance of 2.28 feet from the wall. The force causes a strain of .05 in/in in rod AB and .09 in/in in rod CB. Points A and C are secured at 2.06 feet from each other. What is the new perpendicular distance between the wall and point B?



(a) 2.28 ft

(b) 2.34 ft

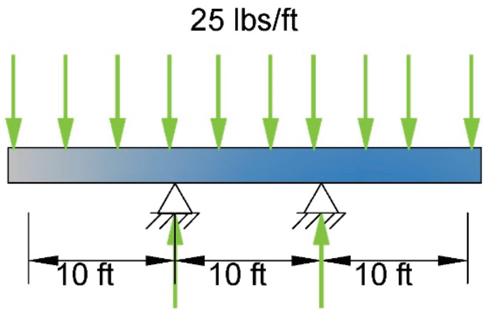
(c) 2.48 ft

(d) 2.67 ft

15.0 SOLUTIONS

15.1 SOLUTION 1 - BEAM

A wood beam is situated as shown in the figure below. The material has strength of 900 psi. The beam shall be designed to have a safety factor of 1.0. What should be the dimension of the height of the beam? Assume the height of the beam is 2 times the width of the beam.

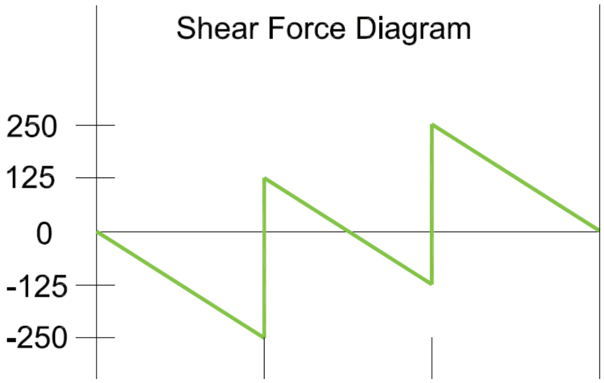


First, solve for the reactionary forces. The reactionary forces should balance out the 25 x 30 = 750 lbs force. Also since the reactionary forces are symmetrical, the forces should be equal.

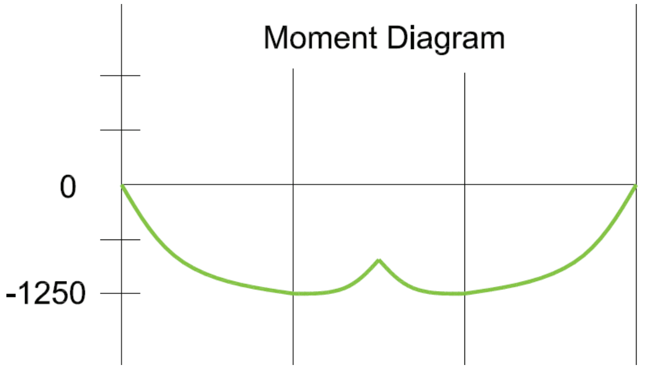
$$R_1 = \frac{25 * 30 * (30 - 2(10))}{2 * 10} = 375\ lbs$$

$$R_2 = \frac{25 * 30 * (30 - 2(10))}{2 * 10} = 375\ lbs$$

Next use these reaction forces to create the shear force diagram.



Next, calculate the area under the curves, to create the moment diagram.



The maximum moment is 1,250 lb-ft.

$$M_{max} = 1{,}250\ lb - ft$$

Finally, use the section modulus equation to solve for the dimensions of the beam.

$$S = \frac{I}{c} = \frac{\mathrm{M}}{\sigma_{\mathrm{allow}}};\mathrm{where\ I} = \frac{\mathrm{bh}^3}{12}\mathrm{and\ c} = \frac{\mathrm{h}}{2}\mathrm{and\ b} = \frac{1}{2}\mathrm{h}$$

$$\sigma_{\mathrm{allow}} = 900\mathrm{psi}$$

$$\frac{1}{12} * \frac{\frac{1}{2}h(h^3)}{h/2} = \frac{1{,}250\ lb - ft}{900\ \mathrm{psi}};$$

$$h^3 = \frac{1{,}250\ lb - ft * \left(12\frac{in}{ft}\right) * 12}{900\ \mathrm{psi}};$$

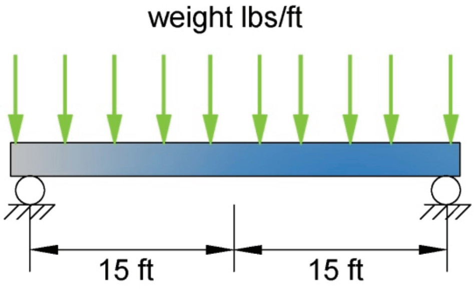
$$h^3 = 200;$$

$$h = 5.84\ in$$

The correct answer is most nearly, **(d) 5.84 in.**

15.2 SOLUTION 2 - BEAM

A W 8 x 10 beam has an allowable stress of 50 ksi and dimensions as shown below. What is the maximum weight per foot that the beam can support?



In order to solve this problem you should use the sectional modulus equation.

$$S = \frac{\mathrm{M}}{\sigma_{\mathrm{allow}}};$$

Make sure you use the x-direction, since the load is perpendicular to the x-direction.

$$S_x = \frac{I_x}{h/2} = \frac{30.8\ in^4}{7.89 in/2} = 7.81\ in^3$$

The maximum moment can also be found.

$$\mathrm{M_{max}} = \frac{\mathrm{wL^2}}{8}$$

But you also know from the sectional modulus equation that the moment is also equal to S*allowable stress.

$$M_{max} = 7.81\ in^3 * 50 ksi$$

$$M_{max} = 7.81\ in^3 * 50{,}000\ psi = 390{,}500\ lb - in$$

$$M_{max} = 32{,}542\ lb - ft$$

Now solve for w (lb/ft).

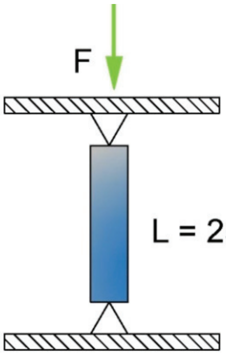
$$\frac{\mathrm{wL^2}}{8} = 32{,}542\ \ lb - ft$$

$$\mathrm{w} = 289\ \mathrm{lb/ft}$$

The correct answer is most nearly, **(c) 289 lb/ft**

15.3 SOLUTION 3 – BUCKLING

A W 6 X 9 steel column with yield strength of 40 ksi and a modulus of elasticity of 29,000 ksi will be used to support an unknown load. What is the critical buckling load of this column? Assume the column is slender.



$$A = 2.68\ in^2; I_{xx} = 16.4\ in^4; I_{yy} = 2.2\ in^4; S = 5.56\ in^3$$

Next use the critical force equation below, since you can assume the column is slender.

$$F_{Euler} = \frac{\pi^2 EI}{(KL)^2}$$

Since the load is pin connected on both sides, K = 1. Convert the length to inches.

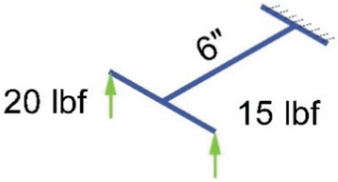
$$F_{Euler} = \frac{\pi^2 \left(29{,}000{,}000 \frac{lbs}{in^2}\right) * (2.2\ in^4)}{(25 * 12)^2\ in^2}$$

$$= 6{,}996\ lbs\ or\ 6.996\ kips$$

The correct answer is most nearly, (a) 9 kips.

15.4 SOLUTION 4 – TORSION

A handle is torqued to close a valve against the point shown in the figure below. What is the maximum shear stress developed at the outer wall of the solid stem? The length of the stem is 6” and the handle has a diameter of 4”. The stem has a radius of 2”. The forces are applied at the end of the handle at equal distances from the center.



First, you need to calculate the torque on the stem. So you need to balance the moments upon the center of the stem at the handle.

$$M = 0 = 20lbf * 2\text{"-15lbf*2"} + T$$

$$T = 10\ lbf - in$$

Then use the maximum shear stress equation.

$$\tau = \frac{TR}{J};\ \ Solid\ shafts \rightarrow J = \frac{\pi}{2}R^4$$

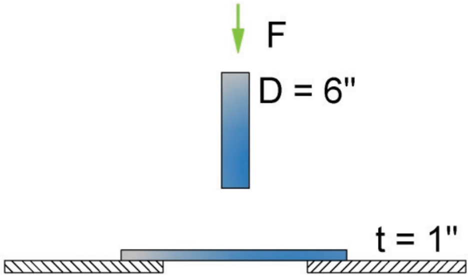
$$\tau = \frac{(10\ lbf - in) * (2\ in)}{\frac{\pi}{2}(2in)^4};$$

$$\tau = 0.8\,psi$$

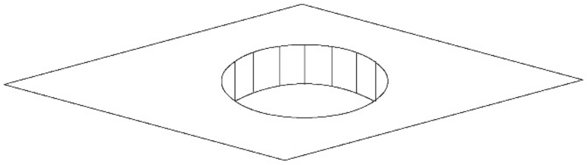
The correct answer is most nearly, **(a) 0.8 psi**.

15.5 SOLUTION 5 – SHEAR STRESS

A hole must be made in a metal plate (shear strength 30 ksi). A rod with a diameter of 6” will be used to puncture the metal plate. The metal plate has a thickness of 1”. What is the minimum force required to puncture the metal plate?



The first step is to find the maximum shear force the metal plate can withstand before breaking. This will be equal to the maximum shear stress multiplied by the applicable area. The area that will be under shear is equal to the thickness multiplied by the circumference of the rod. This may be difficult to picture, but the area that is in shear is shown in the figure below as dashed.



$$\tau = \frac{V}{A}; \tau = maximum\ shear\ stress; V = maximum\ shear\ force$$

$$30{,}000\ lbf/in^2 = \frac{V}{(1\ in) * (\pi * 6\ in)};$$

$$V = 565{,}486.7\ lbf$$

The answer is most nearly, **(d) 570 kips**

15.6 SOLUTION 6 – STRESS STRAIN

A material is tested at various stresses and the strain is measured as shown in the table below. What is most nearly, the modulus of elasticity of the material?

Stress (psi)	Strain (in./in.)
0	0
15	.0004
30	.0008
38	.0012
42	.0016
44	.0020
45	.0024
45.5	.0028
45.8	.0032
46	.0036
46.1	.0040

The modulus of elasticity is equal to the slope of the stress-strain graph in the linear region. The only linear region is at the first two data points.

$$Slope = \frac{y_1 - y_0}{x_1 - x_0} = \frac{15 - 0}{.0004 - 0} = 37{,}500\ psi$$

$$Slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{30 - 15}{.0008 - .0004} = 37{,}500\ psi$$

Just to be sure, check the next two points.

$$Slope = \frac{y_3 - y_2}{x_3 - x_2} = \frac{38 - 30}{.0012 - .0008} = 20{,}000\ psi$$

$$Slope = \frac{y_4 - y_3}{x_4 - x_3} = \frac{42 - 38}{.0016 - .0012} = 10{,}000\ psi$$

The slope is getting flatter, which means you are no longer in the straight line component.

The correct answer is most nearly, (d) 38,000 psi

15.7 SOLUTION 7 – STRAIN ENERGY

The rod is subject to an axial load, so you can use the strain energy equation in the *NCEES FE Reference Handbook* for axial loads.

$$Strain\ Energy\ (U) = \left(\frac{1}{2}\right) * F * \frac{FL}{AE}$$

The force is equal to 10 KN, but you need to convert the units for the other dimensions to metric to find the other terms.

$$6\ in\ x\ (\frac{0.0254\ m}{1\ in}) = 0.1524\ m$$

$$12\ ft\ x\ (\frac{0.3048\ m}{1\ ft}) = 3.658\ m$$

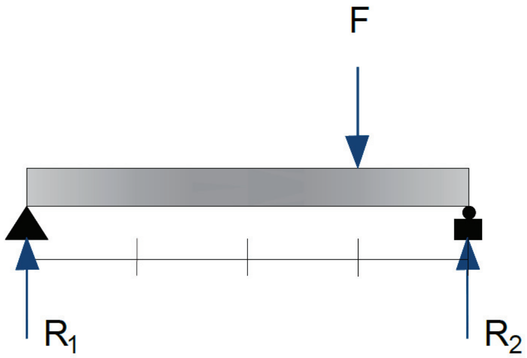
$$U = \left(\frac{1}{2}\right) * (10{,}000\ N) * \frac{(10{,}000\ N) * 3.658\ m}{(\frac{\pi(0.1524\ m\ )^2}{4})(100\ x\ 10^9\ Pa)}$$

$$U = 0.1\ N - m$$

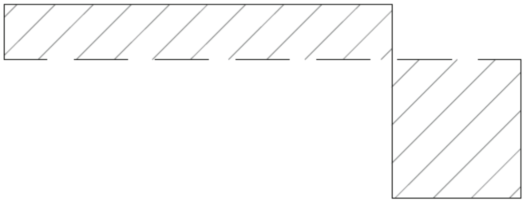
The correct answer is most nearly (A) 0.1 J.

15.8 SOLUTION 8 – SHEAR AND MOMENT DIAGRAMS

First, complete a free body diagram of the beam.



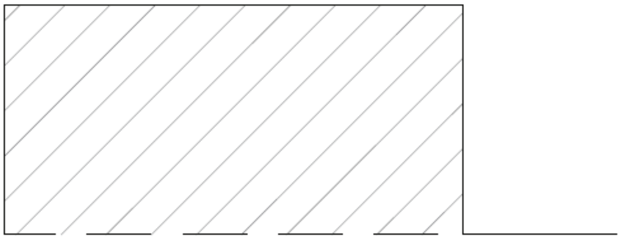
The only solution that has upward forces on the edges is solution C.



The correct answer is most nearly (C).

15.9 SOLUTION 9 – SHEAR AND MOMENT DIAGRAM

First conduct the shear diagram. This shear diagram shows that the moment will increase linearly.



You also know that the ends of the moment diagram will have moment values equal to M1 and M2. So this takes out B & D since the moments are equal to zero at the right ends. Since the force is a point force, then the moment will increase linearly as opposed to in a curve as shown in C.

The correct answer is most nearly (A).

15.10 SOLUTION 10 – STRESS TRANSFORMATIONS

The principal stresses are found with the below equation. Compressive stresses are negative and tensile stresses are positive.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{1,2} = \frac{-10\,psi + 0psi}{2} \pm \sqrt{\left(\frac{-10\,psi - 0psi}{2}\right)^2 + (5\,psi)^2}$$

$$\sigma_{1,2} = -5\,psi \pm 7.1\,psi$$

$$\sigma_1 = 2.1\,psi\ and\ \sigma_2 = -12.1\,psi$$

The correct answer is most nearly (A) -12 psi & 2 psi

15.11 SOLUTION 11 – STRESS FROM AXIAL LOAD

The internal force at segment AB and the internal force at segment BC will both equal the force, F. Solve for the stress at segment BC, with diameter equal to 1.5".

$$\sigma = P/A$$

$$\sigma_{BC} = \frac{45\,kips}{\pi * \left(\frac{1.5in}{2}\right)^2} = 25\,kpsi$$

The correct answer is most nearly (D).

15.12 SOLUTION 12 - STRESS FROM AXIAL LOAD

The force F will be distributed between the bronze and aluminum portions of the rod.

$$F = P_{AL} + P_{BZ}$$

The end cap ensures that the deformation is the same between the bronze and aluminum sections. Therefore, the following relationship can be made.

$$\delta = \frac{P_{AL}L_{AL}}{A_{AL}E_{AL}} = \frac{P_{BZ}L_{BZ}}{A_{BZ}E_{BZ}}$$

Since the lengths are the same, the equation simplifies to

$$\frac{P_{AL}}{A_{AL}E_{AL}} = \frac{P_{BZ}}{A_{BZ}E_{BZ}}$$

Substitute the initial equation to solve for the force acting on the bronze component:

$$P_{AL} = \frac{P_{BZ}}{A_{BZ}E_{BZ}} * A_{AL}E_{AL}$$

$$F = \frac{P_{BZ}}{A_{BZ}E_{BZ}} * A_{AL}E_{AL} + P_{BZ}$$

So,

$$P_{BZ} = \frac{F}{\frac{A_{AL}E_{AL}}{A_{BZ}E_{BZ}} + 1} = \frac{F}{\frac{\cancel{\pi/4}(D_{AL}^2 - D_{BZ}^2)E_{AL}}{\cancel{\pi/4}D_{BZ}^2 E_{BZ}} + 1}$$

From the *NCEES FE Reference Handbook*, the modulus of elasticity for

$$P_{BZ} = \frac{60kips}{\frac{[(18in)^2 - (6in)^2] * 10Mpsi}{(6in)^2 * 14.5Mpsi} + 1} = 9.2kips$$

The correct answer is most nearly (C).

15.13 SOLUTION 13 – AXIAL STRAIN

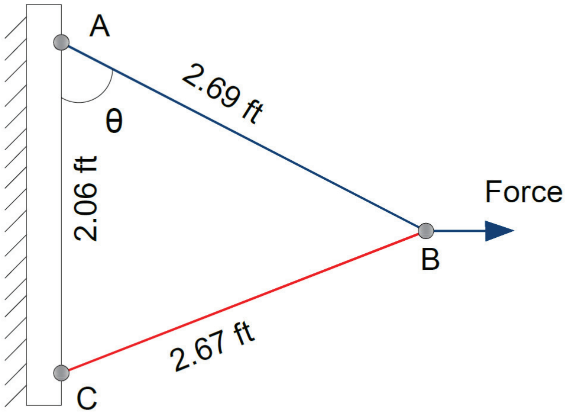
First, solve for the new lengths of AB and CB.

$$\varepsilon = \frac{\Delta L}{L}$$

$$\varepsilon_{AB} = .05\frac{in}{in} = \frac{\Delta L}{2.56\,ft * 12\,(\frac{in}{ft})} \rightarrow \Delta L = 1.536\,in; \rightarrow L_{AB,new} = 32.26\,in\ or\ 2.688\,ft$$

$$\varepsilon_{CB} = .09\frac{in}{in} = \frac{\Delta L}{2.45\,ft * 12\,(\frac{in}{ft})} \rightarrow \Delta L = 2.646\,in; \rightarrow L_{AB,new} = 32.05\,in\ or\ 2.6705\,ft$$

Now you can use the new triangle lengths to solve for an angle within the triangle with the Law of Cosines.

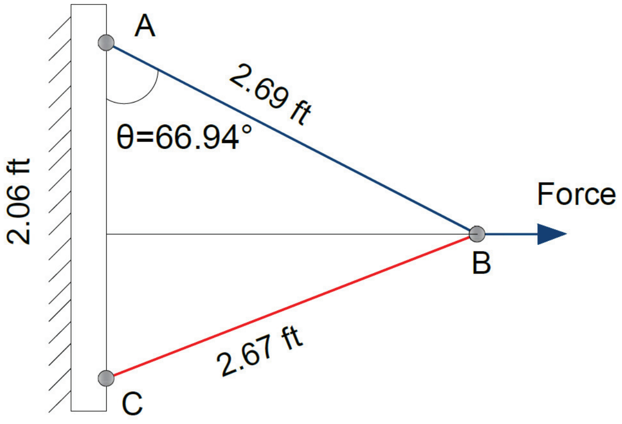


Law of Cosines

$$2.67^2 = 2.06^2 + 2.69^2 - 2 * 2.69 * 2.06 * \cos(\theta)$$

$$\theta = 66.94°$$

Now make a right triangle and solve for the new perpendicular distance between the wall and point B.



$$\sin(66.94) = \frac{Distance}{2.69\ ft}$$

$$Distance = 2.48\ ft$$

The correct answer is most nearly (C) 2.48 ft.

9 – Material Properties & Processing

MECHANICAL FE TEXTBOOK

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Section 9.0 – Material Properties and Processing

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1.0 INTRODUCTION

Material Properties and Processing accounts for approximately 7 to 11 questions on the Mechanical FE exam. This section works in conjunction with *Section 8 Mechanics of Materials*. The Material Properties part covers the mechanical properties covered in Section 8 and other material properties like chemical, electrical and physical types. This section then covers the material processing, material types, phase diagrams and other material issues like corrosion, fatigue and cracks.

There are a few pages on Material Properties & Processing in the *NCEES FE Reference Handbook* that you should be familiar with in order to pass the FE exam. However, in order to use those pages you need to understand the concepts and skills presented in this section.

Section 9.0 Material Properties & Processing (7 to 11 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
9A	Section 2.0	Properties (e.g., Chemical, Electrical, Mechanical, Physical, Thermal)
9B	Section 3.0	Stress-Strain Diagrams
9C	Section 4.0	Ferrous Metals
9D	Section 5.0	Nonferrous Metals
9E	Section 6.0	Engineered Materials (e.g., composites, polymers)
9F	Section 7.0	Manufacturing Processes
9G	Section 8.0 & 9.0	Phase Diagrams, Phase Transformation and Heat Treating
9H	Section 10.0	Materials Selection
9I	Section 11.0	Corrosion Mechanisms and Control
9J	Section 12.0	Failure Mechanisms (e.g., thermal failure, fatigue, fracture, creep)
	Section 13.0	Practice Exam Problems

2.0 PROPERTIES (CHEMICAL, ELECTRICAL, MECHANICAL, PHYSICAL, THERMAL)

2.1 CHEMICAL

A chemical property is the property of a material that undergoes change in its chemical structure during a chemical reaction. An important chemical property is corrosion, which will be discussed later in this section.

2.2 ELECTRICAL

The important electrical properties of materials are conductivity and capacitance. The conductivity describes the material's ability to conduct electricity and capacitance describes the material's ability to hold a charge. Conductivity is often expressed as its inverse, which is called resistivity. This is the material's ability to resist an electrical current.

$$Conductivity(C) = \frac{1}{Resistivity\ (\rho)}$$

Resistivity is used to calculate the resistance of an object like a copper wire. Resistance is given in units of ohms and is a function of the material's resistivity, the cross sectional area of the material and the length of the material.

$$R\ (\Omega) = \frac{\rho L}{A}$$

$$A = area\ (m^2); l = length\ (m)$$

$$\rho = resistivity\ (\Omega - \mathrm{m}); C = conductivity\ (\frac{1}{\Omega - \mathrm{m}})$$

2.3 MECHANICAL

The mechanical properties of materials are found by a series of tests. These tests include the tension or compression test, which determines the Stress-Strain diagram. The Rockwell or Brinell tests determine the hardness of the material. There are also many more tests that determine the other properties like creep, thermal expansion, thermal conductive, specific heat capacity and visco-elasticity. Each of the mechanical material properties will be discussed in the next paragraphs.

2.3.1 Strength

Strength is covered in *Topic 3.0 Stress-Strain Diagrams*.

2.3.2 Hardness

Hardness is the property that describes a material's ability to withstand abrasion, scratching and indentation. Hardness is measured in terms of MPa or is a dimensionless value. The hardness measurements are useful when comparing one material to another.

Hardness is measured by any one of the following tests, (1) Rockwell, (2) Brinell, (3) Meyer, (4) Vickers, (5) Knoop or (6) Scleroscope. Each of these tests use a different object of varying shape and material to impact the subject material. The effect of this impact due to a known force is then measured and a hardness value is assigned. The two main tests that you should know for the FE exam are Rockwell and Brinell. You should also know that hardness values are not absolute, these tested values are relative to each other. Hardness values are used to compare the hardness between different materials.

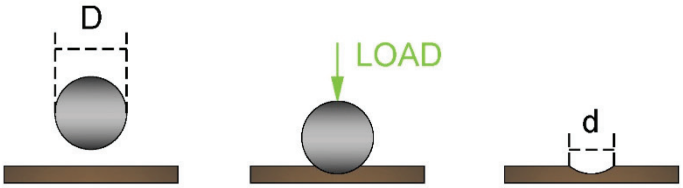


Figure 1: Hardness tests involve a known load impacting a material's surface and the measurement of the load and the result of the impact upon the material's surface.

2.3.1 Rockwell

The Rockwell hardness test is the most popular test. In this test, an indenter is first used as a pre-load to remove any effects of the surface finish and then the full load is used to indent the material. The depth of the indentation determines the Rockwell hardness value. This test has different scales based on the size of the indenter used to impact the test material. The scales range from A through G. Each scale is used for different types of materials. A summary of the scales are shown below.

Rockwell Hardness Scale	Full Load Value (kg)	Indenter Size	Typical Use
A	60	120° cone, 0.2 mm radius diamond	Extremely hard materials
B	100	1/16"-D Steel Sphere	Soft steel
C	150	120° cone, 0.2 mm radius diamond	Hard steel, iron
D	100	120° cone, 0.2 mm radius diamond	Medium hardness
E	100	1/8"-D Steel Sphere	Medium Copper, aluminum
F	60	1/16"-D Steel Sphere	Soft Copper, aluminum
G	150	1/16"-D Steel Sphere	Softest materials

The hardness number is then determined by measuring the thickness of the indent made into the material and inserting this value into one of the following equations.

$$Rockwell\ A, C\ and\ D\ Hardness\ \# = 100 - 500t$$

$$Rockwell\ B, E, F, G, and\ H\ Hardness\ \# = 130 - 500t$$

2.3.2 Brinell

The Brinell hardness test is another popular hardness test method. This test uses a 10-mm diameter steel ball and a 3,000 kg force. As the materials get softer, a smaller force is used and for harder materials, a tungsten carbide ball is used instead of steel.

$$Brinell\ Hardness\ (BHN) = \frac{2P}{\pi D\left(D - \sqrt{D^2 - d^2}\right)}$$

$$Brinell\ Hardness\ \left[\frac{kgf}{mm^2}\right]; P = kgf;$$

$$D = diameter\ of\ indenter\ [mm]; d = diameter\ of\ indentation[mm]$$

There is a rough relationship between the Brinell Harness Number and the tensile stress of a material, which is shown with the equation below.

$$Tesile\ Stress\ (psi) = 500 * (BHN)$$

$$Tesile\ Stress\ (MPa) = 3.5 * (BHN)$$

2.4 PHYSICAL

During the exam you will need to be able to find and use material properties to complete many problems. You should be very familiar with your resources and where to find these material properties. As you go through these descriptions of the important material properties, look through the *NCEES FE Reference Handbook* so you can become familiar with the available properties and the units.

2.4.1 Density

The density of a substance is its mass per unit volume, basically how heavy is something in one cubic foot or one cubic meter.

$$density, \rho = \frac{mass}{Volume}\left[\frac{lbm}{ft^3}, \frac{lbm}{in^3}, \frac{kg}{m^3} or \frac{g}{cm^3}\right]$$

Common Conversion Factors:

$$1\frac{lbm}{ft^3} = 0.000578\frac{lbm}{in^3}$$

$$1\frac{kg}{m^3} = 0.001\frac{g}{cm^3} = 0.0624\frac{lb}{ft^3}$$

The density is typically used to calculate the overall weight of a material based on its volume. The table below lists densities of common metals. You may find that different sources have varying densities for metal alloys. Therefore, it is likely that the test will provide you with the density values when referencing an alloy to avoid possible discrepancies.

Material	**Density (lb/in^3)**
Aluminum Base	
Aluminum (Pure)	0.097
Aluminum 6061	0.098
Copper Base	
Copper (Pure)	0.323
Cupronickel, 30%	0.323
Commercial Bronze, 90%	0.318
Bronze (Aluminum, 9%)	0.276
Yellow Brass	0.306
Naval Brass	0.304
Nickel Base	
Nickel	0.321
Monel	0.305
Cast Iron/Steel Base	
Gray Cast Iron	0.250
Ductile Iron	0.250
Cast Steel, 3% Carbon	0.250
Stainless Steel	
304SS	0.290
316SS	0.290
Titanium	
Titanium (Pure)	0.163
Zinc	
Zinc (Pure)	0.258
Other	
Water	3.6×10^{-5} ($0.0624\ lb/ft^3$)

Specific volume is the inverse of density and is measures as a volume per unit mass.

$$specific\ volume, \nu = \frac{1}{\rho}\left[\frac{ft^3}{lb}\ or\ \frac{m^3}{kg}\right]$$

2.4.2 Melting Point

The melting point is the temperature at which a solid will turn into a liquid at atmospheric pressure. An alloy may have two melting points, solidus and liquidus. *Solidus* is the temperature at which the metal turns from completely solid to a solid and liquid mixture. *Liquidus* is the temperature at which the metal turns from a solid and liquid mixture to completely liquid. The *eutectic* point is a specific composition where the metal transitions from completely solid to completely liquid.

The following figure is a phase diagram for a copper-silver alloy and depicts an example of the solidus and liquidus melting temperatures for various alloy compositions. Observe the eutectic point at approximate 28% copper. The phase diagram will be discussed more later on in the heat treatment topic.

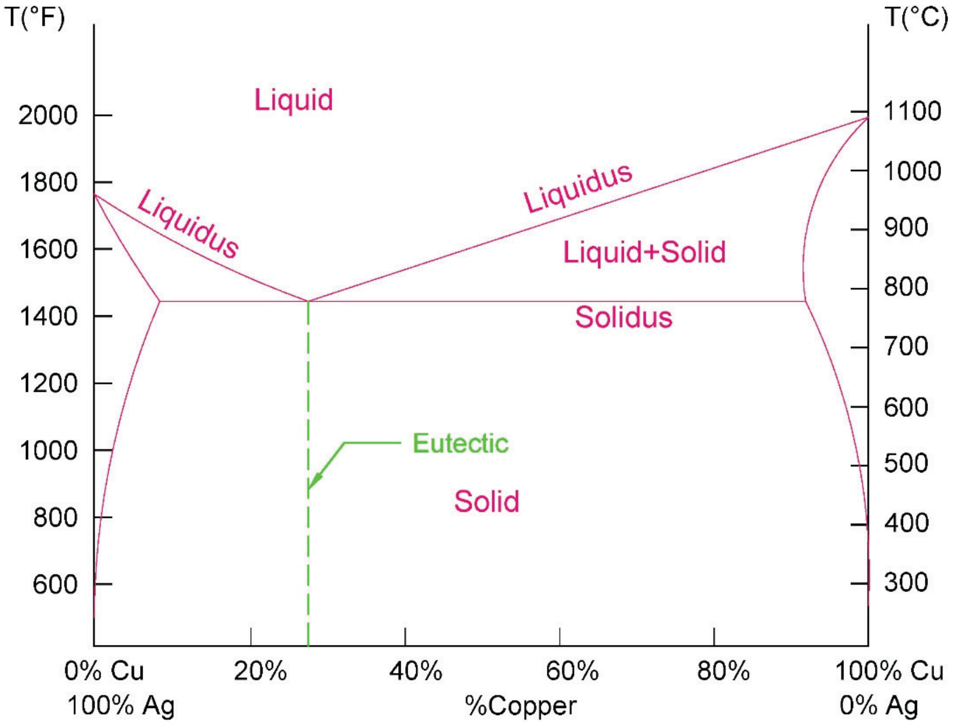


Figure 2: Phase diagram of Solidus and Liquidus Melting Points

The following table list melting points for common metals. Alloys have a solidus and liquidus melting point as described above.

Material	**Melting Point (ºF)**	
	Solidus	**Liquidus**
Aluminum Base		
Aluminum (Pure)	1220	
Aluminum 6061	1080	1200
Copper Base		
Copper (Pure)	1984	
Cupronickel, 30%	2032	2255
Commercial Bronze, 90%	1870	1910
Bronze (Aluminum, 9%)	1905	1915
Yellow Brass	1660	1710
Naval Brass	1630	1650
Nickel Base		
Nickel	2651	
Monel	2370	2460
Cast Iron/Steel Base		
Gray Cast Iron	2100-2200	

Ductile Iron	2100-2200	
Cast Steel, 3% Carbon	2640	
Stainless Steel		
304SS	2550	2650
316SS	2500	2550
Material	**Melting Point (ºF)**	
	Solidus	**Solidus**
Titanium		
Titanium (Pure)	3034	
Zinc		
Zinc (Pure)	787	
Other		
Water	32	

2.4.3 Optical

There are four major optical properties of a material: reflectance, transmittance, absorbance, and emissivity. When a beam of light hits a solid object, it is either reflected, transmitted or absorbed. The sum of the reflectance, transmittance, and absorption is equal to the incoming light, also known as *incident light*. Emissivity is how much light can be radiated by an object and is expressed as a ratio to a perfect black body.

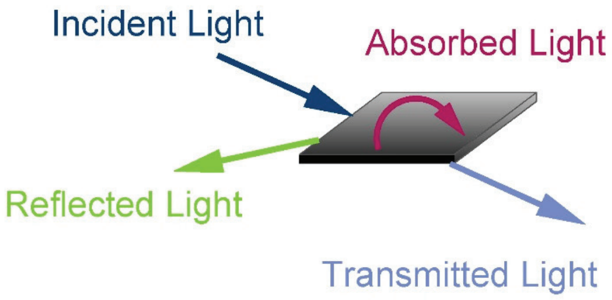


Figure 3: Reflected, Transmitted, and Absorbed Light

2.4.3.1 REFLECTANCE

Reflectance is the amount of light that is reflected back from the object. A mirror has high reflectivity, low transmissivity, and low absorptivity

2.4.3.2 TRANSMITTANCE

Transmittance is the amount of light that goes through the material. For example, glass has a high transmittance, but low reflectivity and low absorptivity.

2.4.3.3 ABSORBANCE

Absorptivity is the amount of light that gets absorbed into a material. Opaque and dark objects have higher absorption. Asphalt has a high absorptivity.

2.4.3.3 EMITTANCE

The incident light or energy that is absorbed into an object can then be emitted from it. Emissivity is how much thermal radiation in the form of light and infrared radiation is emitted from an object compared to a perfect black body. A perfect black body is an object that can absorb 100% of the incoming light, and also emit 100% of this absorbed energy.

2.5 THERMAL

2.5.1 Thermal Expansion

When heat is added to most materials, the average amplitude of the atoms' vibrating within the material increases. This, in turn, increases the separation between the atoms causing the material to expand. If the material does not go through a phase change, the expansion can be easily related to the temperature change. The linear coefficient of thermal expansion (α) describes the relative change in length of a material per degree temperature change. As shown in the following equation, α is the ratio of change in length (ΔL) to the total starting length (L_0) and change in temperature (ΔT).

When a pipe is heated or cooled, the length of the pipe will change based on the temperature and a coefficient of thermal expansion which is dependent on the material.

$$\Delta L = L_0 * \alpha * (T_1 - T_0)$$

$$where\ \Delta L = change\ in\ length\ of\ pipe\ (m\ or\ ft);\ L_0 = \ Initial\ Length\ (m\ or\ ft);$$

$$\alpha = thermal\ expansion\ coefficient\ (°C^{-1}\ or\ °F^{-1}); T_0 = Initial\ Temperature(°C\ or\ °F)$$

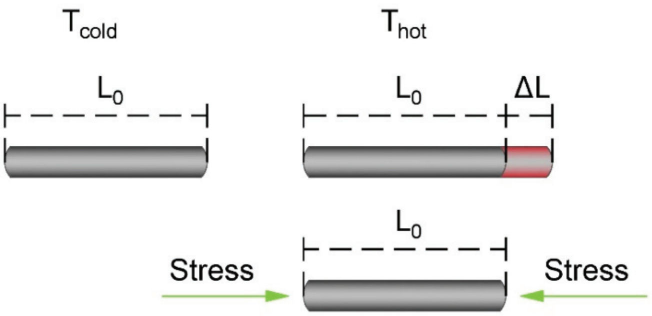


Figure 4: Increasing the temperature causes thermal expansion.

Bending stress: The bending stress of a pipe due to external forces or the weight of a pipe can be found using simple free body diagrams as shown in the following section.

2.5.2 Thermal Conductivity

Thermal conductivity is the ability of a material to conduct heat with a given temperature difference. Thermal conductivity is the material property that measures the rate of change in heat per unit distance per unit temperature difference.

$$q = k * \frac{dt}{dx}$$

$$q = heat\ flow\ per\ unit\ time\ per\ unit\ distance\ \left(\frac{W}{m}\ or\ \frac{Btu}{h-ft}\right)$$

$$k = thermal\ conductivity\ \left(\frac{W}{m-°K}\ or\ \frac{Btu}{h-ft-°F}\right)$$

More commonly on the FE exam you will use the following equation to conduct an energy balance as energy is transferred from one material to another material. The first step is to find the heat flow rate through an area of the subject material, as shown below.

$$Q = \frac{k}{x}\Delta T * Area; \quad where\ x = thickness\ (m\ or\ ft);\ \Delta T = temp.\ diff.; Q = total\ heat\ \frac{Btu}{h};$$

$$Q = UA\Delta T; \quad where\ U = heat\ transfer\ coefficient \frac{Btu}{h-ft^2}\ or\ \frac{W}{ft^2}; A = area; Q = total\ heat \frac{Btu}{h}$$

You will most likely be given the thermal conductivity on the FE exam, if there is an energy balance type equation. However, you should also be aware of the thermal conductivity values that are provided in the *NCEES FE Reference Handbook*. The following table is a list of some of the common materials and their thermal conductivity constants, just to give you some reference as to how the values of the materials presented in the *NCEES FE Reference Handbook* compare to other materials.

Material	*Thermal Conductivity (Btu/ft-hr-F)*
Aluminum	*118*
Air	*0.02*
Cast Iron	*33.5*
Calcium Silicate	*0.03*
Cement	*1*
Chromium	*54.3*
Concrete, lightweight	*0.15*
Copper	*232*
Carbon Steel	*31*
Fiberglass	*0.02*
Glass	*0.61*
Gypsum Board	*0.09*
Iron	*46*
Perlite	*0.18*
PVC	*0.11*
Stainless Steel	*9.25*

Timber	*0.09*
Water	*0.34*
Zinc	*67*

Based on the previous table you can see that metals have high thermal conductivities and plastics, gases, liquids and wood have low thermal conductivities. Insulators like calcium silicate and fiberglass also have low thermal conductivities.

One you have the thermal conductivity and the heat flow rate through a certain area, then you can calculate the total heat flow rate by multiplying by the area of the material.

2.5.3 Specific Heat Capacity

The specific heat describes the ease of a fluid or solid to increase in temperature when heat is applied. Specific heat is also known as heat capacitance and can be thought of as an object's ability to hold and gain heat. For solids and liquids, specific heat is shown as the variable, c_p.

$$heat\ capacity = c_p \left(\frac{Btu}{lbm\ °F}\right)$$

Water has a specific heat of 1.0 $\left(\frac{Btu}{lbm\ °F}\right)$, while aluminum has a specific heat of 0.23 $\left(\frac{Btu}{lbm\ °F}\right)$. As heat is added to water, water will increase in temperature at a slow rate. Since aluminum has a lower specific heat, it needs less energy to raise its temperature.

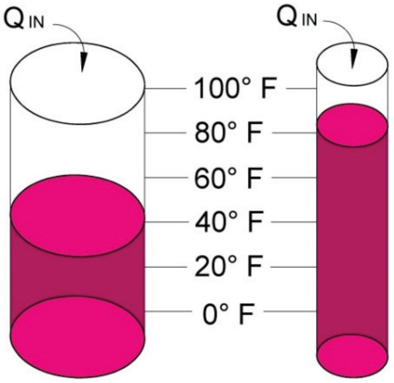


Figure 5: The heat capacity on the left is much larger, thus the same input of heat only raises the temperature by a small amount. The heat capacity on the right is smaller, thus the same amount of heat, greatly raises the temperature.

For gases, the heat capacity value is given in two different forms, one for a gas undergoing a constant-volume process and one for a gas undergoing a constant-pressure process.

$$constant\ volume = c_v$$

$$constant\ pressure = c_p$$

One of the most common specific heats used on the FE exam is air. You should have these values memorized.

$$c_{p,air} = 0.240 \frac{Btu}{lbm\ °R}$$

$$c_{v,air} = 0.171 \frac{Btu}{lbm\ °R}$$

If you encounter a different gas and need the specific heat value, then you should check the *NCEES FE Reference Handbook.*

3.0 STRESS-STRAIN DIAGRAMS

The tension and compression tests determine the strain at various stress levels. This results in the stress-strain diagram. A sample of the stress strain diagram is shown below.

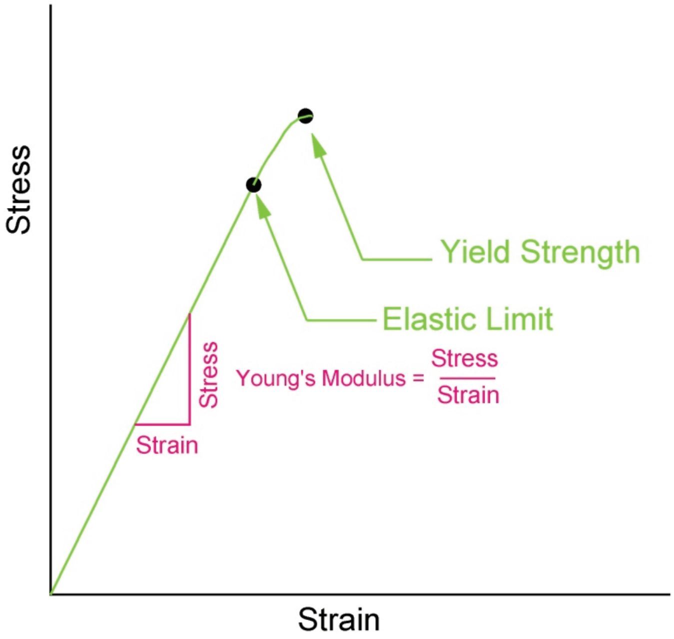


Figure 6: The stress-strain graph is produced through a series of test. A known force is applied to a known area and the strain is measured. The stress is found by dividing the force by the area. Once certain requirements like yield strength and ultimate strength are met, this stress level is recorded. This graph is for a typical brittle material.

3.1 YOUNG'S MODULUS

The modulus of elasticity, which is also known as the Young's Modulus, describes the slope of the stress to strain in the linear portion of the stress-strain graph. This modulus of elasticity has units of psi or MPA.

$$\sigma = E\varepsilon$$

$$\sigma = stress\ (psi);\ E = Young's\ modulus\ (psi);\ and\ \varepsilon = strain(dimensionless)$$

3.2 ELASTIC LIMIT

The elastic limit is the stress level at which permanent deformation will occur. This point is not often used due to the difficulty in measuring this point. Elastic limit is the first stress level when bonds are broken and the first point at which permanent deformation occurs. Since it is difficult to measure this exact stress point, yield strength is often used.

3.3 YIELD STRENGTH

The yield strength is similar to the elastic limit, but instead of any permanent deformation a specified permanent deformation must be reached in order to be called the yield strength. The typical specified permanent deformation amount is 0.2%. This value was used often in *Section 9.0 Mechanics of Materials*.

3.4 PROPORTIONAL LIMIT

The proportional limit is not used as often, but you should still be familiar with this point on the stress-strain graph. This point describes the point at which the stress-strain graph no longer moves in a straight line.

3.5 SHEAR MODULUS

Another stress-strain diagram is the shear stress-strain diagram. This diagram is constructed similar to the stress-strain diagram, except shear stress is applied to the material.

The shear modulus is given the variable, “G”. It describes the material’s ability to resist shearing. It also is the slope of the line on the shear stress-strain diagram, remember that this slope is only at the linear portion of the graph.

$$\tau = G\gamma$$

$$\tau = shear\ stress\ (psi); G = shear\ modulus\ (psi) and\ \gamma = shear\ strain(dimensionless)$$

3.6 POISSON’S RATIO

The Shear Modulus and Young’s Modulus described the relationship between an applied stress and the strain in either the plane normal to the surface (Young’s Modulus) or plane parallel to the surface (Shear Modulus). These two strains are related by the Poisson’s ratio. When a force is applied in the axial direction in either compression or tension, then there will be a relationship between the strain developed in the lateral and axial direction. Please see the following figures that show an object under tension and compression.

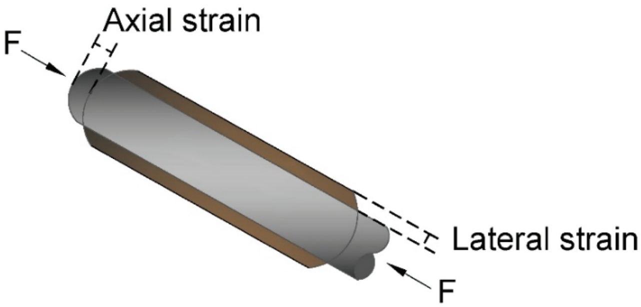


Figure 7: When an object is compressed the length of the object in the axial direction and the lateral direction will be changed. The gray colored object is the original object dimensions and the brown object is the new objected after it undergoes a compression force. Please remember the axial strain and lateral strain shown in the diagram is only ½ the total strain.

The relationship between the axial strain and lateral strain is a property of the material called the Poisson's ratio.

$$Poisson's\ ratio \rightarrow v = -\frac{\varepsilon_{lateral(radial)}}{\varepsilon_{axial(longitudinal)}}$$

$$v = Poisson's\ Ratio\ (dimensionless);\ \ \varepsilon_{lateral} = lateral\ (or\ radial)\ strain\ (m/m\ or\ ft/ft);$$
$$\varepsilon_{axial} = axial\ (or\ longitudinal)\ strain\ (m/m\ or\ ft/ft)$$

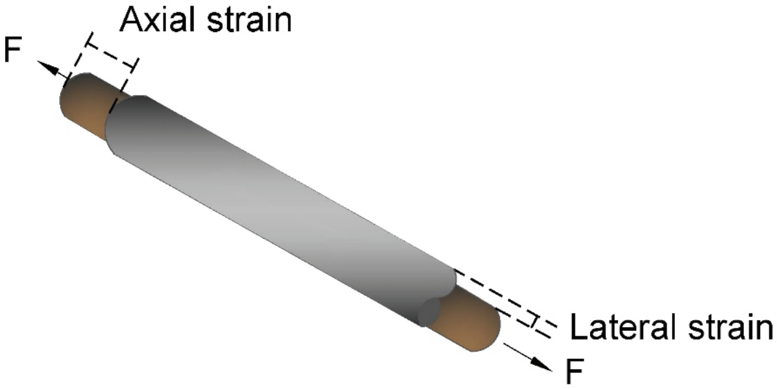


Figure 8: When an object undergoes tension the length of the object in the axial direction and the lateral direction will be changed. The gray colored object is the original object dimensions and the brown object is the new objected after it undergoes a tension force. Please remember the axial strain and lateral strain shown in the diagram is only ½ the total strain.

The Poisson's ratio is also related to the Young's Modulus and the Shear Modulus by the following equation.

$$G = \frac{E}{2 * (1 + v)}$$

$G = shear\ modulus\ (psi);\ E = Young's\ modulus\ (psi);\ v = Poisson's\ Ratio\ (dimensionless);$

3.7 ULTIMATE TENSILE STRENGTH

Ultimate tensile strength (UTS) is defined as the maximum amount of stress due to tension prior to breaking. Brittle materials and ductile materials both have ultimate strength values, but UTS occurs much closer to yield strength for brittle materials. Most often, yield strength can be approximated as the ultimate tensile strength. Ductile materials have UTS values that are much higher than their yield strength values. Once a ductile material yields, then it can still withstand stress until it breaks.

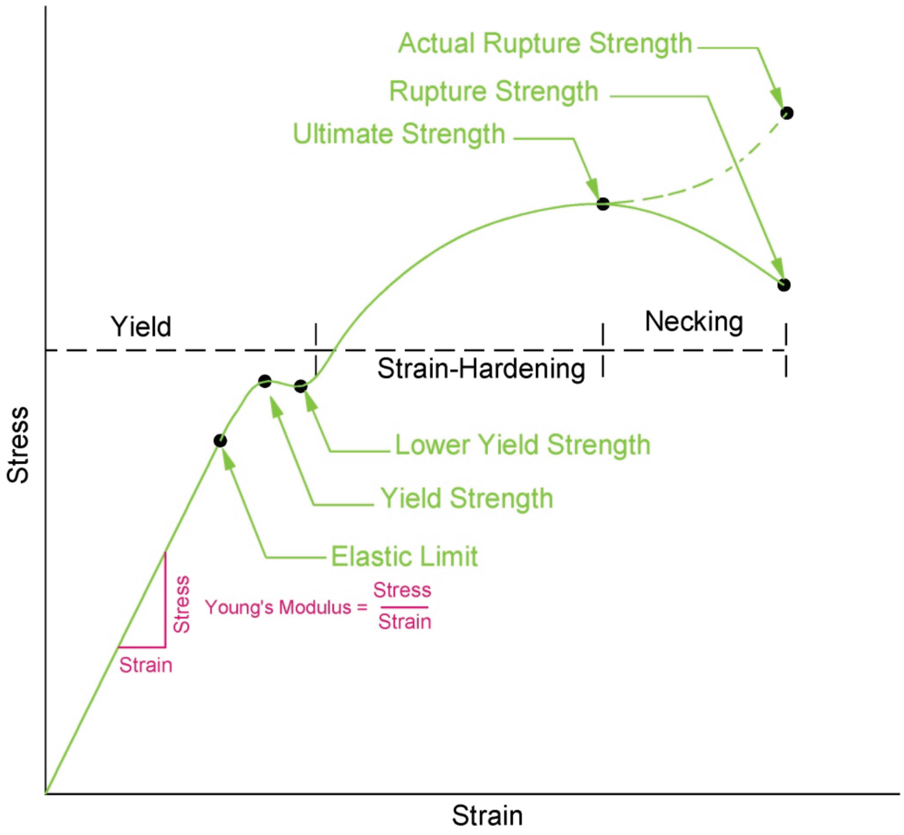


Figure 9: This graph shows the various material properties for a ductile material.

Rupture strength is shown on the previous figure. The difference between rupture strength and ultimate strength is that ultimate strength assumes that the stress applied is solely tensile. The individual fibers within the material are all subject to the same tensile stress. The individual fibers do not all have the same strength capabilities, thus the weakest fiber will fail at a lower stress level and this will determine the ultimate tensile stress.

Rupture strength is the complete breakage of the material, indicating that all the fibers are broken. In actuality this may occur at a large stress value, but in design it is assumed that the rupture strength is less than the ultimate strength because once the weak fibers have failed, the other fibers will have more stress and will fail as well.

4.0 FERROUS METALS

Ferrous metals contain iron (ferrite). These metals include steel, carbon steel, cast iron, etc. The iron makes these metals harder and much more durable. However, the iron also causes the metal to be more susceptible to corrosion. Ferrous metals are also magnetic. Ferrous metals are also brittle and are used in compression/tension. The brittle property means that the ferrous metal will fracture before it bends. Ferrous metals also have a high melting point, typically greater than 1000 °C. Ferrous metals are also have high thermal conductivities and electrical conductivity.

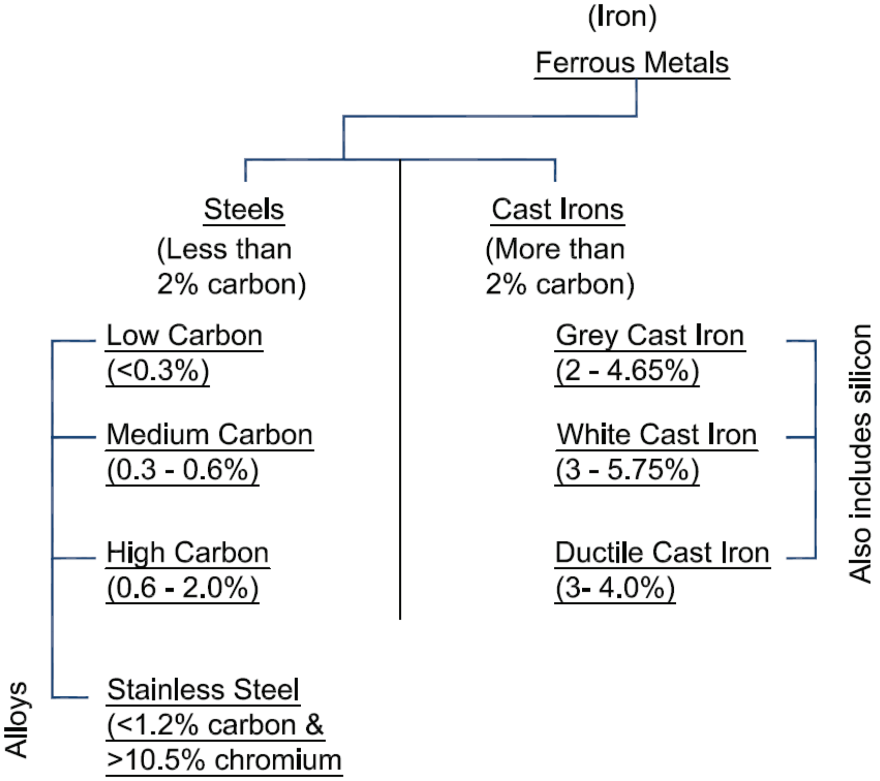

(Iron)
Ferrous Metals
Steels
(Less than
2% carbon)
Cast Irons
(More than
2% carbon)
Low Carbon
(<0.3%)
Medium Carbon
(0.3 - 0.6%)
High Carbon
(0.6 - 2.0%)
Stainless Steel
(<1.2% carbon &
>10.5% chromium
Alloys
Grey Cast Iron
(2 - 4.65%)
White Cast Iron
(3 - 5.75%)
Ductile Cast Iron
(3- 4.0%)
Also includes silicon

Figure 10: Ferrous metals' primary component is iron. There are two main types of ferrous metals, which are steels and cast irons. Low carbon, ferrous metals are called steels. Steel alloys like stainless steel are created by adding in chromium. Cast irons have higher carbon content.

5.0 NONFERROUS METALS

Nonferrous metals do not contain iron (ferrite). Common nonferrous metals include aluminum and copper. These metals are softer and less durable, but these metals are not as susceptible to corrosion.

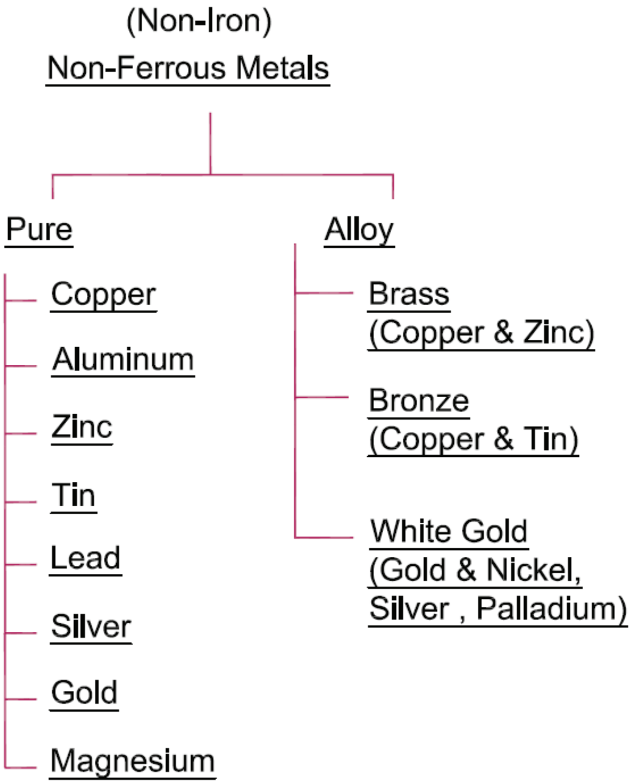

(Non-Iron)
Non-Ferrous Metals
Pure
Copper
Aluminum
Zinc
Tin
Lead
Silver
Gold
Magnesium
Alloy
Brass
(Copper & Zinc)
Bronze
(Copper & Tin)
White Gold
(Gold & Nickel,
Silver , Palladium)

Figure 11: There are two main types of non-ferrous metals, pure types and alloys. These metals are non-ferrous because they do not include iron. Pure non-ferrous metals are solely made up of a single element like copper, aluminum and zinc.

6.0 ENGINEERED MATERIALS

6.1 ALLOYS

Alloys are created to produce a material with improved characteristics for a specific application. Alloys consist of a mixture of metals and sometimes employ other elements. A common FE problem will be to determine the correct alloy for a desired requirement, like meeting a physical or chemical requirement, such as strength or corrosion resistance. Since steel alloys are the most common in the Machine Design & Materials field, you will most likely have a question on steel alloys with respect to their differing material properties.

6.1.1 Steel Alloys

Steel was created by combining iron, carbon, and other elements to create a strong, economical material. By varying the composition of metals, various types of steels with different hardness, toughness, corrosion resistance, and strength are produced.

For the FE exam you should understand the AISI/SAE steel coding system. The last two or three digits (right side) stand for the carbon content in 1/100%. Thus, 1040 steel has a carbon content of 40/100% or 0.40% carbon. The first two digits represent the allow category. There are many different categories but the main ones are 10_ _, where 10 represent plain carbon steel.

AB	Description
10XX	*Plain carbon steel*
13XX	*Manganese steel*
23XX	*Nickel steels*
31XX	*Nickel-Chromium steels*
40XX	*Molybdenum steels*
50XX	*Chromium steels*
61XX	*Chromium-vanadium steels*
72XX	*Tungsten chromium steels*
81XX	*Nickel-chromium-molybdenum steels*
92XX	*Silicon-manganese steels*

The second two digits represent the amount of carbon present in the steel.

	Description
XX40	0.40% carbon
XX10	0.10% carbon

Stainless steels are represented differently. Stainless steels are alloys of iron and chromium. Stainless steels have a minimum Chromium weight percentage of 10.5%.

AISI stands for American Iron and Steel Institute and SAE stands for the Society of Automotive Engineers.

7.0 MANUFACTURING PROCESSES

For the purposes of the FE exam, you should not need to be familiar with the details of the manufacturing processes as if you were in the manufacturing field. This is because the manufacturing processes category is vast and there is little information on manufacturing processes in the *NCEES FE Reference Handbook*. The three main Manufacturing Processes discussed in this book are, (1) Machining, (2) Molding and (3) Heat Treatment.

7.1 MACHINING

Machining is a broad term used to encompass any machine process that removes material from a component. The most common machining processes include turning, drilling and milling. Questions on the FE exam could be general questions that test your knowledge of the various machining processes and when they are used. However, there could be some quantitative questions that are shown in the list below.

1. Turning: Turning is used to reduce the diameter of a cylinder. The machine used to carry out this process is called a lathe. (The next discussion makes the assumption that you are familiar with a lathe.) One possible quantitative FE exam question on this topic could ask you to find the new diameter of the cylinder, based on the depth of the cut. The original diameter will be reduced by 2x the depth of the cut.

$$D_{new} = D_{old} - 2 * (depth\ of\ cut)$$

2. Boring: Boring is used to enlarge existing holes. A similar type of diameter question can be asked for boring as was asked for turning.

3. Drilling: Drilling is used to create a round hole.

4. Milling: Milling is the process where a machine component is fed past a rotating cutting tool. The tool removes material along the direction of the feed.

7.2 MOLDING

Molding is the process where metal is heated to liquid form and then poured into a mold. The liquid turns to a solid in the shape of the mold. Professional engineers rarely encounter the need to create molds and it is also less likely that you will need to personally oversee the mold creation and melting process.

7.3 HEAT TREATMENT

Heat treatment is discussed more in topic 9.0.

8.0 PHASE DIAGRAMS

A basic skill that you should know for the FE exam is how to read phase diagrams. A phase diagram graphically represents the state of a material at varying conditions (i.e. temperature, pressure, composition, etc.). For machine design, the most relevant type of phase diagram is the *binary phase diagram*. Binary phase diagrams are used to identify the properties of an alloy based on the temperature and composition of the material. The image below is an example of the simplest phase diagrams: Copper-Nickel.

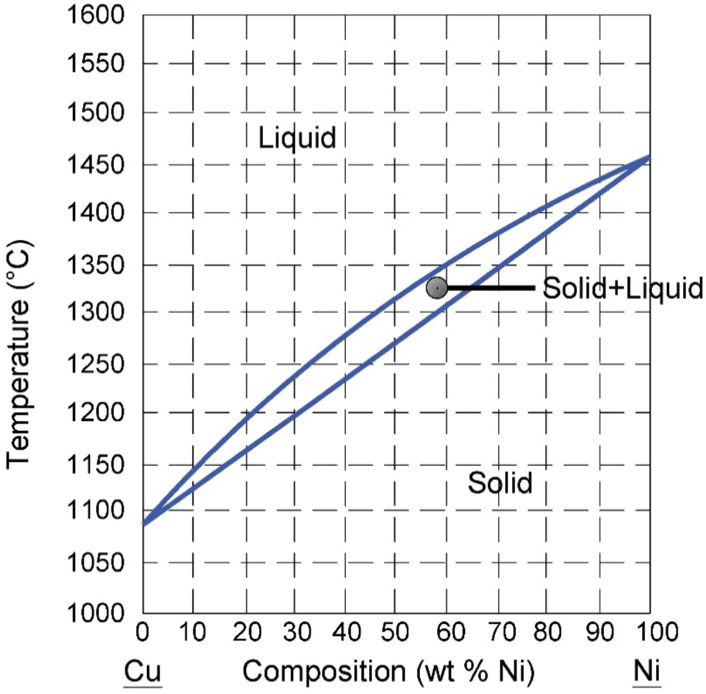


Figure 12: Copper nickel phase diagram

Coordinates: The graph plots the composition of two different metals, in this case the percentage of copper and the percentage of nickel, along the x-axis, against the temperature on the y-axis.

Boundary Line: Each line represents the boundary around a phase region.

Phase: Within each boundary is a unique phase of the metal. In this case, there are only three phases: liquid, solids, and a center region where both solid and liquid exist.

8.1 READING A PHASE DIAGRAM

Typically in manufacturing, the composition of an alloy is known (i.e. the x-axis is known) and you want to know how the alloy will change with temperature and will move vertically along the graph. Assume you have an alloy comprised of 60% nickel, melted to 2600°F. Locate this point, A, on the graph. Following this point down vertically, you can see that the solid + liquid region begins at approximately 1350°C and the alloy will transition to completely solid at approximately 1300°C.

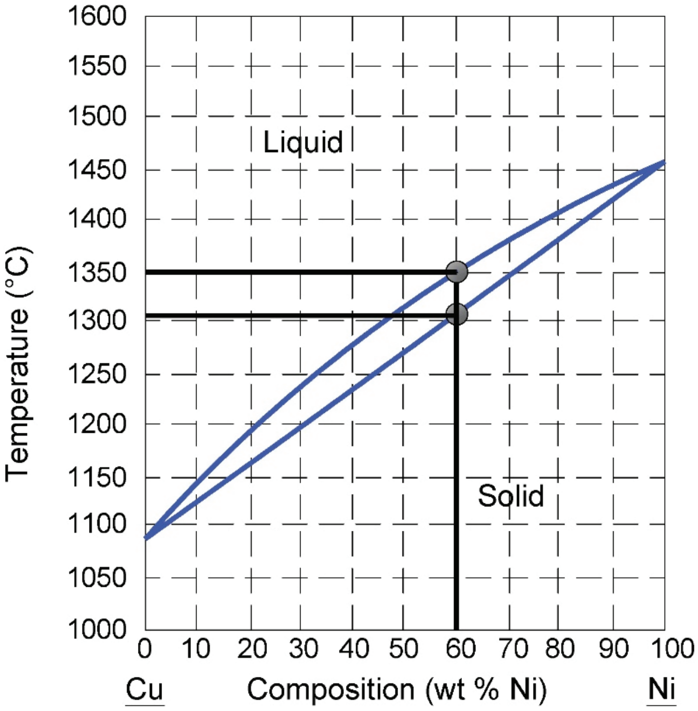


Figure 13: This figure shows the solid and liquid temperatures for a 60% nickel composition in a copper-nickel alloy.

8.2 LEVER RULE

The solid + liquid region contains a mixture of solid and liquid components. Assume you are located at a point "x" inside of this region, with coordinates, "C_x, T_x". The percentage of solid and the percentage of liquid can be determined with the lever rule. Conceptually you can see that the point is closer to the liquid region (left). Therefore, there will be more liquid components than solid components in the mixture.

To use the lever rule, draw a horizontal line through the point B, extending from one boundary line to the other. This is known as the *tie line*. Find the composition at each intersection point of the boundary line and at point "x". The composition shall either be in terms of Nickel or Copper, this does not matter; it just matters that you are consistent.

Finally calculate the percent weight of the liquid and solid.

$$\%Weight_{Liquid} = \frac{Composition_{solid} - Composition_x}{Composition_{solid} - Composition_{liquid}} x100\%$$

$$\%Weight_{Solid} = \frac{Composition_x - Composition_{liquid}}{Composition_{solid} - Composition_{liquid}} x100\%$$

$$100\% - \%Weight_{Liquid} = \%Weight_{Solid}$$

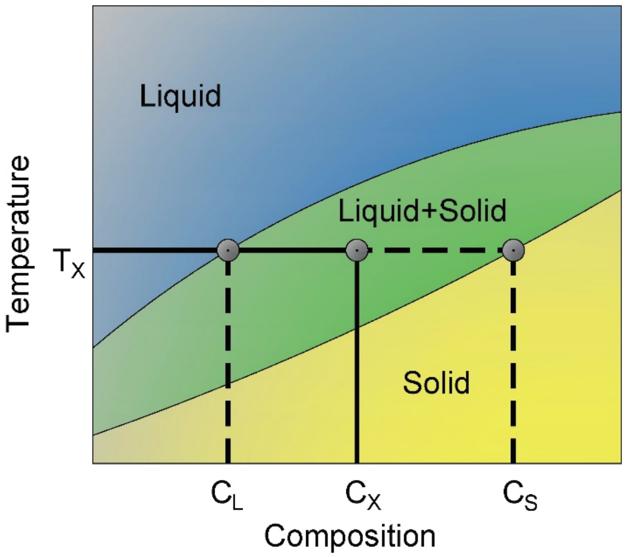


Figure 14: In the above figure the point C_x, T_x is comprised of both solid and liquid. The percentage of solid and liquid can be found by using the lever rule.

8.3 IRON-CARBIDE DIAGRAM

The most important binary phase diagram is the Iron-Carbide diagram, since this diagram represents the phases for the most common metals in manufacturing: steel and cast iron. The iron carbide diagram describes the alloy material based on the percent carbon (C) and iron (Fe) in the alloy. The temperature of the material will also affect the alloy material.

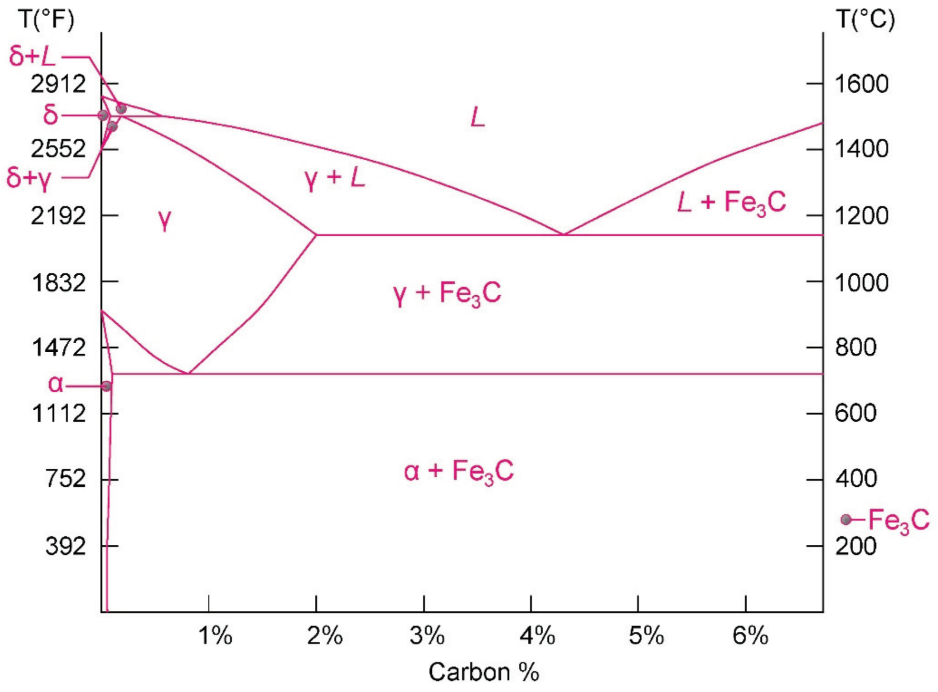


Figure 15: The above diagram shows the various phases and mixed regions as a function of the % carbon and temperature of the Carbon-Iron mixture.

The following table will define each area of the phase diagram.

Symbol	Symbol Name	Material Name	Notes
α	Alpha	Ferrite (solid)	Stable form at iron at room temperature
δ	Delta	Ferrite (solid)	Stable above 1,394 C.
γ	Gamma	Austenite (solid)	Not stable below 727 C.
Fe_3C	Iron-carbide	Cementite	
L		Fe-C Liquid	

Between the major phases are the mixed regions. In the mixed regions you must use the lever rule to find the percentage of ferrite, austenite (alpha or delta), cementite or liquid. In addition, each of the mixed regions has special names as shown in the following figure. However, you should not need to know each definition for the FE exam, since it would be very detailed.

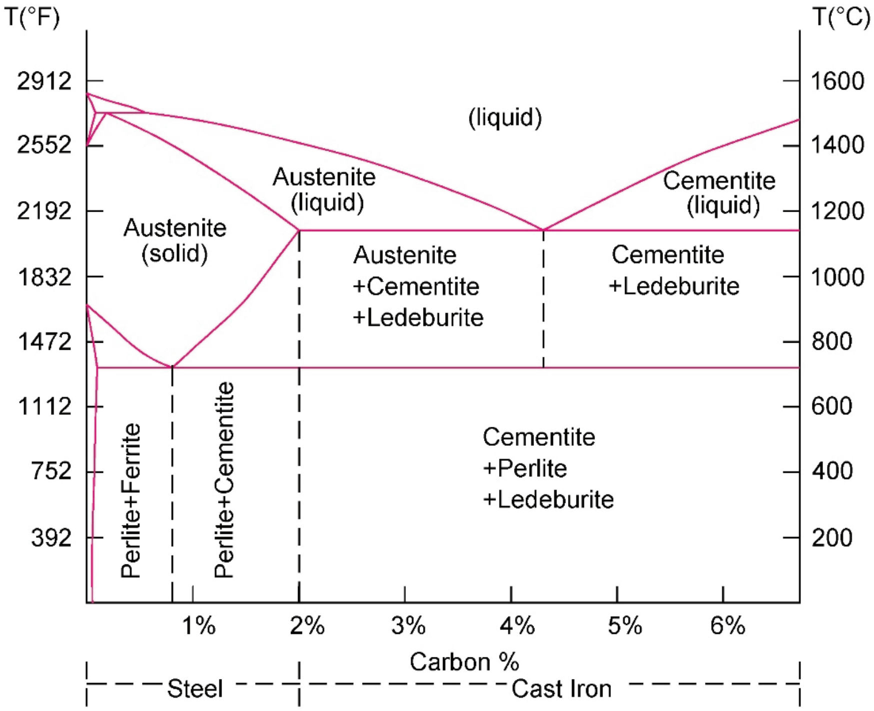

T(°F)
T(°C)
2912
2552
2192
1832
1472
1112
752
392
1600
1400
1200
1000
800
600
400
200
(liquid)
Austenite (liquid)
Cementite (liquid)
Austenite (solid)
Austenite +Cementite +Ledeburite
Cementite +Ledeburite
Perlite+Ferrite
Perlite+Cementite
Cementite +Perlite +Ledeburite
1%
2%
3%
4%
5%
6%
Carbon %
Steel
Cast Iron

Figure 16: Sample Iron-Carbide diagram

The graph above appears to show only two classifications of alloys, Cast Iron and Steel. Cast iron is classified as Iron-Carbon mixtures with around 2 to 7% carbon (the remaining percentage is iron). Steel is classified as having a carbon percentage between .01% to around 2%, with the remaining percentage as iron. There is a third classification at less than .01% carbon and it is simply iron, since nearly 100% of the alloy is iron.

9.0 PHASE TRANSFORMATION AND HEAT TREATING

This topic focuses on the process of heat treatment. Heat-treating is the process of heating metals to specific temperatures at specific rates in order to change the performance and material properties of the metal. This is done by changing the microstructure of the metal and/or the composition. The chemistry that takes place and the details in the microstructure is outside the scope of the FE exam. For the exam, you should be focused on the most popular heat treatment methods and should have a general concept of these processes. The major heat treatment processes are annealing, hardening, strengthening, tempering and quenching. These processes generally either make materials softer or harder, stronger or weaker, ductile or brittle.

9.1 ANNEALING

The process of annealing involves heating a material to a specific temperature and keeping the material at that temperature for a set time. Finally, the temperature is slowly reduced back to room temperature. The purpose of annealing is to change the metal to make it more ductile and workable for machining and bending.

For the purposes of the FE exam, it is important to know the below diagram. In the annealing process, steel is heated to a specific temperature suitable for the type of annealing.

<ins>Full Annealing</ins>: The metal is raised to about 50° C above the austenitic temperature and then slowly cooled by slowly reducing the temperature of the heat source. This is shown as the downward sloping curved band. This produces a coarse pearlite, which is described as a ductile material, but with a relatively lower yield and tensile strength.

<ins>Normalizing Annealing</ins>: Normalizing occurs at temperatures above the critical temperature. Then the temperature is held for a long period of time and then cooled in ambient air, which cools the metal at a much quicker rate. This produces a fine pearlite, which is described as hard and stronger material but with lower ductility.

<ins>Process Annealing</ins>: The metal is heated to a temperature just below the ferrite-austenite or austenite-cementite line. The temperature is held for a long time and then slowly cooled by slowly reducing the temperature of the heat source. This produces a soft and ductile material for machining. This is done to allow for the cold working of the material. Cold working is the bending and machining of the material at room temperature.

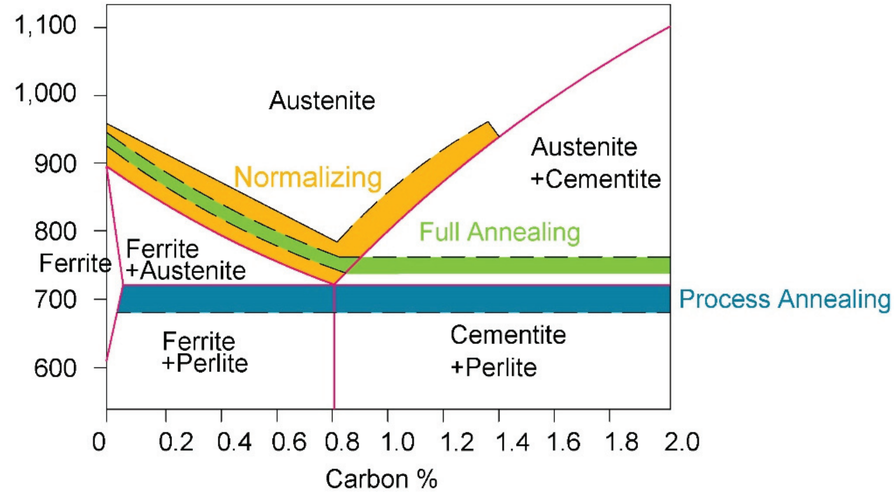


Figure 17: The three main types of annealing differ based on their heating temperatures and time to cool. Process annealing occurs below the austenite temperature critical point, as shown in blue. Normalizing occurs slightly above the transformation curve to austenite, as shown in orange. Full annealing occurs slightly above the transformation curve to austenite+cementite and austenite, as shown in green.

9.2 TIME TEMPERATURE TRANSFORMATION GRAPH

The previous figure showed an important graph of temperature vs. carbon %. This is another important graph that must be understood for heat treatment.

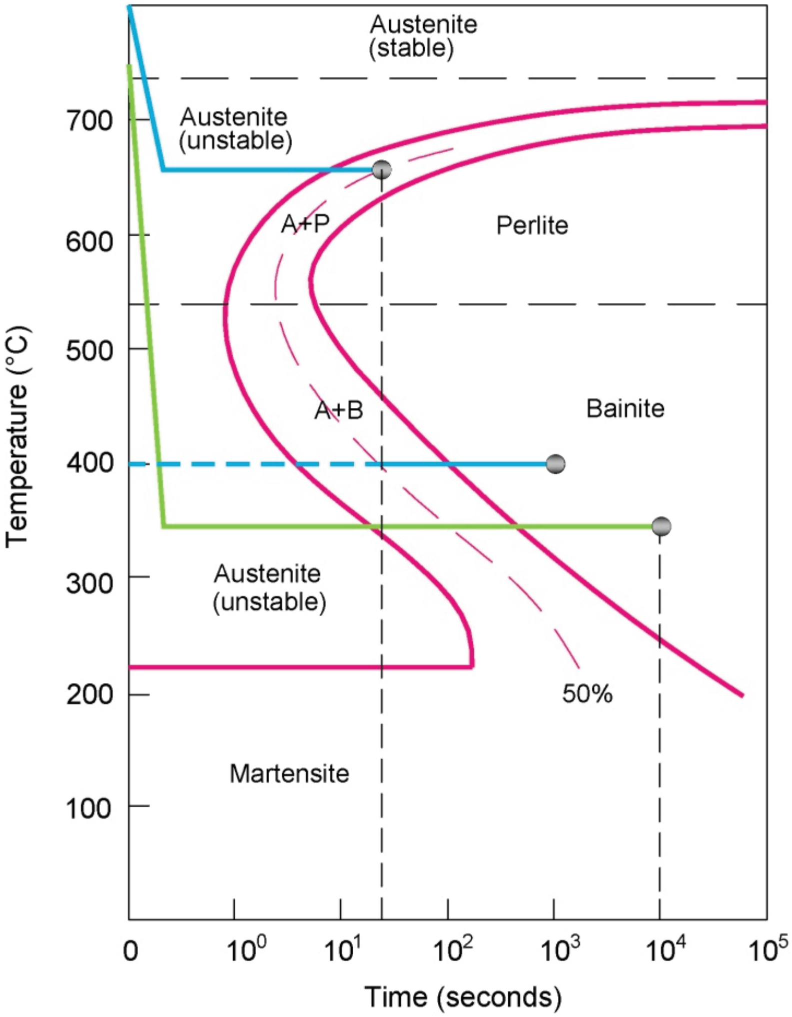


Figure 18: The time temperature transformation (TTT) graph shows how the material changes based on how long the material is held at a certain temperature. This graph is only true for a specific carbon percentage. Also be sure to recognize that the time axis is in a logarithmic scale.

On the graph, the term martensite is introduced. Martensite is achieved through rapid cooling, much faster than cooling in ambient air. This rapid cooling is called quenching, which will be discussed later. Martensite is a hard, brittle material with superior strength.

The previous figure showed two paths that will help you to understand how to read the TTT graph.

Green Path: The first path is in green. The metal is at a temperature of 800° C, where it is 100% austenite and stable. It is then rapidly cooled to about 350°C, as shown as the steep downward slope. At this point, it is still 100% austenite but it is now unstable. Then it is held at this temperature till 10^4 seconds. At this point, it is 100% bainite and when it is cooled back down to room temperature, it will be 100% bainite.

Blue Path: In this path, the metal is at a temperature of 800° C, where it is 100% austenite and stable. It is then rapidly cooled to about 650°C, as shown as the steep downward slope. At this point, it is still 100% austenite but it is now unstable. Then it is held at this temperature till about 20 seconds. At this point, it is 50% austenite and 50% perlite. Next, the metal is rapidly cooled down to 400°C. At this point, the 50% perlite is stable, but the 50% austenite is unstable. It is then held at this temperature till 10^3 seconds. The 50% unstable austenite is transformed into bainite, which results in 50% perlite and 50% bainite. When it is cooled back down to room temperature, the result will be the stable 50% perlite and 50% bainite.

On the FE exam, you should be able to navigate TTT graphs. These graphs also make it easier to understand the effect of time on the final material property. The next topics will give names to the various ways of cooling metal after it has been heated.

9.3 HARDENING & QUENCHING

The most common type of hardening is case hardening. This occurs when carbon diffuses into the metal, which causes an improved and harder material that is able to resist wear and tear. The term, "case" refers to the outer surface of the metal. The inside of the metal remains the same, but the outer surface is hardened.

Hardening is conducted by heating a metal to temperatures similar to the full annealing temperature, such that the pearlite is transformed into austenite. Then the heated austenite must be immediately and rapidly cooled through quenching. This can be seen on the TTT graph that was shown previously. Rapidly cooling through quenching will result in martensite.

In the next figure there is a solid blue line, which shows the hardening process that produces 100% martensite. However, if the quenching is stopped at 150 C and then held at that temperature, then after a sufficient time the material will change to 100% bainite. This is shown as the dashed blue line at the 150 C temperature.

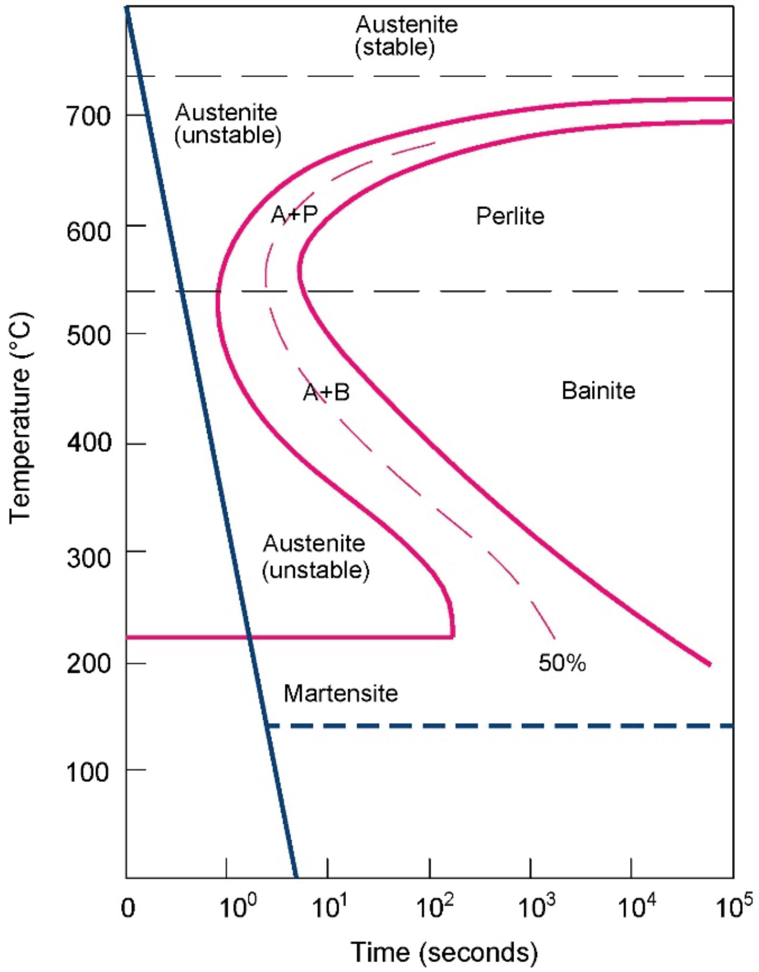


Figure 19: This figure shows quenching from 800 C down to room temperature as the blue solid line. This produces 100% martensite.

Martensite is very hard, strong and brittle. Hardening is typically used to create metals that are able to cut other metals. The following graph shows the hardness levels of martensite as a function of carbon percentage. In order to give a sense of scale, pearlite which is produced through annealing has a maximum Rockwell C hardness of around 20.

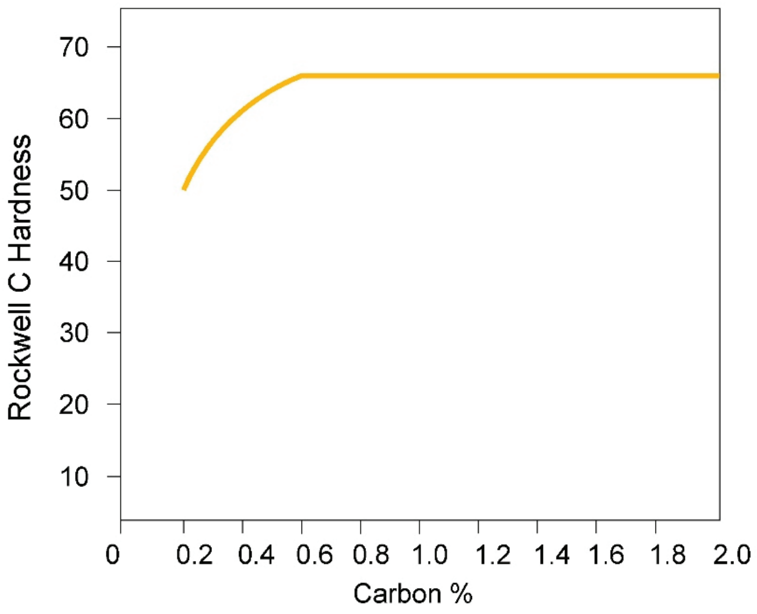


Figure 20: After hardening, martensite is created. This figure shows the hardness of martensite as a function of the carbon %.

9.4 TEMPERING

After a material has been hardened through the previous process, then the material can be tempered to increase its ductility and reduce its brittleness. Martensite is so brittle, that any impact will cause fracture. Tempering involves heating the steel to a temperature that corresponds to the hardness required. Tempering at lower temperatures will cause a lesser reduction in hardness, but will require a longer amount of time to achieve the ductility. Tempering at higher temperatures will cause a slightly higher reduction in hardness, but the tempering process will be finished at a faster rate.

10.0 MATERIALS SELECTION

For materials selection type questions, you should be familiar with the material properties presented throughout this section. You need to be able to determine which of the materials presented in the problem or in the *NCEES FE Reference Handbook* have the properties necessary to meet the requirements of the question.

11.0 CORROSION MECHANISMS AND CONTROL

Corrosion is the deterioration of a material over time. Corrosion is accelerated when exposed to hostile environments or in contact with certain materials. Being able to understand which metals are more or less susceptible to corrosion is important for selection of materials in varying machine design applications.

11.1 CORROSION RATES

The corrosion rate is determined by the thickness surface deterioration over time, typically measured in mils/year, where 1 mil = 0.0254 mm or one-thousandth of an inch (.001 in)

Corrosion rates are measured by finding the weight difference of a metal over time to calculate the material loss.

$$Corrosion\ Rate\ \left(\frac{mils}{year}\right) = \frac{1\ mil}{8.33e-5ft} * \frac{Weight\ (lbs)}{Density\ (\frac{lbs}{ft^3}) * Area\ (ft^2) * Time\ of\ Exposure\ (year)}$$

11.2 GALVANIC SERIES

The most common types of corrosion occur when metals are exposed to salt laden or acidic air. Corrosion can also be accelerated when dissimilar metals are in contact with each other. This is known as *galvanic corrosion*. The farther apart the materials are on the galvanic series chart, the more likely the metals are to exchange electrons between each other, causing corrosion.

The *galvanic series* tables rate which metals are more likely to undergo corrosion. It is measured based on a metals electric potential compared to a reference material. A metal that is anodic (a metal that loses electrons, i.e. negative voltage potential) is more vulnerable to corrosion, while a cathodic material (the metal that receives electrons, i.e. positive voltage potential) is more stable and is better at resisting corrosion. For example, steel and cast iron will corrode faster than titanium and stainless steel. Furthermore, 316 stainless steel resists corrosion better than 304 stainless steel.

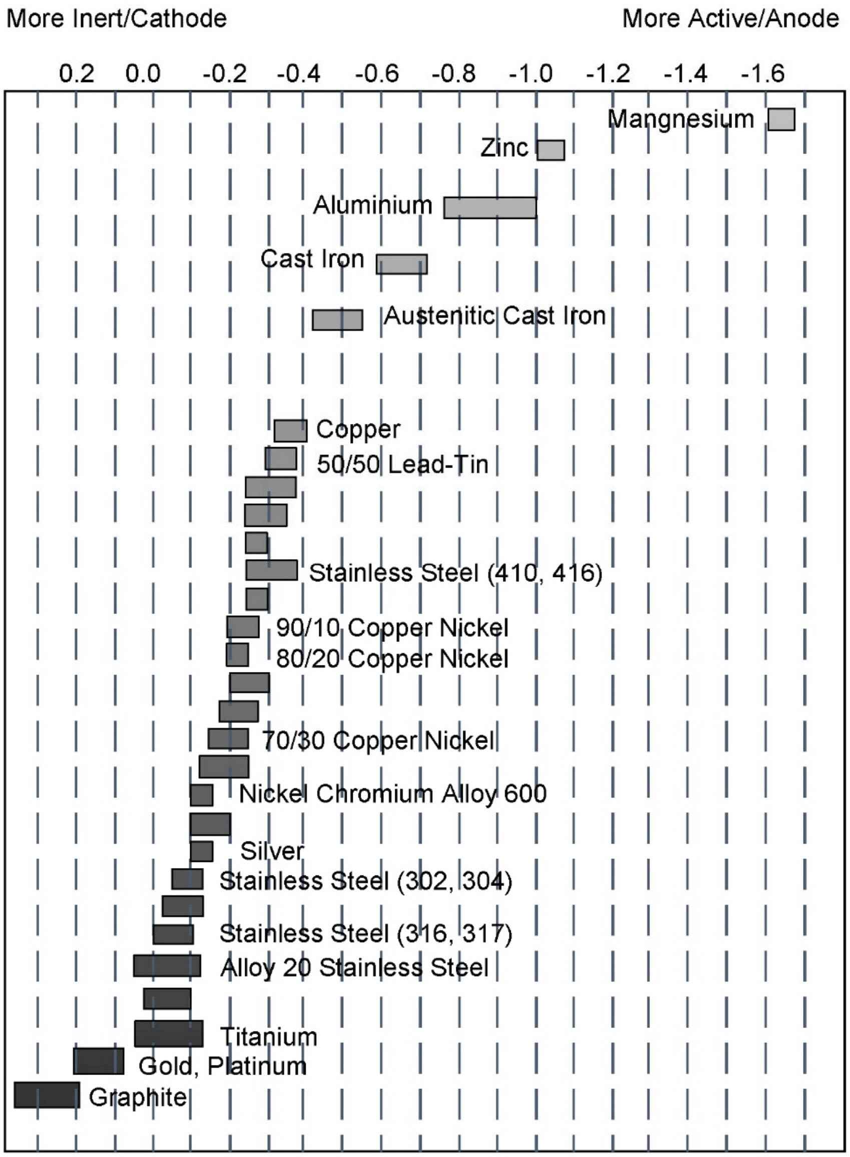


Figure 21: Galvanic Series in Salt Water Chart

11.3 CORROSION PROTECTION

Sometimes it is not feasible or cost effective to select a material that is resistant to the corrosive agents of your application. For example, a certain material strength is required and cannot be sacrificed by replacing one type of metal for another or an inert metal is much too costly for your project. If this is the case, then you must pursue other options of corrosion protection. Some of these methods are shown in the next few paragraphs.

11.3.1 Paints and Coatings

For metals exposed to harsh environments, paints or coatings can be applied. Common coating types include epoxy, phenolic, zinc, and polyurethane. The specific types of coatings are beyond the scope of this book. The application and testing methods for coatings are governed by ASTM, the National Association of Corrosion Engineers Standards (NACE), and the Society for Protective Coatings (SSPC).

11.3.2 Isolation Kits/Dielectric Unions

Galvanic corrosion between dissimilar metals is prevented by the following methods: (1) using metals that are similar in electrode potential on the galvanic series chart, (2) coatings, and (3) dielectric unions or isolation kits.

Isolation kits use gaskets and sleeves to isolate one metal from the other and prevent the electrode transfer that causes the galvanic corrosion.

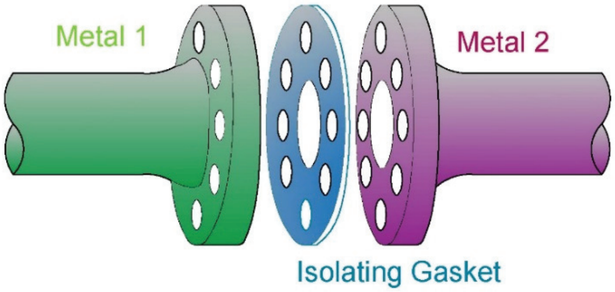


Figure 22: An isolation kit uses a non-conductive gasket to stop the electrochemical reaction.

Dielectric Unions act in a similar manner. A dielectric union is a fitting with two different metals on opposite ends and an internal material that isolates the dissimilar metals from each other.

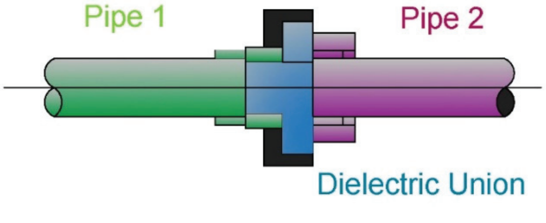


Figure 23: A dielectric union separates two connected pipes of dissimilar metals with a non-conductive washer shown in blue. Normally an electrochemically reaction will occur from the more noble metal to the less noble metal with the water serving as the path connecting the two metals. The dielectric union stops this reaction by physically separating the metals.

11.3.3 Cathodic Protection

Cathodic protection uses a sacrificial anode that has a more negative electrode potential than the metal it is protecting. The two metals are connected, typically by a wire, and ultimately the corrosion occurs at the sacrificial anode and not the protected metal. Typical sacrificial anode materials are zinc, aluminum, or magnesium.

Another method of cathodic protection is by *galvanization*. In this process, a metal is coated with zinc, typically by *hot dip galvanizing*, which submerges the metal into liquefied zinc. The protective coating then acts as the sacrificial anode to the metal beneath it. Even if part of the metal is exposed, the zinc will still act as the anode and protect the adjacent metal. It is important to note that galvanization does not protect against acidic corrosion.

11.4 OXIDATION

Another type of corrosion occurs when metals start to oxidize, this is what causes rusting of iron. Oxidation is a chemical or electrochemical reaction where a molecule loses electrons to another molecule. For the purpose of the exam, the focus is on the oxidation between metal and air. The metal oxidizes (loses electrons and corrodes) in the presence of air (the oxidizing agent that receives the electrons). It will be difficult to test this topic numerically, but you should be familiar with the overall concept of oxidation and how oxidation chemically occurs.

Each element has a typical oxidation state that represent number electrons transferred. See the table in the Chemistry chapter of the *NCEES FE Reference Handbook* for standard oxidation states.

A metal (M) with two valence electrons (divalent) will likely use all of its electrons and be combined with ½ oxygen (O_2) molecule to form a metal-oxide.

$$M + \frac{1}{2}O_2 \rightarrow MO$$

If the metal loses more electrons, then the same ratio will be used to complete the above reaction. For example, the following shows the reaction for aluminum.

$$2Al + \frac{3}{2}O_2 \rightarrow Al_2O_3$$

Aluminum has 3 electrons, thus the least common multiple is 6 electrons, which requires 2 aluminum molecules and 3 oxygen molecules. More examples of the oxidation reaction are shown in the table below.

Metal		*Oxygen*	*Reaction*
Aluminum	$2Al \rightarrow 2Al^{3+} + 6e^-$	$\frac{3}{2}O_2 + 6e^- \rightarrow 3O^{2-}$	$2Al + \frac{3}{2}O_2 \rightarrow Al_2O_3$
Chromium	$2Cr \rightarrow 2Cr^{3+} + 6e^-$	$\frac{3}{2}O_2 + 6e^- \rightarrow 3O^{2-}$	$2Cr + \frac{3}{2}O_2 \rightarrow Cr_2O_3$
Nickel	$Ni \rightarrow Ni^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Ni + \frac{1}{2}O_2 \rightarrow NiO$

Iron	$Fe \rightarrow Fe^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Fe + \frac{1}{2}O_2 \rightarrow FeO$
Beryllium	$Be \rightarrow Be^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Be + \frac{1}{2}O_2 \rightarrow BeO$
Cobalt	$Co \rightarrow Co^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Co + \frac{1}{2}O_2 \rightarrow CeO$
Manganese	$Mn \rightarrow Mn^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Mn + \frac{1}{2}O_2 \rightarrow MnO$
Silver	$Ag \rightarrow Ag^{2+} + 2e^-$	$\frac{1}{2}O_2 + 2e^- \rightarrow O^{2-}$	$Ag + \frac{1}{2}O_2 \rightarrow AgO$
Silicon	$Si \rightarrow Si^{4+} + 4e^-$	$\frac{2}{2}O_2 + 4e^- \rightarrow 2O^{2-}$	$Si + \frac{2}{2}O_2 \rightarrow SiO_2$
Titanium	$Ti \rightarrow Ti^{4+} + 4e^-$	$\frac{2}{2}O_2 + 4e^- \rightarrow 2O^{2-}$	$Ti + \frac{2}{2}O_2 \rightarrow TiO_2$
Molybdenum	$Mo \rightarrow Mo^{4+} + 4e^-$	$\frac{2}{2}O_2 + 4e^- \rightarrow 2O^{2-}$	$Mo + \frac{2}{2}O_2 \rightarrow MoO_2$

The following figure shows how a metal oxide film forms on the surface of a metal with 2 electrons.

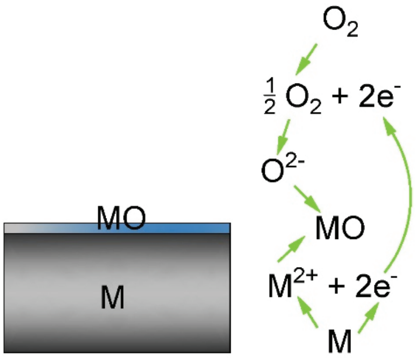


Figure 24: Oxidation occurs when electrons flow from the metal to the air. This causes a film to form on the top layer of the metal. This film is a metal-oxide.

11.4 BATTERY DISCHARGING

Corrosion can also be understood by learning about how a battery operates. You want to be able to identify the anode material and the cathode material. The anode material will be were oxidation occurs, which means that the anode will corrode.

A battery consists of a cathode and an anode that is electrically separated from each other. The electrical separation ensures that charge cannot be transferred between the cathode and anode, unless they are connected via a circuit. The separation between the cathode and electrode does allow for the transfer of ions between the cathode and anode. The cathode and anode are also immersed in an electrolyte. The electrolyte allows the transfer of charge within the battery. The circuit allows the transfer of charge on the exterior of the battery.

When a battery is fully charged, the anode has excess electrons and the cathode has a deficiency of electrons. This difference in charge creates an electric potential aka voltage. The potential difference will depend on the electrode materials. Electrons will want to flow from the anode (excess location) to the cathode (deficient location).

If the battery is not connected to an electric circuit, then there is no path for the electrons to flow, because the cathode and anode are electrically separated.

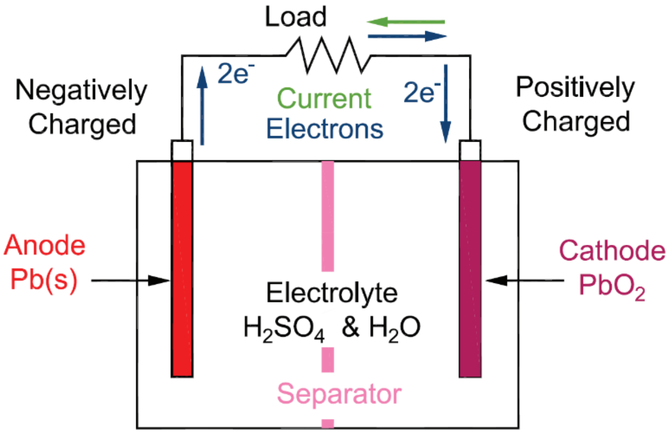


Figure 25: For this figure it is assumed that the battery is fully charged. Then the negatively charged anode and the positively charged cathode are connected via a circuit. Electrons will flow from the negatively charged anode to the positively charged cathode. Current is assigned a positive flow that is always opposite to the flow of electrons. Since electrons flow from negative to positive, then current flows from positive to negative.

In order to describe these steps, an example and figure shall be used below. Please remember that the steps occur simultaneously as opposed to sequentially.

Step 1: Oxidation is Loss (Anode & Circuit). The first step that will be discussed occurs at the anode and the circuit. The anode has an excess of electrons, so electrons will leave the anode and flow to the cathode. One typical anode material is lead (Pb). The lead in its solid form will lose electrons which will cause the following reaction. The lead will break down to an aqueous solution and the electrons will leave through the circuit. The loss of electrons is called oxidation.

$$Pb_{solid} \rightarrow Pb^{2+}_{aqueous} + 2e^{-}$$

Step 2: Oxidation is Loss (Anode & Electrolyte). The next step occurs between the anode and the electrolyte. The aqueous lead solution will react with the electrolyte. The electrolyte in this example is sulfuric acid (H_2SO_4). The sulfate ion SO_4^{2-} will react with the aqueous lead solution and cause the creation of a lead sulfate solid upon the anode.

$$Pb^{2+}_{aqueous} + SO_4^{2-} \rightarrow PbSO_{4\,(solid)}$$

Step 3: Reduction is Gain (Cathode & Circuit): On the other side of the battery, the cathode will receive the electrons. The protons from the electrolyte will reduce the lead oxide (cathode). The lead oxide will produce the same lead aqueous solution and water. This causes a gain in electrons, which decreases the difference in charge between the cathode and anode. This reaction of gaining electrons is called reduction.

$$PbO_{2\,(solid)} + 4H^{+} + 2e^{-} \rightarrow Pb^{2+}_{aqueous} + 2H_2O$$

Step 4: Reduction is Gain (Cathode & Electrolyte): The aqueous lead solution will react with the electrolyte, similar to the anode side. The sulfate ion SO_4^{2-} will react with the aqueous lead solution and cause the creation of a lead sulfate solid upon the anode.

$$Pb^{2+}_{aqueous} + SO_4^{2-} \rightarrow PbSO_{4\,(solid)}$$

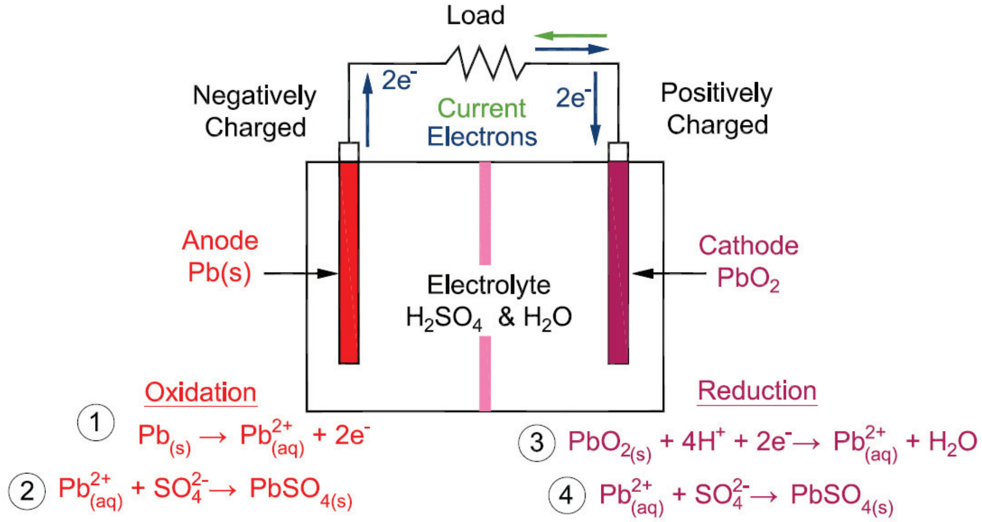

Load
2e-
Current
Electrons
2e-
Negatively Charged
Positively Charged
Anode Pb(s)
Cathode PbO2
Electrolyte H2SO4 & H2O
Oxidation
Reduction
① Pb(s) → Pb2+(aq) + 2e-
② Pb2+(aq) + SO42- → PbSO4(s)
③ PbO2(s) + 4H+ + 2e- → Pb2+(aq) + H2O
④ Pb2+(aq) + SO42- → PbSO4(s)

Figure 26: This figure shows the same previous figure but with a summary of the chemical reactions. Oxidation occurs at the anode, where electrons are lost. Reduction occurs at the cathode, where electrons are gained.

Another important thing to know about batteries is that corrosion occurs at the oxidation location. The anode will be corroded, similar to how a sacrificial anode is used to protect more noble metals. The anode will offer itself to be corroded so the other more important metal does not corrode.

The previous figures showed a lead/lead oxide battery, but there are many other batteries. The next table shall show the combined anode and cathode reactions and the corresponding electric potentials. The electric potentials show the tendency of the electrons to flow in the reduction direction. You will see that the reduction potentials are positive and the cathode potential half reactions are negative. The battery consists of both of those chemical reactions and in order to find the electric potential of the batter you must subtract the anode potential from the cathode potential. You must be sure that the potential is in terms of reduction.

$$Battery\ Potential \rightarrow Cathode\ potential_{reduction} - Anode\ potential_{reduction}$$

The anode potential in terms of reduction will be negative. If the anode potential is given in terms of oxidation then it will be positive.

Anode *(negative)* *(Oxidation is Loss)*	Cathode *(positive)* *(Reduction is Gain)*	Electric Potential (V)

Lead (-0.35 V) $Pb_{(s)} + SO_4^{2-} \rightarrow PbSO_{4\,(s)} + 2e^-$	Lead Oxide (1.46 V) $PbO_{2\,(s)} + 4H^+ + SO_4^{2-} + 2e^- \rightarrow PbSO_{4\,(s)} + 2H_2O$	1.46 – (-0.35) +1.81V
Lithium (-3.05 V) $Li_s \rightarrow Li^+ + e^-$	Silver Chloride (0.22 V) $AgCl_s + e^- \rightarrow Ag + Cl^-$	0.22 – (-3.05) +3.27V
Zinc (-1.28 V) $Zn_{(s)} + 2OH^- \rightarrow ZnO_{(s)} + H_2O + 2e^-$	Manganese Oxide (0.15 V) $2MnO_{2\,(s)} + H_2O + 2e^- \rightarrow Mn_2O_3 + 2OH^-$	0.15 – (-1.28) +1.43 V
Cadmium (-0.81 V) $Cd + 2OH^- \rightarrow Cd(OH)_2 + 2e^-$	Nickel (0.49 V) $NiO_2 + 2H_2O + 2e^- \rightarrow Ni(OH)_2 + 2OH^-$	0.49 – (-0.81) +1. 3 V

Example:

A zinc, silver-oxide battery has the following chemical reactions.

$$Ag_2O(s) + H_2O + 2e^- \rightarrow 2Ag(s) + 2(OH)_{aq} \quad 0.34\,V$$

$$Zn(s) + 2(OH)^-{}_{aq} \rightarrow ZnO(s) + H_2O + 2e^- \quad 1.25\,V$$

What material is the anode?

The anode is where the loss of electrons occurs. The anode is the Zinc.

What material is the cathode?

The cathode is where the gain of electrons occurs. The cathode is the Silver-Oxide.

What is the electric potential of reduction?

Reduction occurs at the cathode. So the electric potential of reduction is 0.34 V.

What is the electric potential of oxidation?

Oxidation occurs at the anode. So the electric potential of oxidation is 1.25 V.

What is the electric potential of the battery?

In order to find the electric potential of the battery, you need to subtract the electric potential of reduction at the anode from the electric potential of reduction at the cathode.

$$E_{anode,reduction} = -E_{anode,oxidation}$$

$$E_{anode,reduction} = -(1.25\,V) = -1.25\,V$$

$$E_{battery,reduction} = E_{cathode,reduction} - E_{anode,reduction} = 0.34\,V - (-1.25\,V) = 1.59\,V$$

Lastly, here is a summary of battery corrosion. Battery corrosion occurs at the oxidation location, which is the terminal that is anodic or less noble. Within a battery

Metal	Reaction (Anode perspective)	Electrode Potential (V)
Most Anodic (Top)		

Lithium	$Li \rightarrow Li^{+} + e^{-}$	-2.96
Potassium	$K \rightarrow K^{+} + e^{-}$	-2.92
Sodium	$Na \rightarrow Na^{+} + e^{-}$	-2.71
Magnesium	$Mg \rightarrow Mg^{2+} + 2e^{-}$	-2.36
Aluminum	$Al \rightarrow Al^{3+} + 3e^{-}$	-1.66
Zinc	$Zn \rightarrow Zn^{2+} + 2e^{-}$	-0.76
Chromium	$Cr \rightarrow Cr^{2+} + 2e^{-}$	-0.56
Iron	$Fe \rightarrow Fe^{2+} + 2e^{-}$	-0.44
Nickel	$Ni \rightarrow Ni^{2+} + 2e^{-}$	-0.25
Lead	$Pb \rightarrow Pb^{2+} + 2e^{-}$	-0.13
Hydrogen	$H_2 \rightarrow 2H^{+} + 2e^{-}$	0
Copper	$Cu \rightarrow Cu^{2+} + 2e^{-}$	+0.34
Iron	$Fe^{2} \rightarrow Fe^{3+} + e^{-}$	+0.77
Silver	$Ag \rightarrow Ag^{+} + e^{-}$	+0.80
Platinum	$Pt \rightarrow Pt^{4+} + 4e^{-}$	+1.20
Water	$2H_2O \rightarrow O^{2+} + 4H^{+} + 4e^{-}$	+1.23
Gold	$Au \rightarrow Au^{3+} + 3e^{-}$	+1.50
Most Cathodic (Bottom)		

When two metals are placed together, the more anodic metal will corrode.

12.0 FAILURE MECHANISMS

12.1 THERMAL FAILURE

Thermal failure can occur due to the material heating or cooling such that the material properties are outside of the requirements necessary to prevent failure. Another example of thermal failure occurs in the fatigue topic in *Section 8.0 Mechanics of Materials*. A material may repeatedly expand and contract due to thermal changes, which will eventually cause the material to fail due to fatigue.

12.1.1 Creep

Another concept that you should understand which is influenced by temperature is called Creep. The term creep strength, describes a materials ability to withstand a constant load. This term is similar to the fatigue term, which will be discussed later. Fatigue is concerned with repeating, on-off loads, while creep is concerned with steady, always on load. Creep strength is the stress level at which a material will fail at a certain time and certain temperature. This stress is typically less than the yield strength of the material. This is why it is important to understand creep when selecting a material to ensure that your design takes into account the long term effects of creep. You may select a material that meets the yield strength for the current design situation, but over time creep will occur and the material may fail. In practice, creep is taken into account by assigning a safety factor to the yield strength and in more sensitive situations, your design should include the lifetime of the material and the creep strength.

Creep strength is defined as the stress that will result in a certain value of elongation at certain time and temperature. Creep tests are conducted on the materials where a constant stress is applied at a certain temperature and the strain is measured over time.

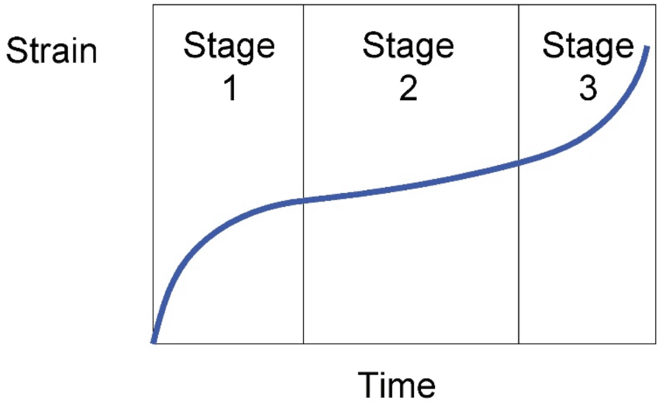


Figure 27: Creep strain over time occurs in three stages. The first stage is rapid acceleration of strain, followed by the second stage of slow and steady increase in strain. The final stage is another rapid acceleration of strain, followed by rupture.

The creep strain graph can also vary based on the stress applied to the material. If the stress is low, then the material will not reach stage 3, as shown in the following graph.

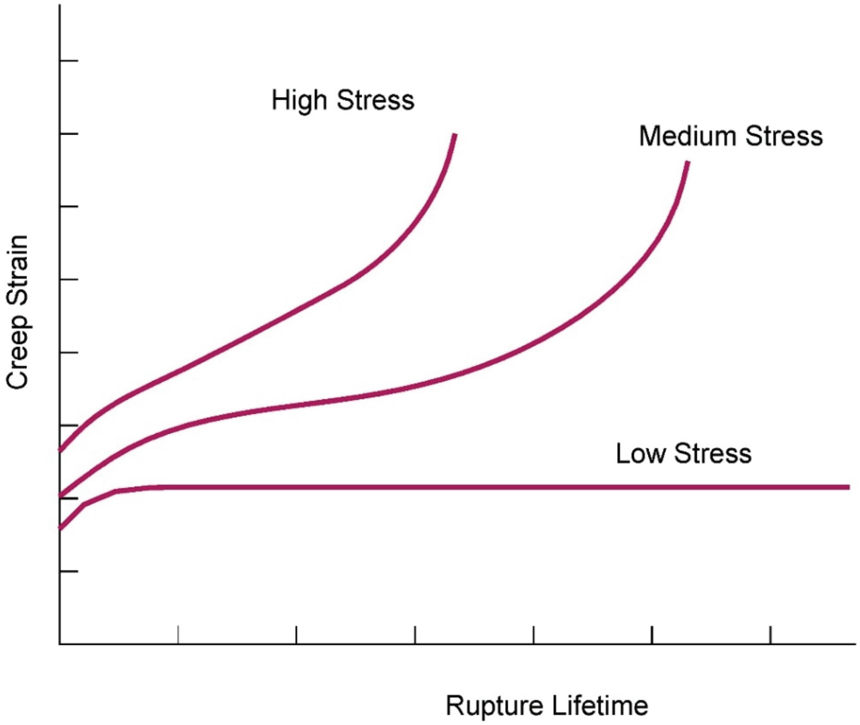


Figure 28: A material that is subject to low stress will experience an instantaneous amount of creep strain when the material is first loaded. Over time the creep strain will only increase slightly, until it finally ruptures at the end of the curve. A material that is subject to a high stress will rupture faster than a material that is subject to medium stress. A material subject to high

stress will have a high instantaneous creep strain and then will quickly reach the ends of its curve, where it will rupture. The medium stress curve is more typical of the three stages presented in the previous figure.

The previous figure also applies to varying temperatures similar to varying stresses. If the temperature is higher, then rupture will be achieved at a faster rate. If the temperature is lower, then the creep strain will only slightly increase and the lifetime of the material will be much greater than at higher temperatures.

12.2 DUCTILE OR BRITTLE FAILURE

Ductile materials will plastically deform as opposed to break when subject to stress. Brittle material will not deform, nor will it absorb any energy prior to breaking. Brittle material will also undergo less strain prior to breaking.

12.3 FATIGUE

The stress, strain, shear, bending, buckling and torsion topics discussed the strength of materials based on a maximum one-time load. However, the strength of materials is not only characterized by this maximum loading but also by the maximum cycles of repeated loading. Fatigue describes the relationship between cycles of loading and the magnitude of the loading. A small load with a small number of cycles of applying this load will not fatigue an object but a large load with a large number of cycles of applying this load will fatigue an object.

This repeatedly applied load will cause the object to fail at typically a smaller value than the ultimate tensile strength or yield point, which was discussed earlier. This is why the study of fatigue in Machine Design and Materials is important. Fatigue typically occurs when objects are subject to cyclic loading throughout its life, where the stress on the object varies in a sinusoidal waveform from a maximum level in one direction and then a maximum level in the opposite direction.

The sinusoidal shape is not as important as is the maximum and minimum values of the shape. These maximum and minimum values determine the alternating stress and mean stress values. Alternating stress is defined as the difference between the maximum and minimum stress values acting upon the component. The mean stress is the average value of the stress values acting upon the component.

$$\sigma_{mean} = \frac{\sigma_{maximum} + \sigma_{minimum}}{2}$$

$$\sigma_{alternating} = \left|\frac{\sigma_{maximum} - \sigma_{minimum}}{2}\right|$$

These two values will be used to construct the Goodman diagram in the next topics.

12.3.1 Goodman Diagram

The Goodman diagram is a graph of the mean stress vs. the alternating stress. The area under curve represents the combination of static stress and alternating stress that will not result in failure. The area outside of the curve represents combinations that will most likely result in failure of the material. The stresses shown on this Goodman Diagram are shown for stresses in the tensile direction.

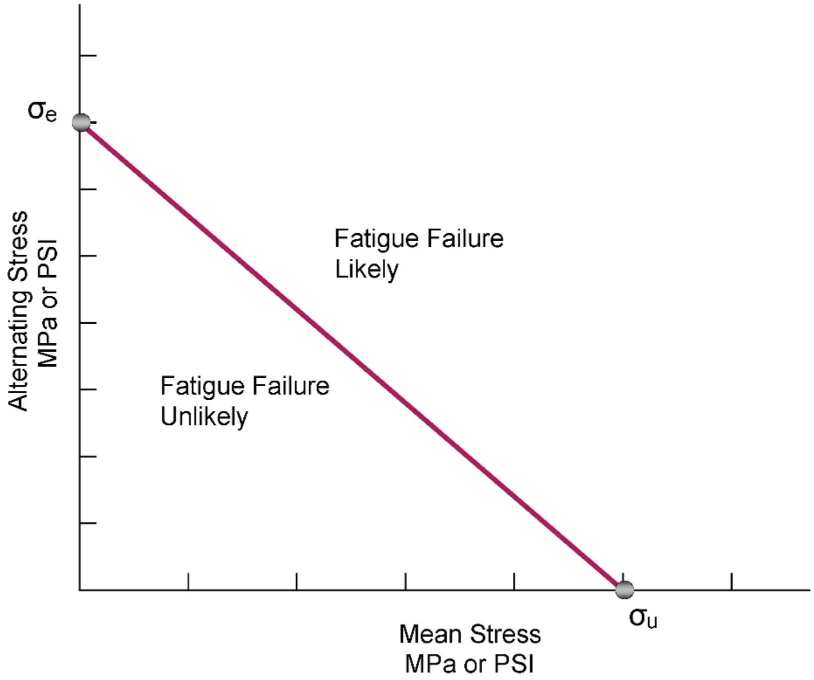


Figure 29: This figure shows the Goodman diagram and the locations of likely and unlikely failure. The diagram indicates that failure can still occur when the stress is below the ultimate tensile stress due to alternating stresses.

The equation for the line is shown below. The maximum mean stress value is equal to the ultimate tensile strength value, only when the alternating stress is equal to zero. This situation occurs when the maximum and minimum values are the same.

$$\sigma_{alternating} = \sigma_{fatigue\ endurance} * (1 - \frac{\sigma_{mean}}{\sigma_{ultimate\ tensile\ strength}})$$

$$Maximum\ mean\ stress\ occurs\ when \rightarrow \sigma_{maximum} = \sigma_{minimum}$$

The maximum alternating stress is equal to the fatigue endurance stress. This value occurs when the mean stress is equal to zero. The maximum alternating stress occurs when maximum stress is equal and opposite to the minimum stress.

$$Maximum\ alternating\ stress\ occurs\ when \rightarrow \sigma_{maximum} = -\sigma_{minimum}$$

The endurance or fatigue stress is determined through a universal tester that causes repeated stress to a material until the material fails. Varying levels of stress are tested and the number of cycles is counted until failure. A graph is then used to show the stress level and its number of cycles until failure. The endurance or fatigue stress occurs when the plot levels out and becomes horizontal.

There are other diagrams similar with different lines/arcs that separate the conditions for fatigue failure. Some of these diagrams include the modified Goodman diagram, ASME elliptic

12.4 CRACK PROPAGATION

Crack propagation mainly deals with a subset of Fracture Mechanics. Fracture mechanics is a vast topic that covers the fact that a lot of failures of mechanical components do not result from a complete break, but from a fracture where a single crack or break in continuity of a component occurs. Once a crack or fracture appears, the strength of the component at that location becomes compromised. A component may have a certain strength value, but at that location the strength becomes much weaker and the stress that can cause failure is much less.

Cracks and fractures appear regularly in components and there was a need to be able to quantify and predict failure based on a fracture or crack. There are a couple of methods that are used to quantify the effect of cracks on the failure of a component, (1) the energy balance approach and (2) the stress intensity approach. The energy balance approach is not covered in this book, because the equations for this approach are not shown in the *NCEES FE Reference Handbook*.

The stress intensity approach is based on three types of stresses applied at various configurations to the crack.

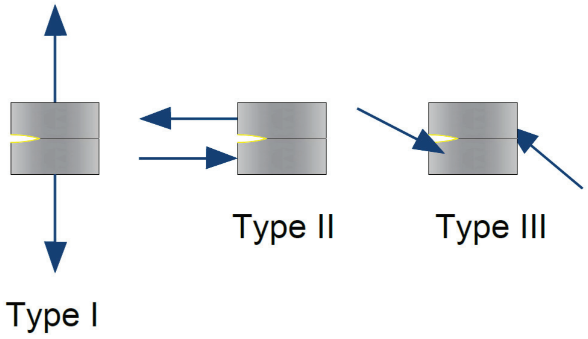


Figure 30: The main stress configuration that you should know is Type I, which is a tension stress that pulls apart the material relative to the crack. Type II and III apply shear stresses relative to the crack.

The stress intensity approach defines a stress intensity factor called K_{IC}. This factor is created from a series of tests on materials in certain stress configurations for various cracks. As previously stated you only need to know Type I. The main crack configurations that you need to know are an edge crack and a crack within a material that can be assumed infinitely long. These configurations result in the following equations for the stress intensity factor.

$$Edge\ Crack \rightarrow K_{IC} = 1.1\sigma_f\sqrt{\pi a}$$

$$a = crack\ length$$

$$Center\ Crack \rightarrow K_{IC} = \sigma_f\sqrt{\pi a}$$

$$2a = crack\ length$$

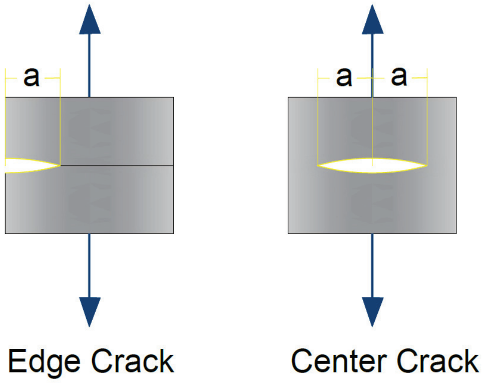


Figure 31: The two main crack configurations that you must know are the edge crack and center crack.

The failure stress can be found by rearranging the stress intensity factor equation.

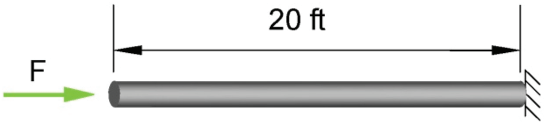
$$Edge\ Crack\ Failure\ Occurs\ when \rightarrow \sigma_f = \frac{K_{IC}}{1.1\sqrt{\pi a}}; a = crack\ length$$

$$Center\ Crack\ Failure\ Occurs\ when \rightarrow \sigma_f = \frac{K_{IC}}{\sqrt{\pi a}}; 2a = crack\ length$$

13.0 PRACTICE EXAM PROBLEMS

13.1 PRACTICE PROBLEM 1 – THERMAL STRESS

A steel rod with a diameter of 3” and at an initial temperature of 75 F is heated up to 180 F. What force must be applied to constrain the rod? $\alpha_{steel} = 6.8\ x\ 10^{-6}\ [\frac{in}{in-F}]$ and the modulus of elasticity equals 30 x 10^6. psi



(a) 91 kips

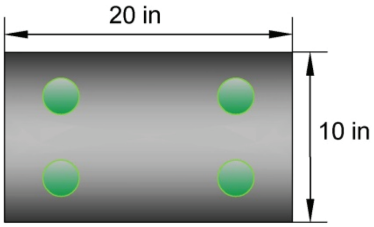
(b) 151 kips

(c) 215 kips

() 276 kips

13.2 PRACTICE PROBLEM 2 – MODULUS OF ELASTICITY

A post consists of 4 identical steel rods of an unknown diameter. The post is filled with concrete and has dimensions as shown below with a height of 10 ft. There is a force of 250 lbs acting upon the post. The modulus of elasticity of the steel is 30 x 10^6 psi and the modulus of elasticity of the concrete is 3 x 10^6 psi. The external force is shared equally between the 4 steel rods and the concrete. What is the diameter of the steel rods?



(a) 2.4 in

(b) 3.1 in

(c) 4.5 in

(d) 6 in

13.3 PRACTICE PROBLEM 3 – THERMAL CONDUCTIVITY

An aluminum plate (k = 120 Btu/h-ft-F) of thickness 3” and an area of 5 square ft separates temperatures of 200 °F and 75 °F. What is the total heat loss per hour?

(a) 250 MBtu/h

(b) 300 MBtu/h

(c) 400 MBtu/h

(d) 505 MBtu/h

13.4 PRACTICE PROBLEM 4 – STRESS-STRAIN

A 6’ long cylindrical rod of unknown diameter must be able to hold 100,000 lbf without elongating more than 2”. The rod is made of a material with a modulus of elasticity of 15 x 10^6 psi. What is the minimum diameter of the rod to meet the maximum elongation requirement?

(a) 0.08 in

(b) 0.17 in

(c) 0.25 in

(d) 0.55 in

13.5 PRACTICE PROBLEM 5 – STRESS-STRAIN

A cylindrical rod has a diameter of 0.5 inches. This rod is stressed in tension with a force of 3,500 lbf. This causes the diameter of the rod to be reduced to 0.4998 inches. The rod is made of a material with a modulus of elasticity of 15 x 10^6 psi. What is Poisson's ratio of this material?

(a) 0.08

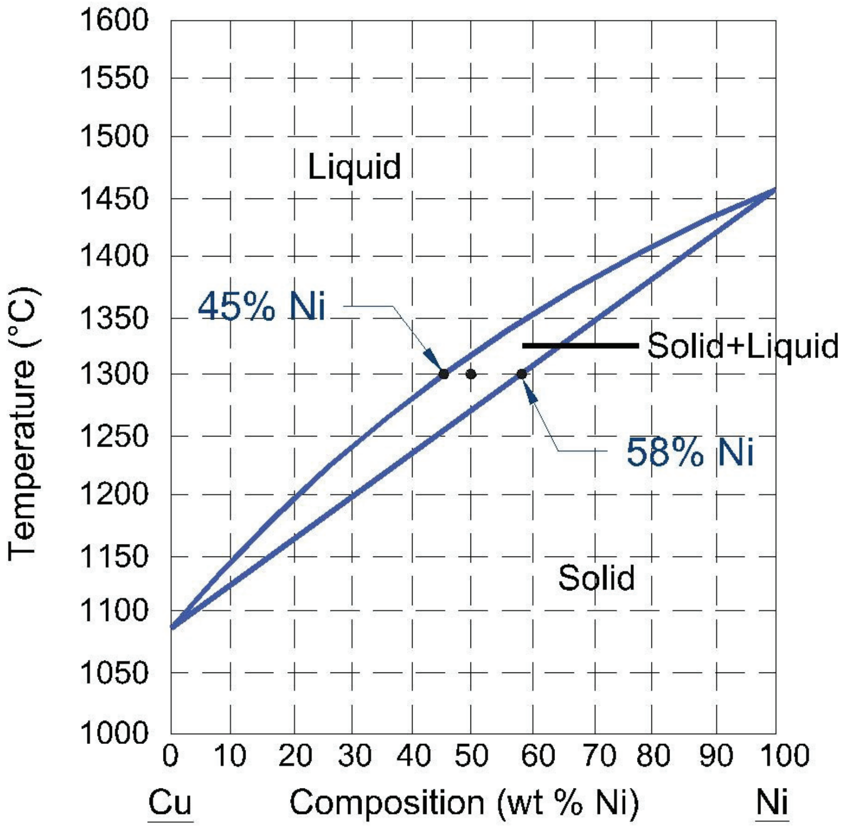
(b) 0.21

(c) 0.29

(d) 0.34

13.6 PRACTICE PROBLEM 6 – ALLOYS

A copper-nickel alloy has 50% nickel and exists as a solid-liquid mixture at 1300°C? What is the percentage by weight of solid at this temperature?



(a) 38%

(b) 50%

(c) 62%

(d) 63%

13.7 PRACTICE PROBLEM 7 – FATIGUE

A material undergoes a cyclical stress with a tensile stress of 50 MPa and a compressive stress of 10 MPa. What is the stress amplitude?

(a) 30 MPa

(b) 40 MPa

(c) 50 MPa

(d) 60 MPa

13.8 PRACTICE PROBLEM 8 – FATIGUE

A rod undergoes a cyclical force, with a tensile force of 10,000 lbs and a compressive force of 50,000 lbs. What is the minimum diameter to ensure that the mean stress is less than 1,600 psi?

(a) 1.4 in

(b) 2.7 in

(c) 4.1 in

(d) 6.0 in

13.9 PRACTICE PROBLEM 9 – FATIGUE

A beam is loaded with a compressive stress of 100 MPa. Upon initial loading, the beam undergoes a strain of 0.002 mm/mm. At an average temperature of 27 C, the creep rate is 10 x 10^{-7}/hr. What is the total change in length of the 1 meter beam after 365 days? Assume the compressive stress is constant for the entire time.

(a) 2 mm

(b) 5 mm

(c) 8 mm

(d) 11 mm

13.10 PRACTICE PROBLEM 10 – POISSON'S RATIO

A rod is subject to a tensile stress of 12 x 10^6 psi. The modulus of elasticity of the material is 50,000 ksi. The diameter of the rod is reduced by 1 inches. The diameter of the rod is originally 9 inches. What is the Poisson's Ratio?

(a) -0.23

(b) 0.003

(c) 0.16

(d) 0.46

13.11 PRACTICE PROBLEM 11 – CRACK PROPAGATION

Which of the following methods does not decrease the risk of crack propagation?

(a) Polishing

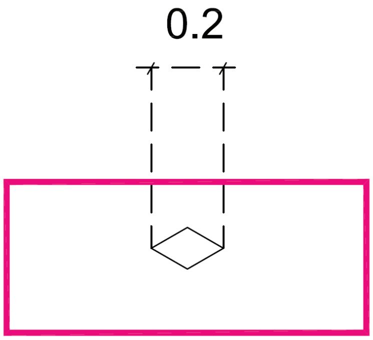
(b) Surface hardening

(c) Reducing stress acting upon the component

(d) Protecting the component from impact.

13.12 PRACTICE PROBLEM 12 – CRACK PROPAGATION

The fracture toughness of a material is 20 ksi-in$^{0.5}$ and there is an existing center crack of 0.2 in as shown in the figure below. What is the maximum amount of stress that can be applied before crack propagation occurs?



(a) 20 ksi

(b) 36 ksi

(c) 142 ksi

(d) 229 ksi

14.0 SOLUTIONS

14.1 SOLUTION 1 – THERMAL STRESS

If the rod is unconstrained, then the expansion distance will be found with the below equation.

$$\Delta l_{temp} = \alpha_{steel} * \Delta T * L$$

$$\Delta l_{temp} = 6.8\ x\ 10^{-6}\left[\frac{in}{in-F}\right] * (180\ f - 75\ f) * 20ft * (12\ in/1ft)$$

$$\Delta l_{temp} = 0.1714\ in$$

However, the rod is constrained by a force, F, and the force can be found with the following equation. The force required to offset the thermal expansion will be equal to the force that compresses the rod.

$$\Delta l_{temp} = \Delta l_{force} = \frac{FL}{AE}$$

$$0.1714\ in = \frac{F * (20\ ft * \frac{12in}{ft})}{\frac{\pi}{4}3^2 * 30x10^6 psi}$$

$$F = 151{,}444\ lbs$$

The correct answer is most nearly, **(b) 151 kips.**

14.2 SOLUTION 2 – MODULUS OF ELASTICITY

Since the concrete and steel are connected together, their deformation shall be equal.

$$\Delta l_{conc} = \Delta l_{steel}$$

$$\frac{\cancel{P_{conc} * L_{conc}}}{A_{conc} * E_{conc}} = \frac{\cancel{P_{steel} * L_{steel}}}{A_{steel} * E_{steel}}$$

The forces and heights of the post are equal.

$$A_{conc} * E_{conc} = A_{steel} * E_{steel}$$

$$\left[20 * 10 - 4\left(\frac{\pi D^2}{4}\right)\right] * 3x10^6 = 4\left(\frac{\pi D^2}{4}\right) * 30x10^6$$

$$20 * 10 * 3x10^6 = 4\left(\frac{\pi D^2}{4}\right) * 30x10^6 + 4\left(\frac{\pi D^2}{4}\right) * 3x10^6$$

$$20 * 10 * 3x10^6 = D^2 * (\pi * 30x10^6 + \pi * 3x10^6)$$

$$D = \sqrt{\frac{20 * 10 * 3x10^6}{\pi * 30x10^6 + \pi * 3x10^6}}$$

$$D = 2.4\ in$$

The correct answer is **(a) 2.4 inches**.

14.3 SOLUTION 3 – THERMAL CONDUCTIVITY

Use the following equation to find the total heat loss per hour.

$$Q = \frac{k}{x} * A * \Delta T$$

$$Q = \frac{120 \frac{Btu}{h - ft - F}}{3\ in * \left(\frac{1\ ft}{12\ in}\right)} * 5ft^2 * (200\ F - 75\ F)$$

$$q = 300{,}000 \frac{Btu}{h}$$

The correct answer is most nearly**, (b) 300 MBtuh**.

Note that the convention for 1,000 Btu is 1MBtu, where the M stands for the Roman numeral value for 1,000. Similarly, 1,000,000 Btu is commonly written as 1MMBtu. Sometimes 1,000 Btu is written as 1kBtu.

14.4 SOLUTION 4 – STRESS-STRAIN

The first step is to calculate the maximum strain.

$$\varepsilon = \frac{2''}{6\ ft * 12 \frac{in}{ft}} = .0278$$

Next use this value to calculate the maximum stress with the modulus of elasticity.

$$E = \frac{\Delta Stress}{\Delta Strain}$$

$$15\ x\ 10^6 = \frac{\Delta Stress}{.0278}$$

$$\sigma = 0.417\ x\ 10^6\ psi$$

Finally, stress is equal to the force per unit area. You are given the design force and the maximum stress value. So you need to use the stress equation to solve for area and ultimately the diameter of the rod.

$$\sigma = \frac{F}{A} = \frac{100{,}000\ lbf}{\frac{\pi D^2}{4}}$$

$$D = \sqrt{\frac{100{,}000\ lbf * 4}{\pi * (0.417\ x\ 10^6\ psi)}}$$

$$D = 0.55\ in$$

The correct answer is most nearly, **(d) 0.55 in**.

14.5 SOLUTION 5 – STRESS-STRAIN

Poisson's ratio is defined by the following equation.

$$v = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$\varepsilon_x = stain\ in\ direction\ of\ tension;\ \varepsilon_y = strain\ perpendicular\ to\ directon\ of\ tension$$

$$\varepsilon_y = \frac{\Delta diameter}{diameter} = \frac{0.4998 - 0.5000}{0.5000} = -.0004$$

In order to solve for the strain in the direction of tension you need to use the modulus of elasticity equation.

$$E = \frac{\sigma}{\varepsilon_x}$$

$$15\ x\ 10^6\ psi = \frac{\frac{F}{A}}{\varepsilon_x} = \frac{\frac{3{,}500\ lbf}{\pi * 0.25^2}\ psi}{\varepsilon_x}$$

$$\varepsilon_x = .001188$$

Now plug in the strains into the first equation.

$$v = -\frac{-.0004}{.001188} = 0.3366$$

The correct answer is most nearly, **(d) 0.34**.

14.6 SOLUTION 6 - ALLOYS

At 50% nickel and 1300°C, the alloy exists within the solid + liquid state. Use the lever rule to determine what percentage of the mixture is solid.

$$\%Weight_{Solid} = \frac{Composition_x - Composition_{liquid}}{Composition_{solid} - Composition_{liquid}} x100\%$$

First, find the composition of the liquidus and solidus points at 1300°C.

$$Composition_{solidus} = 58\%\ wt\ Nickel,\ \ Composition_{liquidus} = 45\%\ wt\ Nickel$$

The solution becomes:

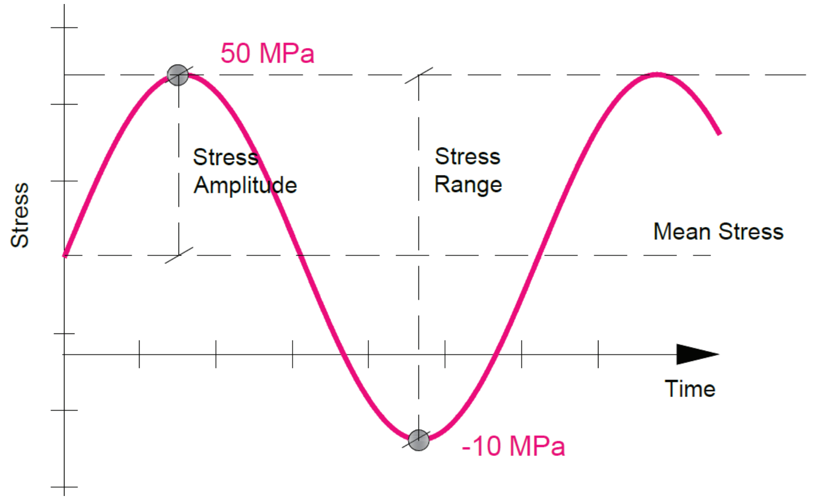
$$\%Weight_{Solid} = \frac{50\% - 45\%}{58\% - 45\%} x100\% = 38\%$$

This makes sense because the point in question is closer to the liquidus line, which means the mixture will contain more liquid than solid. Moreover, from the graph, the point appears to be about 38% away from the liquidus line.

The correct answer is most nearly, (**a) 38%.**

14.7 SOLUTION 7 - FATIGUE

The stress amplitude is one half of the range of the cycle.



The stress range is equal to 60 MPa and the stress amplitude is equal to 30 MPa.

The correct answer is most nearly, **(a) 30 MPa**.

14.8 SOLUTION 8 - FATIGUE

First, find the mean force.

$$F_{mean} = \frac{F_{max} + F_{min}}{2} = \frac{10{,}000\ lb + (-50{,}000\ lb)}{2} = -20{,}000\ lb = 20{,}000\ lb\ compressive\ force$$

Now use the stress equation to find the minimum diameter to ensure the mean stress is less than 1,600 psi.

$$\sigma_{mean} \geq \frac{F_{mean}}{Area}$$

$$1{,}600\ psi \geq \frac{20{,}000\ lb}{0.25 * \pi * D^2}$$

$$D^2 \geq 15.92\ in^2$$

$$D \geq 3.99$$

The correct answer is most nearly **(c) 4.1 in**.

14.9 SOLUTION 9 – FATIGUE

First, calculate the initial change in length of the beam due to the initial loading.

$$\varepsilon = 0.002 \frac{mm}{mm} = \frac{\Delta L}{L}$$

$$\Delta L = 0.002 * 1000\ mm = 2.0\ mm$$

Then calculate the total strain after 1 year. Creep is given in strain per unit time.

$$\varepsilon = \left(10\ x\ 10^{-7} \frac{mm}{mm - h}\right) * 365\ days * (\frac{24\ hr}{1\ day})$$

$$\varepsilon = 0.00876 \frac{mm}{mm}$$

Now calculate the change in length with the strain equation

$$\varepsilon = 0.00876 \frac{mm}{mm} = \frac{\Delta L}{L}$$

$$\Delta L = 0.00867 * 1000\ mm = 8.76\ mm$$

The change in length is approximately 10.76 mm.

The correct answer is most nearly **(d) 11 mm**.

14.10 SOLUTION 10 – POISSON'S RATIO

First, you need to solve for the strain in the axial direction and in the tangential direction.

$$v\ (poisson's\ ratio) = -\frac{\varepsilon_{lateral(radial)}}{\varepsilon_{axial(longitudinal)}}$$

$$\varepsilon_{radial} = \frac{\Delta D}{D} = \frac{-1\ in}{9\ in} = -0.111$$

$$\sigma = E\varepsilon_{longitudinal}$$

$$\varepsilon_{longitudinal} = \frac{\sigma}{E} = \frac{12\ x\ 10^6\ psi}{50{,}000\ x\ 10^3\ psi} = 0.24$$

Now plug in the values back into Poisson's ratio equation.

$$v\ (poisson's\ ratio) = -\frac{-0.111}{0.24} = 0.46$$

The correct answer is most nearly, **(d) 0.46**.

14.11 SOLUTION 11 – CRACK PROPAGATION

Polishing will reduce existing crack lengths and surface hardening will increase the strength of the material which will lessen the risk of crack propagation. In addition, reducing stress acting upon the component will decrease the driving force that increases crack propagation.

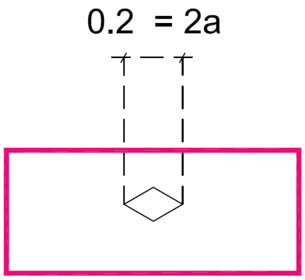
The correct answer is most nearly, **(d) Protecting the component from impact**. Protecting the component will stop crack initiation but will not stop crack propagation.

14.12 SOLUTION 12 – CRACK PROPAGATION

For this question use the fracture toughness equation.

$$K = Y\sigma\sqrt{\pi a}$$

$$2a = crack\ (in), Y = geometrical\ factor\ (1);\ K = fracture\ toughness; \sigma = stress\ (psi)$$



$$a = 0.1\ in.$$

$$20\ ksi - in^{.5} = (1.0)\sigma\sqrt{\pi 0.1\ in}$$

$$\sigma = 35.7\ ksi$$

The correct answer is most nearly **(b) 36 ksi.**

10 – Fluid Mechanics

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Section 10.0 – Fluid Mechanics

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1.0 INTRODUCTION

Fluid mechanics accounts for 10 to 15 problems on the Mechanical FE Exam. The topics range from college fluid mechanics topics like fluid properties, fluid statics, energy, impulse, momentum, internal flow, external flow and compressible flow to the topics that are more often used in practice like incompressible flow, power, efficiency, performance curves and scaling laws for fans, pumps and compressors. As you go through this section, you should also check the fluids topics within the *NCEES FE Reference Handbook*.

Section 10.0 Fluid Mechanics (10 to 15 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
10A	Section 2.0	Fluid Properties
10B	Section 3.0	Fluid Statics
10C	Section 4.0	Energy, Impulse and Momentum
10D	Section 5.0	Internal Flow
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10F	Section 8.0	Compressible Flow
10G	Section 9.0	Power and Efficiency
10H	Section 10.0	Performance Curves
10I	Section 11.0	Scaling Laws for Fans, Pumps and Compressors
	Section 12.0	Practice Exam Problems

2.0 FLUID PROPERTIES

During the exam you will need to be able to find and use fluid properties to complete many problems. You should be very familiar with the *NCEES FE Reference Handbook* and where to find these fluid properties. As you go through these descriptions of the important fluid properties, look through the handbook and you will see that the only fluids mentioned in the handbook with all of these properties are air and water. This should give you an indication that most of the questions on fluids will revolve around air and water and if another fluid is given in a question, then all the properties for that fluid must be provided in the question, except for heat capacity for select fluids like air, water, ethane, methane, mercury, etc. Only select fluids have their densities shown in the handbook.

2.1 DENSITY

The density of a substance is its mass per unit volume. For example, density is typically shown as pound-mass per cubic foot or kilograms per cubic meter.

$$Density = \frac{lbm}{ft^3}[IP] \;\; or \;\; \frac{kg}{m^3}[metric]$$

The density of a fluid is measured as a weight per unit volume. Specific volume is the inverse of density and is measured as a volume per unit mass.

$$Specific\,Volume = \frac{ft^3}{lbm}[IP] \;\; or \;\; \frac{m^3}{kg}[metric]$$

2.1.1 IP Conversions

When performing calculations in English units, it is important to distinguish between pound mass (lbm) and pound force (lbf). The mass of an object is measured in pound-mass, similar to the English units of kilograms (kg). Pound-force, on the other hand, is a measurement of weight. It is a unit of force and is used to describe the mass of an object subject to gravity. Pound-force is comparable to the metric force of Newtons.

$$Weight\;[lbf] = mass * gravity = 1lbm * 32.174\frac{ft}{s^2}$$

$$\boldsymbol{1lbf = 32.174\frac{lbm \cdot ft}{s^2}}$$

To perform calculations between pound-mass and pound force, the conversion factor, g_c is used. Since g_c is merely a unit conversion factor, it can be multiplied or divided anywhere in an equation.

$$g_c\;[conversion\;factor\;] = \frac{ma}{F}$$

$$\boldsymbol{g_c = 32.2\;\frac{lbm \cdot ft}{lbf \cdot s^2}}$$

Finally, mass can be represented in terms of slugs, which simplifies the force equations, essentially internalizing the g_c conversion factor.

$$\boldsymbol{1\;slug = 32.2lbm}$$

$$1lbf = \cancel{1lbm} * \frac{1slug}{\cancel{32.2lbm}} * \cancel{32.2}\frac{ft}{s^2} = 1slug * \frac{ft}{s^2}$$

$$\mathbf{1\ slug = \frac{1 lbf \cdot s^2}{ft}}$$

Example: Density relationships in terms of lbm, slug and lbf.

$$Density, \rho = 1\frac{lbm}{ft^3} = \frac{lbm}{ft^3} * \frac{1}{g_c} = \frac{\cancel{lbm}}{ft^3} * \frac{lbf \cdot s^2}{32.2\cancel{lbm} \cdot ft} = \frac{1}{32.2}\frac{lbf \cdot s^2}{ft^4}$$

$$\mathbf{32.2\frac{lbm}{ft^3} = 1\frac{lbf \cdot s^2}{ft^4} = 1\frac{slug}{ft^3}}$$

Realize that the densities in the *NCEES FE Reference Handbook* properties table are given in terms of $\frac{lbf \cdot s^2}{ft^4}$ or $\frac{slug}{ft^3}$.

2.2 VISCOSITY

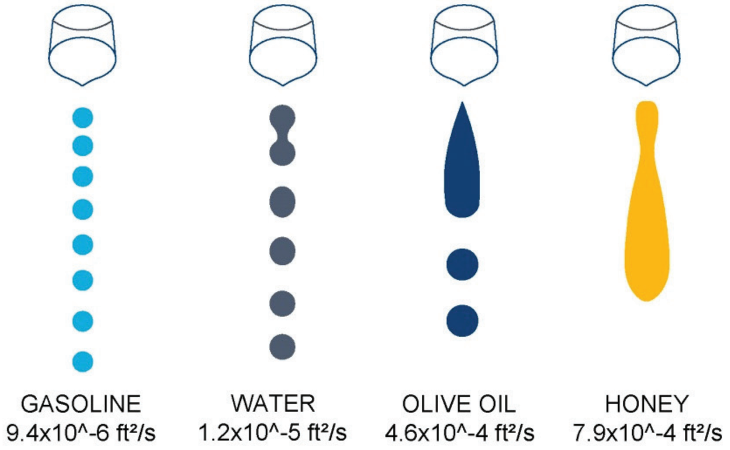


Figure 1: Varying liquids and their kinematic viscosities

The viscosity of a fluid describes the fluids resistance to flow. Viscosity is measured in cP or centipoises and is represented by the variable, μ or mu. Viscosity is measured with a device called a viscometer. There are many different types of viscometers, but each typically has the fluid moving past/through an object or it has the object moving through the fluid. The time of travel will vary based on the viscosity of the fluid. For example, water has a viscosity of ~1.00 cP (centipoises) at 68 °F, while syrup has a viscosity of ~1400 cP and air has a viscosity of ~.01827 cP.

$$\mu = \ viscosity, [cp]$$

$$Alternate\ units\ of\ \mu\text{:}\ \left[\frac{lbm}{ft * s}\right] or \left[\frac{lbf * s}{ft^2}\right] or \left[\frac{g}{cm * s}\right] or \left[\frac{N * s}{m^2}\right]$$

The units described above are related to cP by a factor of 100. 100 cP is equal to 1 $[g/(cm * s)]$. The imperial units are$[lbm/(ft * s)]$ and are related to cP by the following conversion.

$$1 \text{ cP} = 6.72 \text{ x } 10^{-4} \left[\frac{lbm}{ft * s}\right]$$

There are two types of viscosities, dynamic (absolute) viscosity and kinematic viscosity. The previously discussed viscosity μ is dynamic viscosity. Kinematic viscosity describes the ratio of the fluids resistance to flow (dynamic viscosity) to the fluids density. Kinematic viscosity is indicated by the symbol, v or nu.

$$v(nu) = \frac{\mu \left[\frac{lbm}{ft * s}\right]}{\rho \left[\frac{lbm}{ft^3}\right]} = \left[\frac{ft^2}{s}\right]$$

$$where\ \rho = density, \mu = dynamic\ viscosity; v(nu) = kinematic\ viscosity$$

Kinematic viscosity has the units ft^2/s as shown above.

Viscosities of water have been included below for your convenience.

Table 1: This table shows how water's viscosity and density change as the temperature of water increases.

Temperature [F]	Fluid	Viscosity [cP]	Density [lbm/ft^3]	Density [lbf-s^2/ ft^4 =slugs/ft^3]	Kinematic Viscosity [ft^2/sec]
32	Water	1.792	62.42	1.940	1.9291E^{-05}
50	Water	1.308	62.41	1.940	1.4083E^{-05}
68	Water	1.002	62.32	1.937	1.0804E^{-05}
86	Water	0.7978	62.15	1.932	8.6259E^{-06}
104	Water	0.6531	61.94	1.925	7.0853E^{-06}
122	Water	0.5471	61.68	1.917	5.9603E^{-06}
140	Water	0.4668	61.38	1.908	5.1104E^{-06}
158	Water	0.4044	61.04	1.897	4.4519E^{-06}
176	Water	0.355	60.67	1.886	3.9319E^{-06}
194	Water	0.315	60.26	1.873	3.5126E^{-06}
212	Water	0.2822	59.83	1.860	3.1695E^{-06}

2.3 SPECIFIC GRAVITY

Specific gravity is the term used to describe the ratio between a liquid's densities compared to the density of water. Water has a specific gravity of 1.0.

$$SG = \frac{\rho_{fluid}}{\rho_{water}}$$

As an example, the specific gravity of diesel is found below.

$$\rho_{diesel} = 59.9\frac{lbm}{ft^3};\;\; \rho_{water} = 62.4\frac{lbm}{ft^3}$$

$$SG_{diesl} = \frac{59.9}{62.4} = 0.96$$

Specific gravity can also be used to describe the ratio between a gas's densities compared to the density of air. Air has a specific gravity of 1.0.

$$SG = \frac{\rho_{gas}}{\rho_{air}}$$

$$\rho_{methane} = 0.041\frac{lb}{ft^3};\;\; \rho_{air} = 0.075\frac{lb}{ft^3}$$

$$SG_{methane} = \frac{0.041}{0.075} = 0.55$$

2.4 HEAT CAPACITY

Heat capacity describes the amount of energy required to raise a fluid's or solid's temperature by 1 degree. This term will be discussed more in *Section 11.0 Thermodynamics.*

There are two types of heat capacity as shown below. The c_p term is used for processes involving constant pressure and the c_v term is used for processes involving constant volume. The c_p term is used to calculate the change in enthalpy of a fluid or solid as a function of a change in temperature of the fluid or solid. Enthalpy includes the pressure-volume work. For liquids and solids, the terms are very near to one another, because liquids and solids are nearly incompressible. However, for gases the terms will be more different.

2.5 SPECIFIC HEAT RATIO

The specific heat ratio for gases is the ratio of the specific heat capacity at constant pressure and specific heat capacity at constant volume. This is value is used calculate isentropic relationships.

$$Specific\ Heat\ Ratio, k = \frac{c_p}{c_v}$$

$$k_{air} = 1.4$$

3.0 FLUID STATICS

The paragraphs after fluid statics involve fluids in movement. The majority of *Section 10 Fluid Mechanics* is on moving fluids and not fluids at rest, but there may be a few questions on the FE exam that involve fluid statics. The main concepts that you must understand are the pressure due to fluid height, manometers and buoyancy.

3.1 PRESSURE DUE TO A STATIC FLUID

The first concept that you need to understand is that the static pressure at certain points within a fluid will vary based on the height of the fluid above the point. Static pressure is the pressure acting upon a body or point due to a fluid, when a fluid is at rest.

For example, in the figure below the pressure at point 5A will be greater than the pressure at point 2A, which will be greater than the pressure at point 1A. Another concept that you need to understand is that any container that is open to the atmosphere will have a pressure equal to the atmospheric pressure at that location acting upon the open parts of the container. For example, points 1A and 1B, will have a static pressure equal to the atmospheric pressure. If the location of this container is at sea level, then that pressure will be equal to 14.7 psia.

The next concept that you need to understand is that all points at the *same elevation* will have the *same static pressure*. For example, points 2A, 2B, 3A, 3B, 4A and 4B are all at the same static pressure. It is easy to see how 2A, 3A, 4A and 4B are at the same pressure, but it may be less obvious for points 2B and 3B. It looks as though there is not that much fluid acting on points 2B and 3B, but if these points were not at the same pressure as point 4B, then the fluid would want to move towards point 3B. However, this is not the case. The fluid is at rest at points 2B and 3B, which means they have the same static pressure as point 4B.

Lastly, you need to be able to calculate the pressure at each point based on the height of the fluid acting upon that point. Do not forget to include the atmospheric pressure acting upon the top of the fluid, if the container holding the fluid is open.

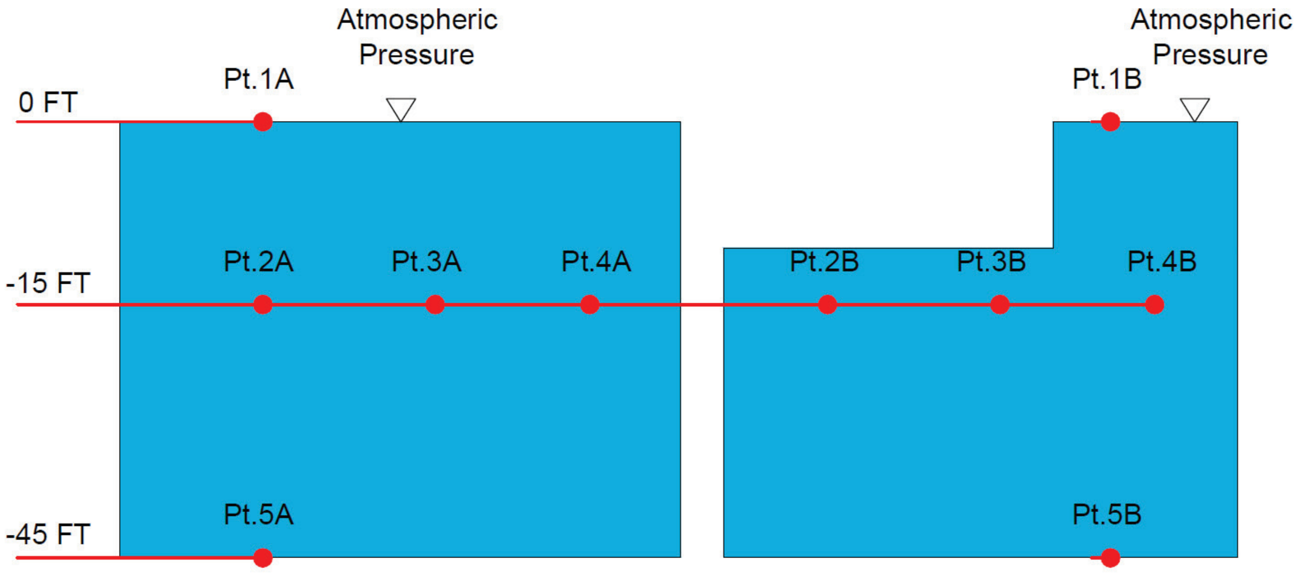


Figure 2: In this figure, all points at the same elevation have the same static pressure. Even though points 2B and 3B do not appear to have the same amount of water acting upon those points, they are at the same pressure as point 4 B. An easy way to think of this concept is that if point 4B was at a higher pressure than point 3B, then the fluid would want to move from point 4B to point 3B, but since the fluid is at rest, then everything at that elevation must be at the same static pressure.

First, you know that the pressure at the top of the container that is open to the surroundings is equal to the atmospheric pressure.

$$P_{1A} = P_{atm} = 14.7\ psi; \quad P_{1B} = P_{atm} = 14.7\ psi$$

Next, points 2 through 4 have 15 feet of fluid above these points. In order to calculate the pressure at this point you must convert the weight of the fluid to a force per unit area. This is done using the density and height of the fluid. The change in the pressure is equal to the following equation.

$$\Delta P = \rho g \Delta h\ \ [SI]; \qquad \Delta P = \rho \frac{g}{g_c} \Delta h\ \ [IP]$$

Note that the gravitational constant, g_c, is needed to convert lbm to lbf. This is only required for English units and is not used in metric problems. To prevent confusion, perform dimensional analysis to ensure all units cancel out to the desired unit.

$$P_{2A\&B,3A\&B,4A\&B} = P_{1A\&B} + \Delta P_{15ft}$$

$$P_{2A\&B,3A\&B,4A\&B} = 14.7\ psi + \rho_{fluid}\left(\frac{\cancel{lbm}}{ft^3}\right) * \frac{g\ (\cancel{32.2}\frac{\cancel{ft}}{\cancel{s^2}})}{g_c\left(\cancel{32.2}\frac{\cancel{lbm} - \cancel{ft}}{lbf - \cancel{s^2}}\right)} * 15\ feet;$$

Assuming the fluid is water, substitute the fluid density to calculate the pressure in terms of pounds per square inch. Again, be sure to write out your units to make sure you also convert square feet to square inches. One of the most common mistakes in fluid mechanics, especially when using English units, is a unit conversion error.

$$p_{water} = 62.4(\frac{lbm}{ft^3})$$

$$P_{2A\&B,3A\&B,4A\&B} = 14.7\ psi + 62.4(\frac{\cancel{lbm}}{ft^3}) * \frac{g\ (\cancel{32.2}\frac{\cancel{ft}}{\cancel{s^2}})}{g_c(\cancel{32.2}\frac{\cancel{lbm} - \cancel{ft}}{lbf - \cancel{s^2}})} * 15\ feet;$$

$$P_{2A\&B,3A\&B,4A\&B} = 14.7\ psi + 936\ (\frac{lbf}{ft^2}) * (\frac{1\ ft^2}{144\ in^2});$$

$$P_{2A\&B,3A\&B,4A\&B} = 21.2\ psi$$

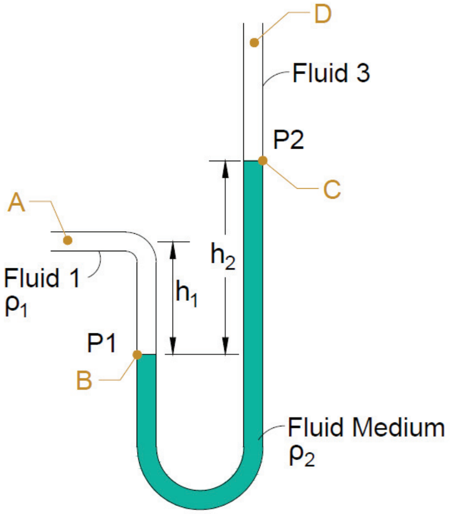
You can then calculate the pressure at the bottom of the container through the same method.

$$P_{5A\ \&\ 5B} = 14.7\ psi + 62.4 * 45\ (\frac{lbf}{ft^2}) * (\frac{1\ ft^2}{144\ in^2});$$

$$P_{2A\&B,3A\&B,4A\&B} = 34.2\ psi$$

3.2 MANOMETERS

Manometers are used to measure pressures using a tube and a fluid. A fluid medium is placed inside the tube, shown in green in the figure below. One end of the tube is exposed to a fluid of known pressures, indicated as Fluid 3 in the figure below. Typically this fluid is air at atmospheric pressure. The other end of the tube is exposed to the fluid being measured, indicated as Fluid 1 in the figure below. The pressure differential between fluid 1 and 3 can be calculated based on the fluid-medium height difference, using the fluid static principals explained above.



In the figure above, use the equation, ΔP=ρgh to calculate pressure changes from one point to another. Starting from point A and traveling to point B of the tube. Using the density of fluid 1, the pressure difference from point A to point B can be calculated.

$$P_B = P_A + \rho_1 g h_1$$

Notice that the height at point B is lower than point A, so the pressure will be greater at point B and therefore the pressure shall be added. Now, traveling up from point B to point C, the pressure at point C can be found as the following.

$$P_C = P_A + \rho_1 g h_1 - \rho_2 g h_2$$

If fluid 3 is air exposed to atmosphere, then fluid 3 and therefore point C and D have a pressure of 14.7psia. The pressure at point A becomes the following.

$$P_A = 14.7psia + \rho_2 g h_2 - \rho_1 g h_1$$

Manometers can have various configurations, but the principals behind the calculation are the same. Start at one point and add or subtract the pressures based on the vertical height difference of the fluid multiplied by the fluid density and gravity.

3.3 FORCES ON SUBMERGED SURFACES

The following topic discusses how to find the resultant force due to the pressures of a fluid when a flat plat is submerged. The first concept to understand is that the fluid creates a pressure gradient along the plate, which linearly increases as the depth of the submergence, y increases. See the figure below for an illustration of the pressure gradient along a flat plate.

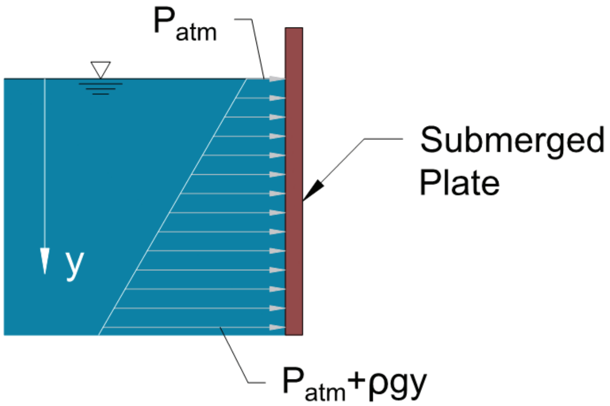

P_atm
Submerged Plate
y
P_atm+ρgy

Figure 3: Pressure gradient of fluid acting on a submerged surface

The pressure gradient can be equated to a single resultant force acting on the plate. The magnitude and the location of this resultant force, F_R, must be found. The magnitude of resultant force, F_R, the moment is taken about a center of rotation, 0. Since the pressure from the fluid is perpendicular to the plate, so is the resultant force. The derived equation for the resultant force is as follows, where P_0 is the pressure at the surface and y_C is the distance along the plate to the centroid of the area.

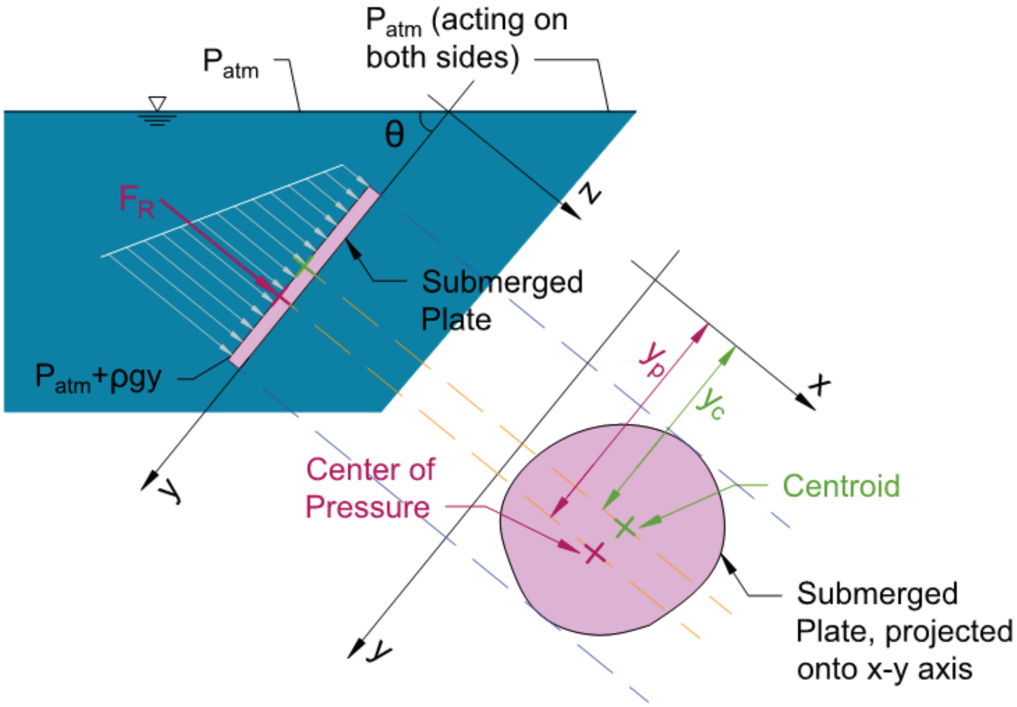

P_atm
P_atm (acting on both sides)
θ
z
F_R
Submerged Plate
P_atm+ρgy
y_p
y_c
x
Center of Pressure
Centroid
Submerged Plate, projected onto x-y axis
y
y

Figure 4: Solving for Resultant Force F_R along Submerged Flat Plate

$$F_R = (P_0 + \rho g y_c sin\theta) * A$$

If P_0 is atmospheric pressure and the plate is exposed to atmosphere on both sides, then an atmospheric force will be pushing back against F_R and the equation becomes:

$$F_{R,atm\ on\ both\ sides} = (\rho g y_c sin\theta) * A$$

The position of the reactive force will be located at the center of pressure, and not the area centroid. This is calculated as:

$$y_p = y_c + \frac{I_{x,c}}{\left(y_c + \frac{P_0}{\rho g sin\theta}\right) * A}$$

For atmospheric pressure acting on both sides of the surface, P_0=0, therefore,

$$y_{p,atm\ on\ both\ sides} = y_c + \frac{I_{x,c}}{y_c * A}$$

Where $I_{x,C}$ is the area moment of inertia, taken about the centroid along x axis. Find the simplified area moment of inertia equations for typical geometries in the *NCEES FE Reference Handbook* under statics.

3.4 BUOYANCY

Buoyancy is defined as how a mass sits in a fluid, it either floats or it sinks. The buoyancy force is the force equal to the weight of the volume of displaced fluid. The buoyancy force may be on the exam, but most likely because of how important fluid density is in Thermal & Fluids and how buoyancy force questions test your understanding of fluid density.

$$B = \rho_{fluid} * V_{fluid\ displaced} * [g]$$

$$where\ B = buoyancy\ force;\ \rho_{fluid} = fluid\ density;\ V_{fluid\ displaced} = volume\ of\ displaced\ fluid$$

$$g = \frac{9.81m}{s^2}\ for\ SI\ units$$

When an object is submerged, the buoyancy force is found through the above equation. Notice that the object's mass is not in this equation, thus when an object is submerged, the buoyancy force is only dependent on the volume of the object and the density of the fluid.

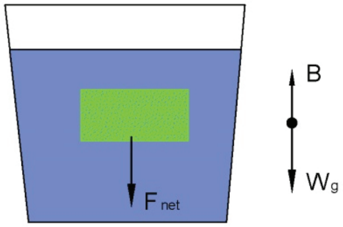


Figure 5: This shows an object in a fluid, where the object's weight is greater than the buoyancy force. This causes the object to sink.

$$B_{submerged} = \rho_{fluid} * V_{fluid\ displaced} * g = \rho_{fluid} * V_{object} * g$$

When the buoyancy force is equal to the weight of the object, then the object will float.

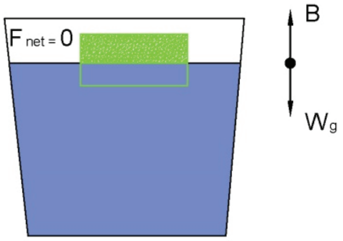


Figure 6: This shows an object in a fluid, where the object's weight is equal to the buoyancy force. This causes the object to float.

$$B_{floating}\ when, \qquad \rho_{fluid} * V_{fluid\ displacted} * g = \rho_{object} * V_{object} * g$$

When the buoyancy force is greater than the weight of the object, then the object will rise.

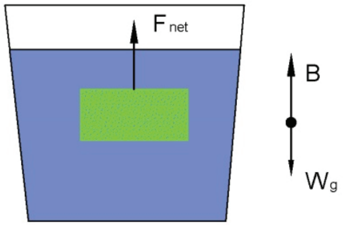


Figure 7: This shows an object in a fluid, where the object's weight is greater than the buoyancy force. This causes the object to rise, until it reaches equilibrium, where the object will float.

$$B_{rising}\ when, \qquad \rho_{fluid} * V_{fluid\ displacted} * g > \rho_{object} * V_{object} * g$$

3.5 WEIGHT & MASS COMPUTATIONS

3.5.1 SI Units

Mass and weight computations are easily done in SI units. Mass is the inertial property of an object and is measured in grams or kilograms in SI units. Mass does not change regardless of the location of the object. Weight on the other hand will depend on the gravity affecting the object. Weight can also be called the force due to gravity and is given units of Newtons. Weight is found by multiplying the mass by the gravity, which for the FE exam will always be 9.8 m/s^2, because it is highly unlikely that any problem will have a situation where the gravity is different.

$$Mass \rightarrow m = [grams]\ \&\ [kilograms]$$

$$Weight \rightarrow W = F_g = m * g\ ; \quad g = 9.8\frac{meters}{s^2}$$

$$Weight\ Units \rightarrow W = F_g[N] = m[kg] * g\left[\frac{meters}{sec^2}\right];$$

The NCEES outline also specifically calls out the unit of force called dyne. One dyne is equal to 0.00001 Newtons.

$$[1\ N] = [100{,}000\ dynes]; \quad [1\ dyne] = [0.00001\ Newtons]$$

3.5.2 USCS Units

In US units, the weight computations get slightly more complex. Mass is given in units of slugs. In order to calculate the force due to gravity aka weight, you must multiply slugs by the gravity term in units of ft^2 per second.

$$Weight = F_g = m\,[slugs] * g\left[32.2\frac{ft}{s^2}\right]$$

However, slugs are not often given in practice. The common mass term is pound-mass. Pound mass first be converted with the g$_c$ factor. This is a conversion factor that will convert mass to force (weight).

$$g_c = 32.2\,\frac{lbm - ft}{lbf - s^2}$$

When you multiply pound-mass by acceleration, you must divide by the above conversion factor.

$$Weight = F_g = m\,[\cancel{lbm}] * \frac{g\left[32.2\frac{\cancel{ft}}{\cancel{s^2}}\right]}{g_c\left[\frac{\cancel{lbm} - \cancel{ft}}{lbf - \cancel{s^2}}\right]}$$

$$Weight = F_g[lbf]$$

3.5.3 Example 1: Find Weight OF 1 KG

As an example, find the weight of 1 kg with slugs, lbm and kg. The following table summarizes all of these calculations.

$$Convert\ Mass \rightarrow \ 1\ kg * \left(2.2\frac{lbm}{kg}\right) = 2.2\ lbm; \ \ 1\ kg * \left(\frac{1\ slug}{14.5939\ kg}\right) = 0.068\ slugs;$$

	USCS Units	USCS	SI	1 KG
Mass	*0.068 slug*	*2.2 lbm*	*1 kg*	
Weight (Equation)	$(slug\ x\ 32.2\frac{ft}{s^2})$	$(lbm\ x\ \frac{32.2\frac{ft}{s^2}}{32.2\frac{ft}{s^2}})$	(kg x 9.8 m/s^2)	
Weight (Calculation)	*2.2 lbf*	*2.2 lbf*	*9.8 N*	

Figure 8: This figure and table shows the relative mass for one kilogram. One kilogram is roughly equal to a mass of 2.2 lbm or 0.068 slugs.

3.5.4 Example 2: Find Weight of 1 LBM

As an example, find the weight of 1 lbm with slugs, kg and lbm. The following table summarizes all of these calculations. First, convert the pound-mass to kilograms and slugs.

$$Convert\ Mass \rightarrow \quad 1\ lbm * \left(\frac{1\ kg}{2.2\ lbm}\right) = 0.454\ lbm; \quad 1\ lbm * \left(\frac{1\ slug}{32.2\ lbm}\right) = 0.031\ slugs;$$

	USCS Units	USCS	SI	1 LBM
Mass	*0.068 slug*	*1 lbm*	*0.454 kg*	
Weight (Equation)	$(slug\ x\ 32.2\frac{ft}{s^2})$	$(lbm\ x\ \frac{32.2\frac{ft}{s^2}}{32.2\frac{ft}{s^2}})$	(kg x 9.8 m/s^2)	
Weight (Calculation)	*1 lbf*	*1 lbf*	*4.5 N*	

Figure 9: This figure and table shows the relative weight due to one pound-mass. One pound-mass has a weight of 1 pound-force or 4.5 Newtons.

3.5.5 Example 3: Find Weight of 1 SLUG

As an example, find the weight of 1 slug with slugs, kg and lbm. The following table summarizes all of these calculations. First, convert slugs to pound-mass and kilograms.

$$Convert\ Mass \rightarrow \quad 1\ slug * \left(32.2\frac{lbm}{slug}\right) = 32.2\ lbm; \quad 1\ slug * \left(\frac{14.5939\ kg}{1\ slug}\right) = 14.59\ kg;$$

	USCS Units	USCS	SI	1 SLUG
Mass	*1 slug*	*32.2 lbm*	*14.59 kg*	
Weight (Equation)	$(slug\ x\ 32.2\frac{ft}{s^2})$	$(lbm\ x\ \frac{32.2\frac{ft}{s^2}}{32.2\frac{ft}{s^2}})$	(kg x 9.8 m/s^2)	
Weight (Calculation)	*32.2 lbf*	*32.2 lbf*	*143 N*	

Figure 10: This figure and table shows the relative weight due to one slug. One slug is roughly equal to 32.2 pound-force or 143 N.

4.0 Energy, Impulse, Momentum

4.1 Energy

There are three main types of energy used in fluid calculations, (1) Pressure Energy, (2) Kinetic Energy, and (3) Potential Energy. For steady, incompressible flow, the energy at one point in a system will contain one or more of these energy types. Adding all three components together will yield the total energy at any point in the system, which is found using the following equation known as Bernoulli's Equation.

$$\frac{P}{\rho}[Pressure\ Energy] + \frac{v^2}{2}[Kinetic\ Energy] + gz[Potential\ Energy] = Constant \quad [SI\ units]$$

$$\frac{Pg_c}{\rho}[Pressure\ Energy] + \frac{v^2}{2}[Kinetic\ Energy] + gz[Potential\ Energy] = Constant \quad [IP\ units]$$

Where

$P = Pressure\ (Pa\ or\ psi)$
$\rho = Density\ (kg/m^3\ or\ lbm/ft^3)$
$v = Velocity\ (m/s\ or\ ft/s)$
$g = gravity\ (9.81\ m/s^2\ or\ 32.2\ ft/s^2)$
$g_c = graviational\ constant\ (32.2\ lbm \cdot ft/lbf \cdot s^2)$
$z = Elevation\ (m\ or\ ft)$

Using the conservation of energy principal, the energy at one point can be equated to the energy at another point. Since friction losses are experienced along the way, energy losses due to friction, denoted by the term h_f, are also added.

$$\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f \quad [SI]$$

$$\frac{P_1 g_c}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2 g_c}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f \quad [IP]$$

In typical application problems, a pump is sized based on the modified equation below, where the energy added from the work of the pump equals the final conditions of the system.

$$E_{mech,in} = E_{mech,out} + E_{losses}$$

$$H_{pump,in}(m) + \frac{P_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f \quad [SI]$$

$$H_{pump,in}(ft) + \frac{P_1 g_c}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2 g_c}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f \quad [IP]$$

$$where\ H_{pump,in} = Input\ pump\ pressure\ requirement, in\ terms\ of\ m\ or\ ft.$$

See the Incompressible Flow topic on how to calculate friction losses and the Power and Efficiency topic on how to calculate pump power.

4.2 IMPULSE-MOMENTUM

The impulse-momentum equations use the principals of Newton's Second Law, which states that the change in momentum is equal to the external forces acting on an object. For steady flow fluids, this can be analyzed by defining a control volume around the fluid. The sum of all the external forces acting on the control volume will equal the momentum of the fluid entering and leaving the control boundary.

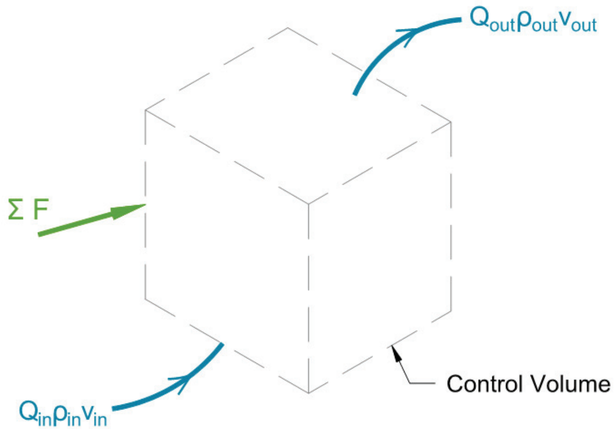


Figure 11: For steady flow fluids, the sum of the changes in momentum in a control volume equals the sum of the external forces acting on the control volume.

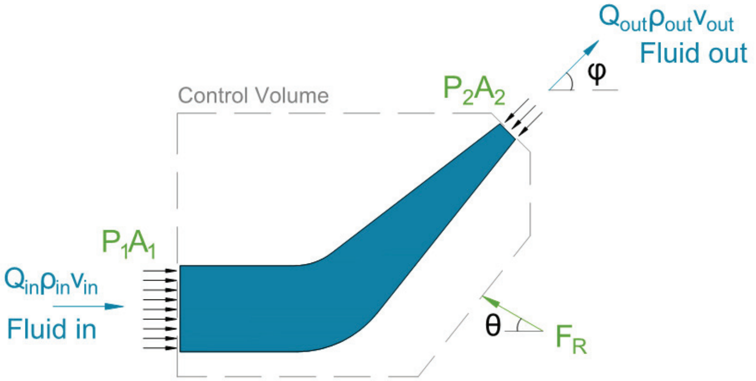
$$\sum F_{ext} = \sum Q_{out} \rho_{out} v_{out} - \sum Q_{in} \rho_{in} v_{in}$$

In the equation above, Q is the volumetric flow rate (m^3/s or ft^3/s), ρ is the density (kg/m^3 or $(lbm/ft^3)/g_c$), and v is the velocity (m/s or ft/s). The equation can also be represented in terms of mass flow rate, $\dot{m}$ (kg/s or $(lbm/s)/g_c$). *Remember that density or mass in terms of lbm in English units need to be divided by g_c in order to convert the units to lbf.*

$$\sum F_{ext} = \sum \dot{m}_{out} v_{out} - \sum \dot{m}_{in} v_{in}$$

In these problems, break out the forces and momentum equations into each coordinate and solve for the resultant forces.

Example: Water enters a reducing elbow that diverts the incoming fluid by angle φ. Find the reactive force acting on the elbow in order to keep it from moving.



To solve this problem, a control volume is taken around the elbow as shown above. The sum of all the forces acting on the control volume will equal the momentum entering and leaving the elbow. The force on the intake is found from the pressure acting on the intake of the control volume, which is positive, and the pressure on the outlet is found from the pressure acting on the outlet of the control volume, which is negative. The force from the pressure is equal to the pressure multiplied by the area that the pressure is acting on. Therefore the sum of the external forces can be calculated below.

$$\sum F_{external} = P_1 A_1 - P_2 A_2 - F_R$$

Then equating the external forces to the difference in momentum gives the following.

$$P_1 A_1 - P_2 A_2 - F_R = Q_{out} \rho_{out} v_{out} - Q_{in} \rho_{in} v_{in}$$

To solve the problem, separate the equation into x, y components.

X-Component:

$$P_1 A_1 - P_2 A_2 \cos\varphi - F_{R,x} = Q_{out} \rho_{out} (v_{out} \cos\varphi) - Q_{in} \rho_{in} v_{in}$$

Since the flow entering the control volume must equal the flow leaving the control volume, $Q_{in}\rho_{in} = Q_{out}\rho_{out}$. Therefore the equation simplifies to:

$$P_1A_1 - P_2A_2\cos\varphi - F_{R,x} = Q\rho[(v_{out}\cos\varphi) - v_{in}]$$

Y-Component:

$$-P_2A_2\sin\varphi + F_{R,y} = Q\rho(v_{out}\sin\varphi) - 0$$

With known inlet/outlet pressures, areas, and velocities and known fluid density, solve for the reactive force, $F_{R,x}$ and $F_{R,y}$ in the equations above, then find the total reactive force on the control volume.

$$F_R = \sqrt{F_{R,x}^2 + F_{R,y}^2}$$

5.0 INTERNAL FLOW

Internal flow is flow contained within an enclosed boundary, like a pipe or a duct. Internal flow on the FE exam will most likely revolve around finding the pressure drop through these pipes or ducts. In the figure below, fluid enters a pipe with a uniform velocity profile, indicated by the velocity vectors, u, at the entrance. As friction is experienced on the inner surface of the pipe, a velocity profile will be created, where the velocity is 0 at the pipe surface due to friction, and is largest at the center-line of the pipe, which is farthest away from the pipe wall and therefore is able to move more freely. This profile develops after traveling a certain distance in the pipe. When the profile no longer continues to change, the flow is said to be fully developed. This image illustrates how fluid will flow through a pipe.

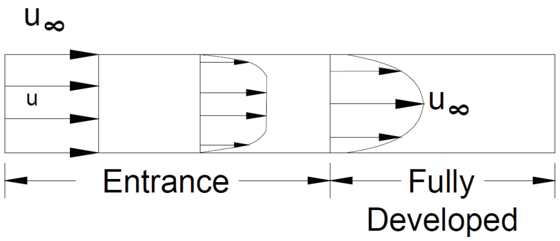


Figure 12: Internal flow is characterized by a fluid flowing within a pipe or duct.

Pressure drop within a pipe will depend on how rough the inner surface of the pipe is: the rougher the pipe, the more surface friction will be experienced, the diameter of the pipe: in other words, how much area the fluid comes into contact with, the length of the pipe: the longer the pipe, the more the friction loss will be, the velocity of the fluid, the type of the fluid, the density of the fluid, and any obstructions that the fluid will encounter, such as valves, elbows, reducers, and branches.

5.1 REYNOLDS NUMBER

The Reynolds number (Re) is used to determine the flow characteristic of the fluid, i.e. whether it is laminar (smooth) or turbulent (rough). The Reynolds number is a unitless number that is found by multiplying the velocity of the fluid through the pipe by the diameter of the pipe and dividing by the kinematic viscosity of the fluid.

$$Reynolds\ number\ (Re) = \frac{v * D}{\nu(nu)} = \frac{\rho * v * D}{\mu}$$

There are some variations based on what resource you use as to what values of Reynolds number describe a certain flow characteristics. The *NCEES FE Reference Handbook* use the following values to define laminar and turbulent flow through a pipe.

$$Re < 2100\ \ (Laminar\ Flow)$$
$$2100\ \leq\ Re\ \leq\ \ 10{,}000\ \ (Transitional\ Flow)$$
$$Re\ >\ \ 10{,}000\ \ (Turbulent\ Flow)$$

(Note: to distinguish between velocity v and kinematic viscosity ν, kinematic viscosity will follow with $\nu(nu)$ for clarification. Be sure to do all unit conversions so that the Re is unit less.)

$\nu(nu) = kinematic\ viscosity\ (m^2/s\ \ or\ \ ft^2/s)$
$\mu(nu) = dynamic\ viscosity\ (kg/(m * s)\ \ or\ \ lbm/(ft * s))$
$v = average\ velocity\ of\ fluid\ (m/s\ \ or\ \ ft/s)$
$D = internal\ diameter\ of\ circular\ tube\ (m\ \ or\ \ ft)$

If a non-circular tube or duct is used, use the following equation to find the diameter, to be used in the Reynolds Number calculations. This is called the hydraulic diameter, D_h.

$$D_h = \frac{4 * Cross\ Sectional\ Area}{Wetted\ Perimeter}$$

The wetted permitted is the perimeter of the tube that is in contact with the fluid. For a rectangle, the wetted perimeter is 2*interior length + 2*interior width. For an open channel, the wetted perimeter is 1*interior length + 2*height of fluid.

5.2 LAMINAR FLOW

The fully developed fluid flow can be characterized as either laminar or turbulent. The way the fluid flows will determine how the calculations should be performed. Laminar flow is the smooth movement of fluid, which is characterized by the Reynolds Number indicated above. Since the fluid moves in a more predictable pattern, these flows are easier to calculate.

For laminar flow, the fully developed velocity profile in a tube or between flat plats can be found with the following equation, where r is the location along the radius of the pipe or distance from the center of the flow, R is the inner radius of the tube or the distance from the center of the plates to the plate, v(r) is the velocity at the distance r, and v_{max} is the max velocity in the flow.

$$v(r) = v_{max} * \left[1 - \left(\frac{r}{R}\right)^2\right]$$

For the following conditions, the max velocities can be found as:

$$v_{max} = 2 * average\ velocity, pipes, laminar\ flow$$
$$v_{max} = 1.5 * average\ velocity, parallel\ plates, laminar\ flow$$

The discussion of friction losses through pipes with laminar flow fluids are described in the Incompressible Fluid topic.

5.3 TURBULENT FLOW

Turbulent flows, on the other hand, are erratic and have swirls in its flow pattern, causing the instantaneous velocities to continuously vary. The discussion of friction losses through pipes with laminar flow fluids are described in the Incompressible Fluid topic.

5.4 BERNOULLI'S EQUATION

The fundamental equation for incompressible, steady flow fluids, is Bernoulli's equation, which essentially says that the energy will be conserved between two points in a system. The fluid energy will either be present as pressure, kinetic, or potential energy. The total energy in state 1 will equal the energy in state 2 plus any losses due to friction in the fluid.

$$[Pressure_1 + Kinetic_1 + Potential_1] = [Pressure_2 + Kinetic_2 + Potential_2] + Friction\ Losses$$

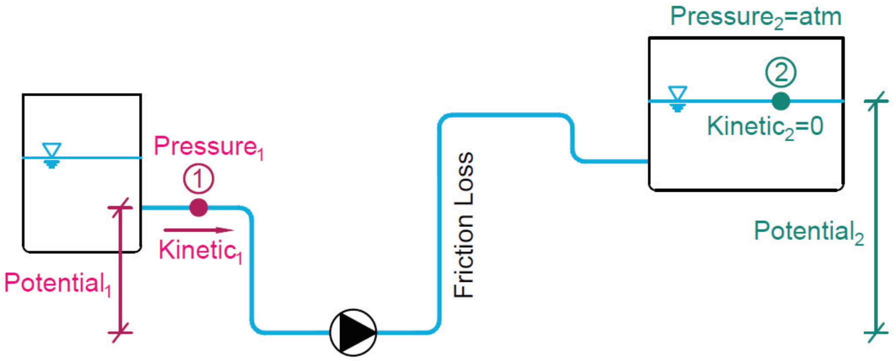


Figure 13: Illustration of Bernoulli's Principle. Energy in state 1 is conserved in state 2 in the form of pressure, kinetic, or potential energy after any losses due to friction.

This equation and its principles are the basis of many equations used in this chapter. The full equation is written out as follows. The English units (IP) version is shown below.

$$\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f + h_{f,fitting}\ \ [SI]$$

$$\frac{P_1 g_c}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2 g_c}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f\ + h_{f,fitting}\ \ [IP]$$

Notice that the IP version requires the gravitational constant conversion factor, g_c, in order to convert lbf in the pressure units (lbf/in^2) to lbm in the density units (lbm/ft^3). Since g=32.2 ft/s^2 and g_c = 32.2 (lbm/lbf)*(ft/s^2), the g term essentially cancels out to P/ρ.

$$\frac{P_1}{\rho}+\frac{v_1^2}{2g}+z_1=\frac{P_2}{\rho}+\frac{v_2^2}{2g}+z_2+h_f+h_{f,fitting} \quad [IP, simplified]$$

$$where\ P=psi(lbf/in^2), \rho=lbm/ft^3$$

The SI version does not require the g_c factor and the pressure will be in Newtons per square meter, which will adjust the location of the density factor. The SI version is shown below.

$$P_1+\frac{1}{2}v_1^2\rho+\rho g z_1=P_2+\frac{1}{2}v_2^2\rho+\rho g z_2+P_f+P_{f,fitting} \quad [SI, simplified]$$

$$where\ P=Pa(N/m^2), \rho=kg/m^3$$

5.5 PITOT TUBE

The next concepts that you need to know for the FE exam are the concepts governing the Pitot tube. There are two equations presented in the *NCEES FE Reference Handbook*, the first one is based on Bernoulli's Equation. The variation of Bernoulli's equation that you need to know for the exam is that the sum of the static pressure, velocity pressure and elevation pressure within a duct or pipe is constant throughout the pipe or duct.

$$\frac{P_{1,static}g_c}{\rho_{air}g}+\frac{1}{2g}v_1^2+z_1=\frac{P_{2,static}g_c}{\rho_{air}g}+\frac{1}{2g}v_2^2+z_2 \quad [IP]$$

$$P_{static}=static\ pressure\ at\ a\ point; v=velocity; z=elevation\ at\ a\ point$$

For the purposes of the Pitot tube, the elevation pressure is assumed to be the same. The elevation pressure is the pressure that changes as the pipe or duct moves up or down. In terms of energy, this is the potential energy.

Static pressure is the pressure that is acting against the outer walls of the pipe or duct. If the flow were to be completely blocked by a valve or damper, then all the pressure would turn into static pressure. If you were to imagine a pressure vessel, the pressure of the fluid within the vessel acting upon the walls would be the static pressure.

Velocity pressure is the pressure that propels the fluid forward, in the direction of flow within the pipe or duct. The equivalent energy is the kinetic energy.

A pitot tube will measure two pressures, the static pressure and the total pressure (static and velocity pressure). The total pressure is called stagnation pressure in the *NCEES FE Reference Handbook.*

$$Total\ Pressure=Velocity\ Pressure+Static\ Pressure$$

Measuring Total Pressure: A pitot tube will create a location of zero velocity, which will convert the total pressure to just static pressure. The Pitot tube is angled directly towards the flow. This static pressure is then measured with a method similar to the manometer.

Measuring Static Pressure: A pitot tube will create a location where it does not measure velocity pressure. The Pitot tube does this by creating a location where the pressure measuring component is angled perpendicular to the flow rate. This pressure is then measured with a method similar to the manometer.

The difference between total pressure (stagnation pressure) and static pressure is equal to velocity pressure.

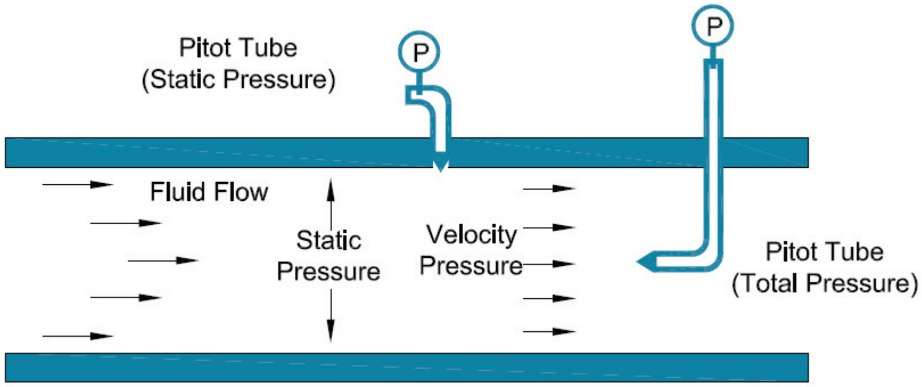


Figure 14: This figure explains the Pitot tube concept graphically. It is important to note that in practice, there most likely will not be two separate tubes. The Pitot tube device will be one singular device with two measurement tubes within that one device. One tube will measure static pressure and the other tube will measure the stagnation pressure (total pressure).

Just to reiterate again, when the Pitot tube is measuring static pressure, the velocity pressure still exists, but the Pitot tube cannot pick up the velocity pressure.

$$Measuring\ Static\ Pressure \rightarrow P_{static} = Pitot\ Tube\ Presure\ Reading$$

When the Pitot tube is measuring total pressure aka stagnation pressure, it is measuring both velocity pressure and static pressure.

$$Measuring\ Stagnation\ Pressure\ \left(\frac{P_{stagnation}g_c}{\rho_{air}g}\right) = \frac{P_{static}g_c}{\rho_{air}g} + \frac{1}{2g}v_1^2 = Pitot\ Tube\ Reading$$

Now, you can use those two readings to calculate velocity. Simply subtract the static pressure from the total pressure (stagnation pressure) reading.

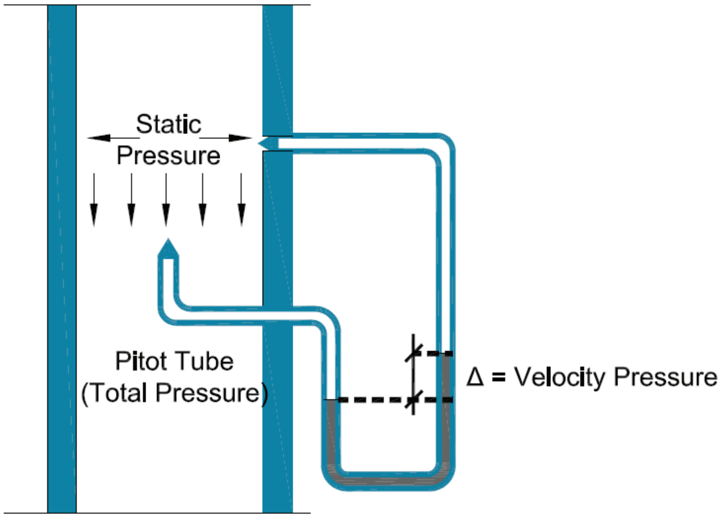


Figure 15: This figure shows graphically how a pitot tube works. In practice, the Pitot tube will look like a single device. One part measures the total pressure and the other part measures the static pressure. As you can see on the left side of the Pitot tube, it is measuring total pressure, so it acts upon the measuring fluid by a greater amount. The right side of the Pitot tube measures the static pressure, which will be less than the total pressure. The difference is equal to the velocity pressure.

This is the equation that governs the Pitot tube.

$$\frac{P_{stagnation} g_c}{\rho_{air}} = \frac{P_{static} g_c}{\rho_{air}} + \frac{1}{2} v_1^2$$

Rearrange the variables, to solve for velocity.

$$v = \sqrt{2 g_c \frac{(P_{stagnation} - P_{static})}{\rho_{air}}}$$

Now, it is just a matter of dimensional analysis to correctly apply this equation, given the various pressure measurements that could arise in a PE exam problem. For example, in English units, the pressure could be given in pounds per square inch. You will also need the density of air or the gas in question, which can be found in the *NCEES FE Reference Handbook* or will be given in the problem.

$$Given\ \Delta P = 2.0\ psi, \rho_{air} = 0.075 \frac{lbm}{ft^3} \rightarrow Solve\ for\ velocity \rightarrow$$

Remember that pressure is given in terms of pound-force and must be converted to pound-mass with the gravitational constant.

$$v = \sqrt{2 * \frac{2 \frac{lbf}{in^2} * g_c}{\rho_{air}}} = \sqrt{2 * \frac{2\ {}^{lbf}/_{in^2} * g_c \frac{lbm - ft}{lbf - s^2}}{0.075\ {}^{lbm}/_{ft^3}}}$$

$$v = \sqrt{2 * \frac{2\ {}^{lbf}/_{in^2} * 32.2 \frac{lbm - ft}{lbf - s^2} * 144\ {}^{in^2}/_{ft^2}}{0.075\ {}^{lbm}/_{ft^3}}} = \sqrt{247{,}296 \frac{ft^2}{s^2}}$$

$$v = 497 \frac{ft}{s}$$

The same process can be done with other units like inches water gauge (in. wg) and Pascals (Pa).

5.6 VENTURI METERS

In the mechanical field, pitot tubes are primarily used to measure air velocity in ducts. Venturi meters are used to measure the velocity of liquids like chilled water, condenser water, domestic water, etc. A Venturi meter operates based on Bernoulli's Theorem. The Venturi meter is shaped with two different areas that are used to change the velocity of the fluid within the pipe. As the velocity increases, the static pressure will decrease. So if you were to compare the difference in static pressure at two points, that difference will be equal to the velocity pressure.

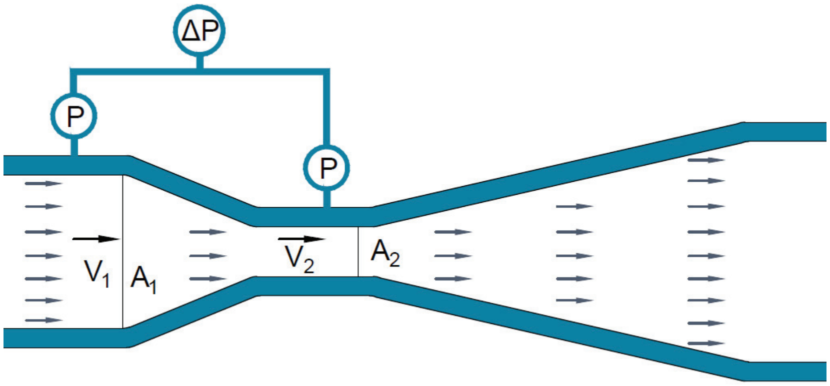


Figure 16: A Venturi meter is installed on a straight length of pipe. The meter consists of a sharp change in area, which changes the velocity pressure and static pressure. The difference between the static pressures at points 1 and 2, will lead to the velocity calculation.

First, start with Bernoulli's Theorem, comparing points 1 and 2.

$$\frac{P_{1,static} g_c}{\rho_{fluid} g} + \frac{1}{2g} v_1^2 + z_1 = \frac{P_{2,static} g_c}{\rho_{fluid} g} + \frac{1}{2g} v_2^2 + z_2$$

$$P_{static} = static\ pressure\ at\ a\ point; v = velocity; z = elevation\ pressure\ at\ a\ point$$

You most likely will not have a change in elevation between points 1 and 2, so you can remove that component. The velocity components can be combined with the conservation of mass formula. If you assume constant density, then conservation mass will simply become conservation of flow rate.

$$A_1 v_1 = A_2 v_2$$

Next, put Bernoulli's Theorem in terms of v_1.

$$v_2 = \frac{A_1 v_1}{A_2}$$

$$\frac{P_{1,static} g_c}{\rho_{fluid} g} + \frac{1}{2g} v_1^2 + \cancel{z_1} = \frac{P_{2,static} g_c}{\rho_{fluid} g} + \frac{1}{2g}\left(\frac{A_1 v_1}{A_2}\right)^2 + \cancel{z_2}$$

$$\frac{P_{1,static} g_c}{\rho_{fluid} \cancel{g}} + \frac{1}{2\cancel{g}} v_1^2 = \frac{P_{2,static} g_c}{\rho_{fluid} \cancel{g}} + \frac{1}{2\cancel{g}}\left(\frac{A_1 v_1}{A_2}\right)^2$$

$$\frac{1}{2} v_1^2 \left(1 - \left(\frac{A_1}{A_2}\right)^2\right) = \frac{P_{2,static} g_c}{\rho_{fluid}} - \frac{P_{1,static} g_c}{\rho_{fluid}}$$

$$v_1 = \sqrt{2 g_c \frac{\left(\frac{P_{2,static}}{\rho_{fluid}} - \frac{P_{1,static}}{\rho_{fluid}}\right)}{\left(1 - \left(\frac{A_1}{A_2}\right)^2\right)}}$$

The Venturi meter is primarily used to determine liquid flow rate, so the velocity term will be multiplied by the area. In addition, there will be some real-life adjustments that will need to be made to the velocity value. This is accounted for by a coefficient of velocity value that is specific to each Venturi meter at specific conditions. This value should be given to you on the exam and if it isn't then you can assume a value of 1.

$$Q = C_v A_1 \sqrt{2 g_c \frac{\left(\frac{P_{2,static}}{\rho_{fluid}} - \frac{P_{1,static}}{\rho_{fluid}}\right)}{\left(1 - \left(\frac{A_1}{A_2}\right)^2\right)}}$$

This equation can also be rewritten in terms of v_2 as opposed to v_1.

$$v_1 = \frac{A_2 v_2}{A_1}$$

$$\frac{P_{1,static} g_c}{\rho_{fluid} g} + \frac{1}{2g}\left(\frac{A_2 v_2}{A_1}\right)^2 = \frac{P_{2,static} g_c}{\rho_{fluid} g} + \frac{1}{2g}(v_2)^2$$

$$\frac{1}{2} v_2^2 \left(\left(\frac{A_2}{A_1}\right)^2 - 1\right) = \frac{P_{2,static} g_c}{\rho_{fluid}} - \frac{P_{1,static} g_c}{\rho_{fluid}}$$

$$v_2 = \sqrt{2 g_c \frac{\left(\frac{P_{2,static}}{\rho_{fluid}} - \frac{P_{1,static}}{\rho_{fluid}}\right)}{\left(\left(\frac{A_2}{A_1}\right)^2 - 1\right)}}$$

The Venturi meter is primarily used to determine liquid flow rate, so the velocity term will be multiplied by the area.

$$Q = C_v A_2 \sqrt{2g_c \frac{\left(\frac{P_{2,static}}{\rho_{fluid}} - \frac{P_{1,static}}{\rho_{fluid}}\right)}{\left(\left(\frac{A_2}{A_1}\right)^2 - 1\right)}}$$

This equation can be rewritten in the same format as shown in the *NCEES FE Reference Handbook*.

$$Q = C_v A_2 \sqrt{2g_c \frac{\left(\frac{P_{1,static}}{\rho_{fluid}} - \frac{P_{2,static}}{\rho_{fluid}}\right)}{\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}}$$

6.0 EXTERNAL FLOW

External flow is the flow of a fluid over an object. Similar to internal flow, fluid will begin at an even velocity profile, this is called the free stream velocity and is shown at the beginning of the flat plate below. As the fluid flows over the object, the friction from the surface of the object will create a velocity profile, where the velocity on the surface is 0, and the velocity continues to increase as you move away from the surface.

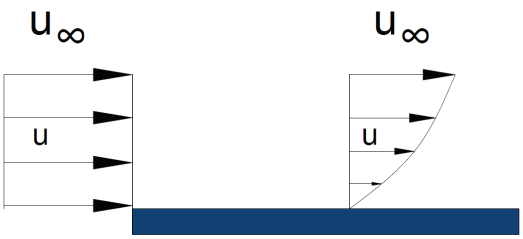


Figure 17: External flow is characterized by a fluid flowing over a surface.

Based on how the fluid hits the object, the external flow of the fluid over a surface will create drag and/or lift forces on the object. The flow over an object can also be defined as laminar, transitional, or turbulent for external flows. For a flat plate, the flow starts off as laminar, then eventually transitions to turbulent flow.

6.1 REYNOLDS NUMBER

The Reynolds number for external flow is similar, except that instead of the internal diameter, D, the Reynolds number is dependent on the length of the surface for a flat plate and the external diameter for a cylinder or a sphere. The Reynolds number is used to calculate the drag coefficient of an object.

For a flat plate, the Reynolds number is solved from the following equation.

$$Reynolds\ number\ (Re)_{Flat\ Plate} = \frac{v * L}{\nu(nu)} = \frac{\rho * v * L}{\mu}$$

The critical Reynolds number may also vary based on external factors, but generally for a flat plate, the transitional periods occur at the values below.

$$Re < 100{,}000\ \ (Laminar\ Flow)$$
$$100{,}000\ \leq\ Re\ \leq\ \ 3{,}000{,}000\ \ (Transitional\ Flow)$$
$$Re\ >\ \ 3{,}000{,}000\ \ (Turbulent\ Flow)$$

For a cylinder or sphere, the Reynolds number is solved from the following equation.

$$Reynolds\ number\ (Re)_{Flat\ Plate} = \frac{v * D_{ext}}{\nu(nu)} = \frac{\rho * v * D_{ext}}{\mu}$$

$$Re < 200{,}000\ \ (Laminar\ Flow)$$
$$200{,}000\ \leq\ Re\ \leq\ \ 2{,}000{,}000\ \ (Transitional\ Flow)$$
$$Re\ >\ \ 2{,}000{,}000\ \ (Turbulent\ Flow)$$

(Note: to distinguish between velocity v and kinematic viscosity, ν, kinematic viscosity will follow with $\nu(nu)$ for clarification. Be sure to do all unit conversions so that the Re is unit less.)

$\nu(nu) = kinematic\ viscosity\ (m^2/s\ \ or\ \ ft^2/s)$
$\mu(nu) = dynamic\ viscosity\ (kg/(m * s)\ \ or\ \ lbm/(ft * s))$
$v = average\ velocity\ of\ fluid\ (m/s\ \ or\ \ ft/s)$
$L = length\ of\ flat\ plate\ (m\ \ or\ \ ft)$
$D_{ext} = exterior\ diameter\ of\ sphere\ or\ cylinder\ (m\ \ or\ \ ft)$

6.2 DRAG

As a body moves through a fluid (air or water), there are resistive forces that act upon the body, similar to friction. There are two types of drag, surface drag and form drag. Surface drag depends on the smoothness of the force, the friction factor. Form drag depends on the shape of the object as in its aerodynamic shape.

$$F_D = C_D * \rho * A * \frac{v^2}{2}$$

$$where\ C_D = coefficient\ of\ drag;\ \rho_{fluid} = fluid\ density;$$

$$A = front\ area\ of\ object; v = velocity\ of\ fluid$$

The area in the equation above is the projected 2D area when and not necessarily the total surface area that is perpendicular to the fluid flow. For example, the area used if the object were a sphere would be equal to the area of a circle and not half the surface area of a sphere.

The projected area of a pipe would equal to the area of a rectangle, the diameter times the length of the pipe.

The drag coefficient can be found with the Reynolds number. The formula to find the drag coefficient of a flat plate can be found in the *NCEES FE Reference Handbook*. For other objects, this number will most likely be provided.

6.3 LIFT

The lift force acts in a direction that is perpendicular to the relative flow. Lift is created due to the difference in pressure on opposite sides of the object. This is because of the difference in velocity between the fluid on the top and the bottom of the object. If the velocity is faster on the top of the object, this will result in a lower pressure and vice versa, if the velocity is slow on the bottom of the object, this will result in a higher pressure. The net pressure effect will then point upwards.

$$F_L = C_L * \rho * A * \frac{v^2}{2}$$

$$where\ C_L = coefficient\ of\ lift;\ \rho_{fluid} = fluid\ density;$$

$$A = area\ of\ object; v = velocity\ of\ fluid$$

The coefficient of lift is dependent on many factors, like angle of attack and the shape of the body. The derivation of this coefficient does not seem applicable to the majority of those practicing in the Thermal & Fluids field and it is assumed that this coefficient will be given in these types of problems.

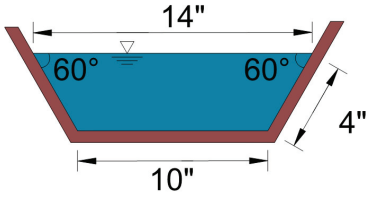
7.0 OPEN-CHANNEL

Open channel flows pertain to flows that are not in a fully enclosed tube and are instead open to atmosphere at the top surface. Since an open-channel flow cannot be pressurized, pressures and flows are mainly dependent on gravity. The Reynolds number for channels follow the same equation as that described in the internal flow topic. One important component to distinguish is that the effective diameter will be calculated based on the hydraulic diameter equation, which is based on the surface that comes into contact with the fluid. The hydraulic radius is one-fourth of the hydraulic diameter.

$$D_h = \frac{4 * Cross\ Sectional\ Area}{Wetted\ Perimeter}$$

$$R_h = \frac{D_h}{4}$$

Example: Find the hydraulic diameter of the following open channel flow.



Solution: the hydraulic diameter is equal to 4 times the area divided by the perimeter in contact with the fluid.

$$D_h = \frac{4 * (10 * 2\sqrt{3} + 4 * 2 * \frac{1}{2})}{4 + 4 + 10} = 16.3\ inches$$

7.1 MANNING EQUATION

For channels with where the depth of the flow is constant, the average velocity is constant. This is characterized as uniform flow. For uniform flow, which is typically obtained with long runs of a channel at a constant slope, the velocity can be found with the Manning equation.

$$V = \frac{a}{n} R_h^{2/3} S^{1/2}$$

$$Q = A * \frac{a}{n} R_h^{2/3} S^{1/2}$$

Where,

$V = velocity\ (m/s\ \ or\ \ ft/s)$
$Q = volumetric\ flow\ (m^3/s\ \ or\ \ ft^3/s)$
$R_h = hydraulic\ radius\ (m\ \ or\ \ ft)$
$S = channel\ slope\ (unitless)$
$a = 1\ m^{1/3}/s\ \ or\ \ 1.49\ \ ft^{1/3}/s, \qquad constant$
$n = Manning\ Coefficient, based\ on\ channel\ surface\ roughness$
$A = cross\ sectional\ area\ of\ channel\ flow\ (m^2\ \ or\ \ ft^2)$

The Manning Coefficient, n, is 0.012 for smooth steel, 0.012 for finished concrete, and 0.014 for unfinished concrete.

8.0 COMPRESSIBLE FLOW

In reality, all fluids are compressible to some extent. A compressible fluid is defined as a fluid that changes in density when the fluid changes in pressure or temperature. On the exam and in practice, a distinction is made between compressible and incompressible fluids because it makes calculations simpler. The majority of the calculations on the exam will be for incompressible flow. The only exceptions will be when the exam clearly indicates compressible flow or when compressible flow is understood to take place in certain conditions. These conditions include any questions on Mach number or nozzles.

8.1 COMPRESSIBLE FLUID

A compressible fluid will reduce its volume in the presence of an external pressure. The quantitative measurement of the compressibility is taken as the relative volume change of the liquid in response for a pressure change.

There are two types of compressibility that you should know for the exam. Compressibility is defined in two forms. *Isothermal compressibility* describes the compressibility of the system when the temperature of the system is constant. This is denoted by βV. *Adiabatic compressibility* refers to the compressibility measured under no energy transfer between the system and the surroundings. This is denoted by βS. Since an adiabatic process is also isentropic, this process is a constant entropy process.

8.2 MACH NUMBER

Mach number is the ratio of speed to sound. It is a value between 0 and infinity. Transonic, subsonic, supersonic, hypersonic, hypervelocity.

$$M = \frac{u}{c}$$

$$M = Mach\,number; u = velocity\left[\frac{ft}{sec}\right]; c = speed\ of\ sound$$

8.2.1 Speed of Sound

The speed of sound in an ideal gas is found through the following equation, which is dependent solely on absolute temperature.

$$c = \sqrt{k * R_s * T}$$

$$M = Mach\,number; k = ratio\ of\ specific\ heats\left[\frac{ft}{sec}\right]; R_s = Specific\ gas\ constant;$$

$$T = temperature\ in\ Rankine$$

Air Properties		Units
R_s	53.35	$\left[\frac{ft-lbf}{lbm-°R}\right]$
k	1.4	N/A

Figure 18: Universal gas constant for air. The gas constant will vary based on the gas.

The speed of sound is important because it helps to determine when a fluid is compressible or incompressible. Once the speed of a fluid is out of the incompressible range, then many of the simplified equations used in Thermodynamics and Fluids cannot be used. Instead, compressible equations must be used.

Description	Mach Number Range
Incompressible	0 to 0.3
Sub-sonic	0.3 to 1
Speed of sound	1
Super sonic	1 to 5

Figure 19: Incompressible fluids have a Mach number less than 0.3.

On the exam, you should assume incompressible fluid unless told otherwise.

8.3 NOZZLES

Nozzles are used to create changes in velocities and pressures of a moving fluid. A nozzle in its simplest form increases the velocity of a fluid by reducing the area, which also increases the fluids pressure.

The nozzle is an important part of the exam and if you can understand what the fluid is doing through the nozzle, then you will be in a good position to get these types of questions correct.

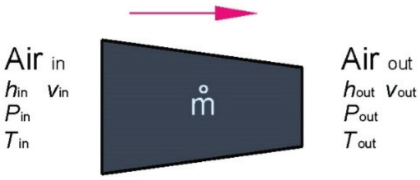


Figure 20: A nozzle increases the velocity pressure of the fluid, decreases temperature and increases the velocity of the fluid. The static pressure decreases.

The total pressure at any location will be the sum of the fluid's static pressure and its velocity pressure. The static pressure is the pressure that acts outward (perpendicular to the direction of flow) and velocity pressure is the pressure that acts in the direction of flow. There is a third type of pressure called elevation pressure which will be discussed with Bernoulli's equation later in this section.

$$\frac{P_{1,static}g_c}{\rho_{air}g}[Static\ pressure] + \frac{1}{2g}v_1^2[Velocity\ pressure] = \frac{P_{2,static}g_c}{\rho_{air}g} + \frac{1}{2g}v_2^2\ \ [IP]$$

$$P_{static} = static\ pressure\ at\ a\ point; v = velocity;$$

You can see that as the velocity increases, the velocity pressure increases. This means that the static pressure has to decrease in order to satisfy Bernoulli's equation between points 1 and 2. The total pressure (static + velocity) must be equal to one another between points 1 and 2. This is also based on the assumption that there is no friction loss between points 1 and 2.

8.3.1 Conservation of energy & mass

One important concept to understand for both nozzles and diffusers is that energy is assumed to be conserved as the fluid passes through the nozzle. Any kinetic energy change (change in velocity) must be accounted for in a change in internal energy. For example, an increase in velocity will result in a decrease in temperature and pressure. When solving nozzle equations, you should use these three equations, (1) conservation of energy, (2) conservation of mass and (3) the ideal gas law.

$$\Delta Q = 0$$

$$q_{in} + h_{in} + \frac{1}{2}v_{in}^2 = q_{out} + h_{out} + \frac{1}{2}v_{out}^2$$

$$where\ q_{in/out} = any\ heat\ entering\ or\ exiting\ the\ system$$

$$h_{in/out} = enthalpy\ at\ the\ entrance\ and\ exit\ of\ the\ nozzle$$

$$v_{in/out} = velocity\ at\ the\ entrance\ and\ exit\ of\ the\ nozzle$$

The second concept is that no mass is created or destroyed through a nozzle or diffuser.

$$\Delta m = 0$$

$$\rho_{in} * A_{in} * v_{in} = \rho_{out} * A_{out} * v_{out}$$

$$A_{in/out} = area\ at\ the\ entrance\ and\ exit\ of\ the\ nozzle$$

$$\rho_{in/out} = velocity\ at\ the\ entrance\ and\ exit\ of\ the\ nozzle$$

The third concept is that the fluid is an ideal gas and follows the following relationship.

$$p = \rho * R * T$$

8.3.2 Converging-Diverging Nozzle

A more complex, converging-diverging nozzle consists of a converging portion that is used to raise the pressure of the fluid and then a throat section, followed by a diverging section. The diverging section is a location of low pressure and in application the fluid in the diverging section is typically released into ambient conditions or a tank.

The pressure in the chamber is the starting point of the fluid. At the chamber, the fluid is at a certain pressure, temperature and density and is labeled with the subscript *"o"*. The chamber in reality can be a tank, pipe or any vessel that is much larger than the nozzle.

The velocity through the nozzle is controlled by the pressure difference between the chamber and the ambient conditions. If the pressure in the chamber is increased then the velocity through the nozzle will be increased, but up to a certain point. This point is Mach 1. Once the fluid reaches this point, any increase in the chamber pressure, will not result in an increased velocity through the nozzle, because the fluid is choked at the nozzle.

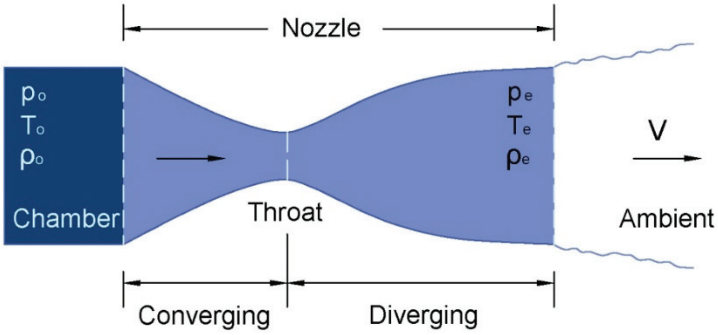


Figure 21: Converging diverging nozzle

8.4 DIFFUSERS

Diffusers are the opposite of nozzles. Diffusers decrease the pressure of the fluid by reducing the velocity.

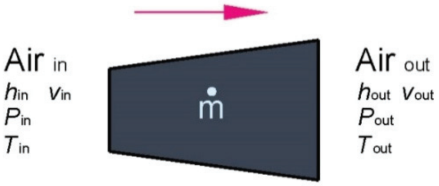


Figure 22: A diffuser decreases the velocity pressure of the fluid, increases temperature and decreases the velocity of the fluid. The static pressure increases.

You can see that as the velocity decreases, the velocity pressure decreases. This means that the static pressure has to increase in order to satisfy Bernoulli's equation between points 1 and 2.

$$\frac{P_{1,static}g_c}{\rho_{air}g}[Static\ pressure]+\frac{1}{2g}v_1^2[Velocity\ pressure]=\frac{P_{2,static}g_c}{\rho_{air}g}+\frac{1}{2g}v_2^2\ \ [IP]$$

$$P_{static}=static\ pressure\ at\ a\ point;\ v=velocity;$$

8.5 BULK MODULUS

The term Bulk Modulus is a property of a fluid that describes the compressibility of the fluid. Bulk modulus, β, is defined in the equation below.

$$\beta = \left(\frac{\Delta P}{\Delta V/V}\right); psi$$

In this equation, V is the original volume of the fluid and delta V is the change in volume, when a pressure change is applied to the fluid. Bulk modulus has units of pressure. For example, if a hydraulic fluid has a bulk modulus of 200,000 psi and a pressure of 1,000 psi applied to the fluid (V = 10 in^3), then the change in volume can be found with the below equation.

$$\beta = \left(\frac{\Delta P}{\Delta V/V}\right); psi$$

$$\Delta V = \left(\frac{\Delta P * V}{\beta}\right) = \left(\frac{1{,}000\ psi * 10\ in^3}{200{,}000\ psi}\right)$$

$$\Delta V = 0.05\ in^3$$

As you can see from the previous example, if a fluid has a smaller bulk modulus then the fluid is more compressible. If the fluid has a larger bulk modulus then the fluid is less compressible. Compressibility and the bulk modulus are inversely related.

9.0 INCOMPRESSIBLE FLOW

As previously discussed, incompressible fluids do not occur in the real world. Incompressible fluids were created to describe a range of fluids, in order to make calculations simpler. The calculations are simpler because incompressible fluids are assumed. An incompressible fluid is a fluid that does not change the volume of the fluid due to external pressure. Most of the basic calculations done in fluid dynamics are done assuming the fluid is incompressible. The approximation of incompressibility is acceptable for most of the liquids as their compressibility is very low. However, the compressibility of gases is high, so gases cannot be approximated as incompressible fluids. The compressibility of an incompressible fluid is always zero.

9.1 FRICTION LOSS – DARCY WEISBACH

Friction loss is found through the use of either the Darcy Weisbach equation or the Hazen-Williams equation. The Darcy Weisbach equation is slightly more involved and will be explained below, starting with the equation. The Hazen-Williams equation is explained in an upcoming topic.

$$h = \frac{fLv^2}{2Dg}\ [Darcy\ Weisbach\ Equation]$$

$$where\ h = ft\ of\ head; f = Darcy\ friction\ factor; v = velocity\ \left[\frac{ft}{sec}\right],$$

$$D = inner\ diameter\ [ft], g = gravity\ [32.2\frac{ft}{sec^2}]$$

For laminar flow through a circular tube, the friction factor is simply found as 64 divided by the Reynolds number.

$$f_{pipe,laminar} = \frac{64}{Re}$$

For turbulent flow, the calculations require a few more steps and are discussed below. During the exam, in order to quickly complete a friction loss question using the Darcy Weisbach Equation, know where all the necessary tables, graphs, and equations are in the *NCEES FE Reference Handbook*.

9.1.1 Inner Pipe Dimensions

In practice, pipes are labeled by their *nominal* diameters, which is the approximate the pipe size. However, the *actual* inner and outer diameters will vary based on the pipe type. Common pipe types are as follows.

- Schedule 40/80 Steel
- Type K, L, and M Copper Piping
- Schedule 40/80 PVC

The friction loss calculations are interested in the inner diameter, since this is what the fluid encounters. The *NCEES FE Reference Handbook* does not include pipe data, so the inner diameter must be given in the problem. If the diameter type is not specified, you can assume they are referencing the inner diameter.

9.1.2 Flow unit conversions.

In English units, volumetric flow rates are typically given in GPM and need to be converted to cubic feet per second for use in the Darcy Weisbach equation.

$$Multiply\ GPM\ by\ \frac{1}{448.83}\ to\ get\ \frac{FT^3}{sec}.$$

Once the volumetric flow rate is converted to cubic feet per second, then simply divide the value by the inner area of the pipe to find the velocity of the fluid through the pipe.

9.1.3 Kinematic viscosity tables of water at various temperatures

The kinematic viscosity is used to find the Reynolds number.

Water Temperature [°F]	Kinematic Viscosity [ft^2/s]
32	1.924×10^{-5}
40	1.664×10^{-5}
50	1.407×10^{-5}

60	1.210 x 10^{-5}
70	1.052 x 10^{-5}
90	0.823 x 10^{-5}
100	0.738 x 10^{-5}
120	0.607 x 10^{-5}
140	0.511 x 10^{-5}
180	0.383 x 10^{-5}
212	0.317 x 10^{-5}

9.1.4 Pipe Roughness

Collect pipe roughness factors ε for common pipe materials, steel, PVC, copper, etc. These values are listed in the Moody diagram in the *NCEES FE References Handbook.*

Material	Roughness (ft)
Steel	.0009833
PVC	.000005
Copper	.000005

Select the roughness factor based on the pipe material, then find the relative roughness factor by dividing the internal diameter of the pipe. Make sure the roughness factor, ε, and the inner diameter, D, are in the same units.

$$Relative\ Roughness\ Factor\ (Unitless) = \frac{\varepsilon}{D}$$

9.1.5 Moody Diagram.

The Moody diagram uses the (1) Reynolds number and the (2) relative roughness factor to determine the friction factor for turbulent flow.

Find the friction factor by locating the intersection of the vertical Reynolds number line shown in black and the Relative Roughness factor curves shown in red.

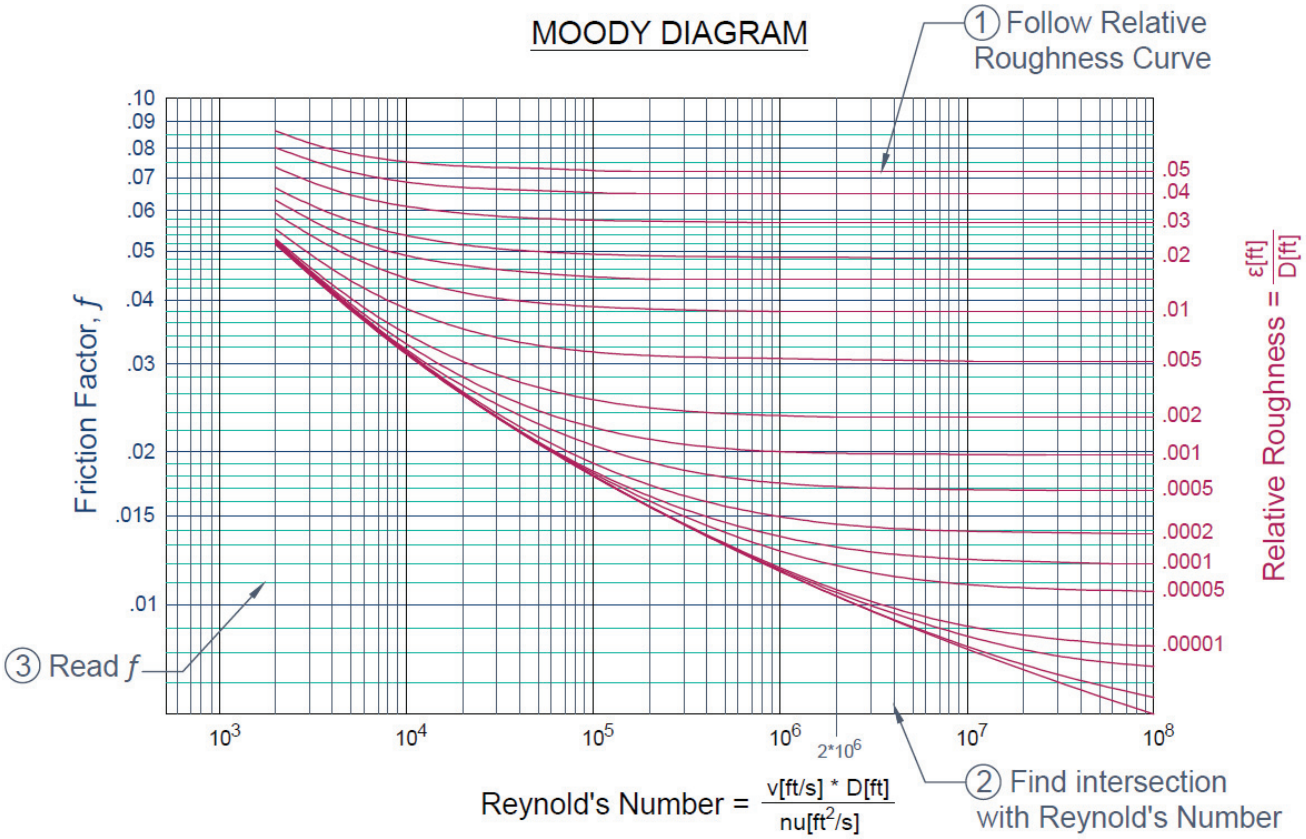


Figure 23: Step 1: Find relative roughness factor, Step 2: Find the intersection of Reynolds number and relative roughness factor. Step 3: Read corresponding friction factor.

Now, that all the variables of the Darcy Weisbach equation have been determined, simply plug in the variables into the equation to determine the friction head.

$$h_f = \frac{fLv^2}{2Dg} \; [Darcy\ Weisbach\ Equation]$$

Friction losses at fittings, entrances, and exits can be calculated either through an equivalent length of pipe that would be added to the total pipe length, L, in the above equation or it can be calculated with a friction coefficient for specific fitting types.

$$h_{f,fitting} = C * \frac{v^2}{2g}$$

The losses due to fittings is added to the losses at the secondary side of Bernoulli's Equation.

$$\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f + h_{f,fitting} \qquad [SI]$$

$$\frac{P_1 g_c}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{P_2 g_c}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f + h_{f,fitting} \qquad [IP]$$

9.2 FRICTION LOSS – HAZEN-WILLIAMS EQUATION

The Hazen-Williams equation is a simplified pressure drop equation that is used in the civil engineering and some mechanical engineering fields. This equation is not accurate for laminar flow and for extremely turbulent flow. The equation is an empirical formulation, meaning that it was created based on experiments and curve fit data. This makes this equation very useful for the typical velocities of 2 to 10 feet per second, because this is the range for which the data fit best.

The equation is a function of the hydraulic diameter, the material roughness coefficient and the flow rate. The roughness coefficients for various materials can be found in the Civil Engineering section in the *NCEES FE Reference Handbook*. Make sure you use the correct version, US or SI units.

US Units:

$$\Delta P_{ft\ head\ per\ 1\ ft}^{0.54} = \frac{V}{kCR^{0.63}}$$

$$\Delta P_{ft\ head\ per\ 1\ ft} = pressure\ drop\ (ft\ per\ ft\ of\ pipe); V = velocity\ \left(\frac{ft}{s}\right)$$

$$C = roughness\ coefficient;\ R = hydraulic\ radius\ (ft); k = 1.318$$

$$Q = volumetric\ flow\ rate\ \left(\frac{ft^3}{s}\right); A = cross\ sectional\ area\ (ft^2)$$

SI Units:

$$\Delta P_{m\ head\ per\ 1\ m}^{0.54} = \frac{V}{kCR^{0.63}} = \frac{Q/A}{kCR^{0.63}}$$

$$\Delta P_{m\ head\ per\ 1\ m} = pressure\ drop\ (m\ per\ m\ of\ pipe); V = velocity\ \left(\frac{m}{s}\right)$$

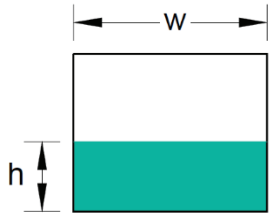
$$C = roughness\ coefficient;\ R = hydraulic\ radius\ (m); k = 0.849$$

$$Q = volumetric\ flow\ rate\ \left(\frac{m^3}{s}\right); A = cross\ sectional\ area\ (m^2)$$

As mentioned in the previous topics, the hydraulic diameter is found by taking the cross sectional area of flow and divided it by the wetted perimeter.

$$Hydraulic\ Diameter = D_h = \frac{4A}{P}$$

$$where\ A = cross\ sectional\ area\ of\ fluid; P = wetted\ perimeter$$



$$Example \rightarrow D_h = \frac{4 * (w * h)}{2w + 2h} = \frac{2wh}{w + h}$$

The hydraulic radius is not one-half the hydraulic diameter, it is actually 1/4 the hydraulic diameter.

$$Hydraulic\ Radius = R_h = \frac{A}{P}$$

$$Example \rightarrow R_h = \frac{(w * h)}{2w + 2h} = \frac{1}{2} * \frac{wh}{w + h}$$

9.3 FLUID POWER

A large portion of the Thermal & Fluids field is using fluids to transmit power in hydraulic and pneumatic industrial systems. Pneumatics is the transfer of energy via compressed air and hydraulics is the transfer of energy via hydraulic fluid like oil. For example, pneumatics is used in the medical field, construction, manufacturing and packaging. Hydraulic fluids are used heavily in the construction and industrial industries to power large equipment like tractors, cranes, excavators, etc. This section of the book provides you a basic understanding of the engineering principles behind both hydraulics and pneumatics.

The first principle is *Pascal's Law*. This law states that pressure is transmitted undiminished in all directions and exerts equal force on all areas for a confined fluid at rest. This means that anywhere within a circuit, pipe, pressure vessel, the fluid's pressure is equal. Also be sure that the fluid is confined at rest. If the fluid is moving, then friction losses will occur.

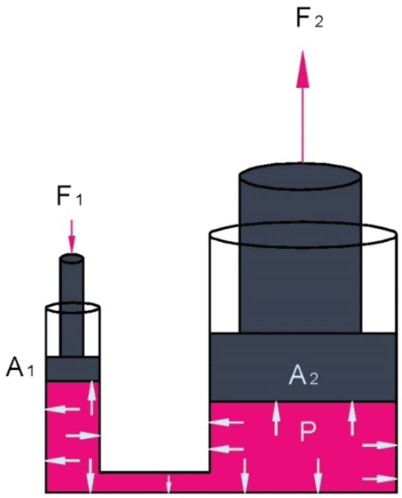


Figure 24: The pressure acting upon all surfaces by the fluid in pink is equal, in accordance with Pascal's law.

$$P = \frac{F}{A}$$

$$P = pressure\left(\frac{lbs}{in^2}\right);\ F = force\ (lbs);\ A = \pi r^2 = area\ (in^2)$$

Here are some conversions that you should be familiar with.

	Pascal (Pa)	Megapascal (MPa)	Bar	Lbs-in^2 (psi)
1 Pa	1	10^{-6}	10^{-5}	$145.04 \text{ x } 10^{-6}$
1 MPa	10^6	1	10	145
1 Bar	10^5	0.1	1	14.5
1 Psi	6,895	$6.895 \text{ x } 10^{-3}$	.06895	1

The second governing law of hydraulic power is Boyle's law. This law states that for a fixed amount of an ideal gas at a fixed temperature that the pressure and volume are inversely related.

$$P_1 * V_1 = P_2 * V_2$$

The final law governing hydraulic power is Charles' law. This law states that if an unconfined volume of liquid or gas is heated, then the volume will expand. If the volume is confined, then instead of the volume expanding, then the pressure will increase.

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

10.0 POWER AND EFFICIENCY

10.1 PUMP POWER

10.1.1 PUMP WORK

The required pump work is calculated from the amount of energy that needs to be added to a system to meet final pressure, velocity, elevation, and to overcome all friction loss requirements. The pump energy is derived from Bernoulli's Equation, where 2 indicates the most hydraulically remote point in a system and 1 indicates the start point of the pumping system. The friction losses will include any friction losses through piping, fittings, equipment, valves, and other miscellaneous losses.

$$H_{pump}(m) = \frac{P_2 - P_1}{\rho g} + \frac{v_2^2 - v_1^2}{2g} + (z_2 - z_1) + h_f + h_{f,fittings} + h_{equip.,misc}$$

$$H_{pump}(ft) = \frac{(P_2 - P_1)g_c}{\rho g} + \frac{v_2^2 - v_1^2}{2g} + (z_2 - z_1) + h_f + h_{f,fittings} + h_{equip.,misc}$$

This equation is simplified by converting all static and dynamic losses, elevation gains, and pressure differentials to find the total *head loss, H,* in terms of feet or meters.

$$H\ (ft\ or\ m) = Final\ Pressure - Starting\ Pressure + Dynamic\ Head + Elevation\ Gains + Friction\ Losses$$

$$1\ psi_{water} = 2.31\ ft_{water}$$

$$9.8\ kPa_{water} = 1\ meter_{water}$$

Finally, solve for the pump work using the simplified version of Bernoulli's Equation. The power produced by the pump, W_{pump}, is called the *Mechanical or Fluid Power*.

$$\boldsymbol{W_{pump,metric} = Q\rho g H; \quad W_{pump,IP} = Q\rho \frac{g}{g_c}\ H * \left(\frac{1\ HP}{550\ lbf \cdot ft/s}\right)}$$

$Q = volumetric\ flow\ rate\ (m^3/s\ or\ ft^3/s)$
$\rho = density\ (kg/m^3\ or\ \ lbm/ft^3)$
$g = gravity\ (9.81\ m/s^2\ or\ 32.2\ ft/s^2)$
$g = gravitational\ constant\ (32.2\frac{lbm-ft}{lbf-s^2})$
$H = total\ head\ (m\ or\ ft)$
$W_{pump} = Pump\ Power\ (W\ = N \cdot m/s\ \ or\ HP)$

This equation is further simplified in terms of gallons per minute and feet of head, where SG is the specific gravity of the fluid.

$$W_{pump} = \frac{SG \cdot Q_{Gallons\ per\ Minute} \cdot H_{feet}}{3{,}956}$$

10.1.2 Pump Efficiency

When calculating pump power, there are two types of efficiencies to consider: (1) Pump Efficiency and (2) Motor Efficiency. Pump efficiency accounts for all the losses from the mechanical components of the pump, such as the inefficiencies from the pump impeller, bearings, seals, etc. The power required to overcome all pump efficiencies is called the *pump or brake power*. Motor efficiencies account for the electrical losses through the motor coils. The power to overcome the motor efficiencies is called the *motor or purchased power*. This is the power supplied by the electrical circuits to the motor-pump assembly.

$$Mechanical\ Power = W_{pump}$$

$$Brake\ (Pump)\ Power = W_{brake} = \frac{W_{pump}}{\eta_{pump}}$$

$$Motor\ (Purchased)\ Power = W_{motor} = \frac{W_{brake}}{\eta_{motor}} = \frac{W_{pump}}{\eta_{motor} \cdot \eta_{pump}}$$

Where η_{pump} is the pump efficiency and η_{motor} is the motor efficiency.

Finally, when selecting a motor, realize that motors come in standard sizes. In terms of horsepower, motors come in [1, 1.5, 2, 3, 5, 7.5, 10, 15, 20, 25, 30, 40, 50, etc]. The motor power is calculated, then rounded up to nearest motor size. If the question explicitly asks for the motor horsepower in standard size then round up to the nearest motor size. If the question does not ask for standard motor size, then simply provide the motor power.

10.2 FAN POWER

10.2.1 Fan Work

Fan work and efficiency follow the same principals as pumps, except that instead of a liquid, the fluid is a gas, typically air. Since air is minimally affected by elevation gains, the potential energy term can be neglected.

$$P_{fan}(Pa) = (P_2 - P_1) + \frac{\rho(v_2^2 - v_1^2)}{2} + \rho g\left[h_f + h_{f,fittings} + h_{equip.,misc}\right] \quad [SI]$$

$$P_{fan}(psi) = (P_2 - P_1) + \frac{\rho(v_2^2 - v_1^2)}{2g_c} + \frac{\rho g}{g_c}\left[h_f + h_{f,fittings} + h_{equip.,misc}\right] \quad [IP]$$

This equation is simplified by converting all friction losses, dynamic losses, and pressure differentials to find the total *pressure loss, ΔP,* in terms of Pa, psi or inches of water gauge. Typically the starting and final pressure in fans will also be the same, atmospheric.

$$\Delta P(Pa, psi, in\ wg.) = Final\ Pressure - Starting\ Pressure + Dynamic\ Head + Friction\ Losses$$

In English units, pressure losses through the fan are typically represented in inches of water (in wg). It is more likely that the FE exam will use metric units or psi for these calculations, but the conversion is presented here for your exposure, since it is an industry standard.

$$1\ psi = 0.036\ in\ wg.$$

Once the total head loss is found, solve for the fan work using the simplified version of Bernoulli's Equation. The power produced by the fan, W_{fan}, is called the *Mechanical or Fluid Power*.

$$\boldsymbol{W_{fan} = \Delta PQ}$$

$Q = volumetric\ flow\ rate\ (m^3/s\ or\ ft^3/s)$
$\Delta P = total\ pressure\ difference\ (Pa, psi, or\ in\ wg)$
$W_{fan} = fan\ power\ (W = Nm/s\ \ or\ \ HP)$

This equation is further simplified in terms of industry standards, using cubic feet per minute for the flow rate and in wg for the pressure drop.

$$W_{fan} = \frac{\Delta P_{in.wg} * Q_{Cubic\ Feet\ per\ Minute}}{6{,}356}$$

10.2.2 Fan Efficiency

Fan efficiencies follow the same process as pump efficiencies. Like pumps, there are two types of efficiencies to consider: (1) Fan Efficiency and (2) Motor Efficiency. Fan efficiency accounts for all the losses from the mechanical components of the fan, such as the inefficiencies from the fan blades, bearings, seals, etc. The power required to overcome all fan efficiencies is called the *fan or brake power*. Motor efficiencies account for the electrical losses through the motor coils. The power to overcome the motor efficiencies is called the *motor or purchased power*. This is the power supplied by the electrical circuits to the motor-pump assembly.

$$Mechanical\ Power = W_{fan}$$

$$Brake\ (Fan)\ Power = W_{brake} = \frac{W_{fan}}{\eta_{fan}}$$

$$Motor\ (Purchased)\ Power = W_{motor} = \frac{W_{brake}}{\eta_{motor}} = \frac{W_{fan}}{\eta_{motor} \cdot \eta_{fan}}$$

Where η_{fan} is the fan efficiency and η_{motor} is the motor efficiency.

Finally, motors for a fan or pump are similar. Installed motors will be of a standard size as indicated in the Pump Efficiency section. When selecting a motor, round up to the next standard size.

10.3 COMPRESSORS

10.3.1 Compressor Work

A common skill, that is required of an engineer, is to determine the work done by the compressor. This work is shown as the difference between the compressor entering enthalpy (H1) and the leaving enthalpy (H2). The equation to determine the work of the compressor is shown below. This equation multiples the mass flow rate by the change in enthalpy between the discharge and suction conditions. This will also be covered in *Section 12.0 Thermodynamics*, since compressors are key components in refrigeration and power cycles.

$$Q\ [Btu/hr] = \left(H_{leaving} - H_{entering}\right)\left[\frac{Btu}{lb}\right] * (mass\ flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_{entering} = entering\ compressor\ enthalpy\ [\frac{Btu}{lb}]$$

$$H_{leaving} = leaving\ compressor\ enthalpy\ [\frac{Btu}{lb}]$$

This work can also be shown by the difference in temperature. Use the equation below to determine the work of the compressor.

$$Q\ [Btu/hr] = (c_p)\left[\frac{Btu}{lbm°\mathrm{F}}\right] * (T_{leaving} - T_{entering})\ [°\mathrm{F}] * (mass\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$T_{entering} = entering\ compressor\ temperature\ [°\mathrm{F}];$$

$$T_{leaving} = leaving\ compressor\ temperature\ [°\mathrm{F}];\ c_p = heat\ capacity\ of\ gas$$

Work can also be expressed in terms of pressure and temperature by using the relationships

The equations above can be changed to non-isothermal conditions, but still assumes adiabatic compression.

$$W\ = \left(\frac{k * \dot{m} * P_{in}}{k-1}\right) * \left(\left(\frac{P_{exit}}{P_{in}}\right)^{\frac{k-1}{k}} - 1\right) \rightarrow assumes\ adiabatic\ compression$$

$$W\ [HP] = 0.0.0642 * \left(\frac{k * CFM}{k-1}\right) * \left(\left(\frac{P_{outgoing}}{P_{incoming}}\right)^{\frac{k-1}{k}} - 1\right) \rightarrow assumes\ adiabatic\ compression$$

$$where\ n = polytropic\ exponent;\ k = 1.4\ when\ the\ process\ is\ adiabatic\ for\ air$$

10.3.2 Compressor Efficiency

Each compressor is provided with power, which is typically electricity. The compressor efficiency is found by dividing the actual work output calculated in the previous section by the input electricity power.

$$Efficiency_{compressor} = \frac{Work\ Out}{Power\ In}$$

The hardest parts about solving for the compressor efficiency is making sure your units are correct and finding the enthalpy values.

11.0 Performance Curves

Performance curves are provided for all centrifugal pumps, fans and compressors. Only fans and pumps are covered in this section, while compressors are covered in more detail in Section *12.0 Thermodynamics*.

11.1 FAN CURVES

The fan curve is a graph depicting the various points that the fan can operate. The curve displays the amount of CFM the fan will provide at a given total static pressure. Fans should be selected to operate at the stable region. The stable region is the area on the fan curve where there is a single flow rate [CFM] value for each pressure value. In the unstable region, a pressure value can have multiple CFM values, which will cause the fan system to surge. The stable region also has very little change in CFM for large changes in total pressure.

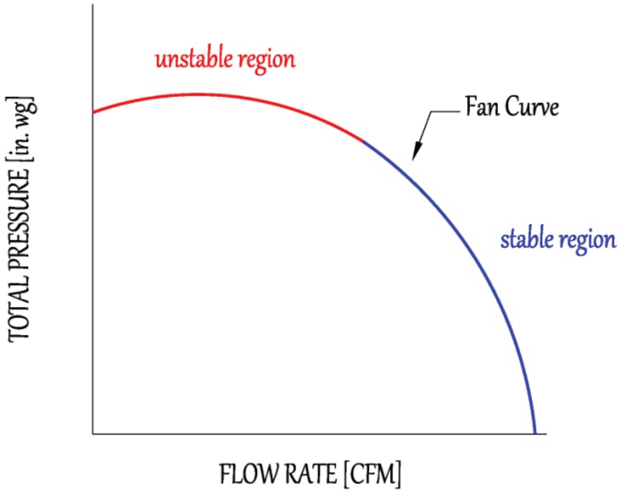


Figure 25: Fan curve

The second curve that works in conjunction with the fan curve is the system resistance curve. This curve is the summation of all the friction losses in the ducting system at varying CFM's. Typically, the friction losses are summed up at the design CFM values, then this design point is connected to the 0,0 point by an upward sloping square polynomial curve, as shown below [green]. If for example, the ducting system has a closed damper or dirty filter, this will cause the curve to shift to the left [red]. If a damper is opened or the dirty filter is cleaned then the curve will shift to the right [blue].

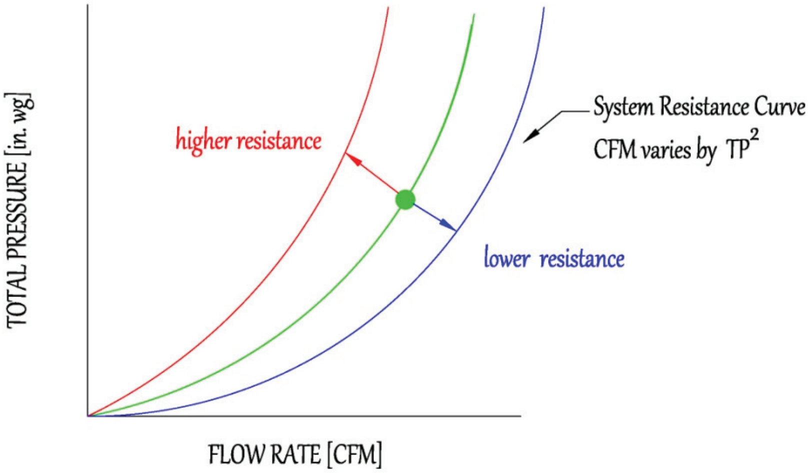


Figure 26: System resistance curve

Combining the system curve with the selected fan curve, determines the operating point of the fan system, indicated in the figure below in green. Following the vertical line down determines the CFM and the horizontal line from that point indicates the operating total pressure. During system operation as dampers close, the system curve shifts toward the left in red. This movement decreases the amount of CFM delivered by the fan. The opposite occurs as dampers open in the system, the amount of CFM delivered by the fan increases.

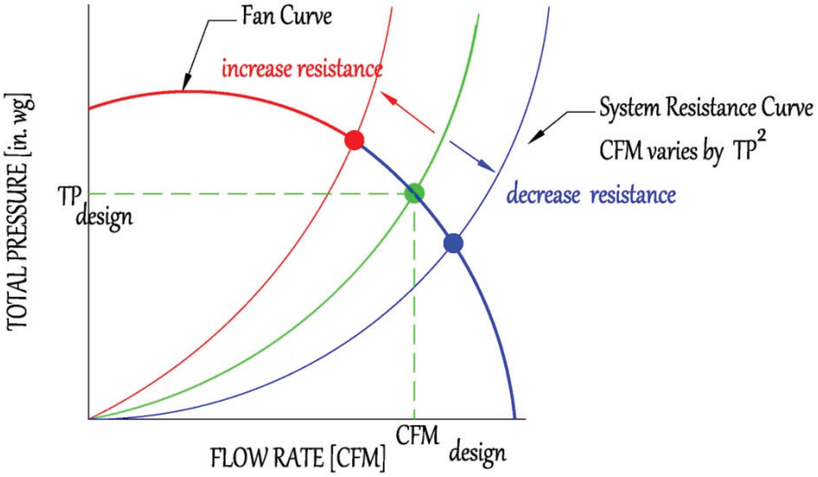


Figure 27: Fan design operating point

It has been shown that the amount of CFM blown by a fan can be changed by shifting the system resistance curve. However, the volumetric flow rate can also be changed by changing the speed of the fan, which shifts the fan curve. Increasing the speed of the fan causes the fan curve to shift to the right. Decreasing the speed of the fan causes the fan curve to shift to the left.

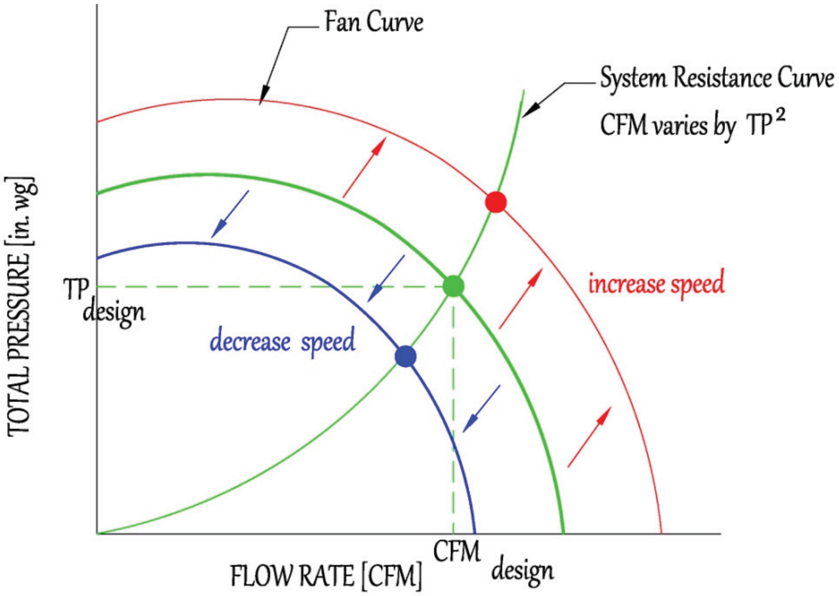


Figure 28: Fan speed effect on cfm and tp

11.2 PUMP CURVES

Pump curves are created by the manufacturers of the pumps through a series of tests and describe the operating points for a specific impeller diameter and pump type. The curve plots the corresponding flow rates at varying pressure, similar to a fan curve. The graph also shows pump efficiency curves. Points within these curves operate at a higher efficiency than the value shown on the curve and points on the curve operate at the stated efficiency.

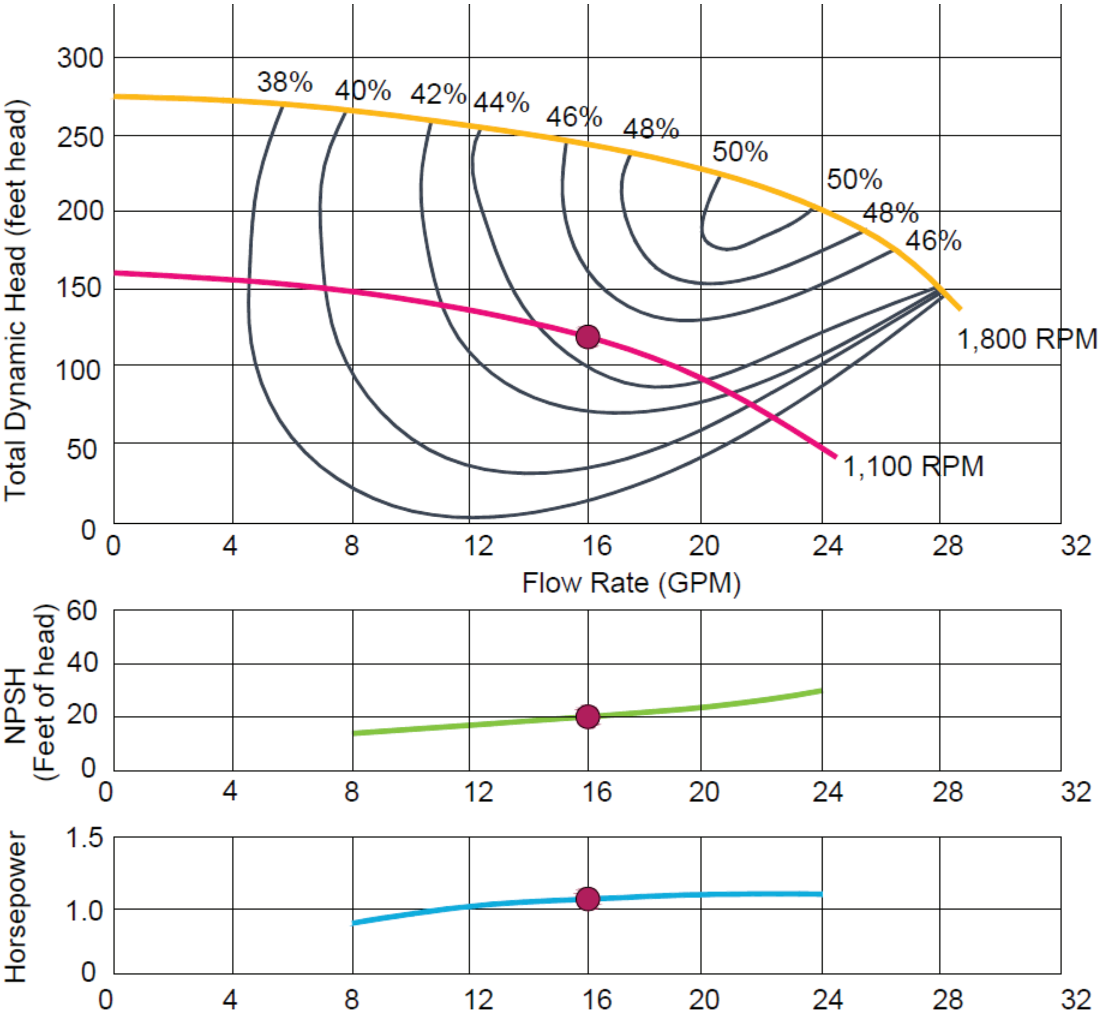


Figure 29: A sample of pump curves

On the exam, you most likely will encounter constant speed pumps. These pumps will always operate at the same speed, which means that the pump will always operate on its curve. The only thing that changes in operation of these pumps is the piping system. The following figure shows how a pump will operate at various system operating conditions. In its design condition, the pump will provide the design flow rate at the design pressure. If the flow rate were less, then the pressure losses through the piping would be less and if the flow rate were more, then the pressure losses through the piping would increase. This change in pressure loss through the system as a function of flow rate is shown as the system resistance curve, which is a squared, parabolic, upward sloping curve.

However, the pump cannot operate at any point along the system resistance curve, it can only operate along its pump curve. So in order to change the pump flow rate, you must change the system resistance curve. This is done by closing and opening valves (changes in the system operating conditions). If valves are opened, this reduces the restrictions, which causes the system resistance curve to change. Now, the intersection between the system resistance curve and the pump curve will be at a point of higher flow and lower pressure. If valves are closed, this increases restrictions, which causes the system resistance curve to change. Now the pump will operate at a point of lower flow and higher pressure.

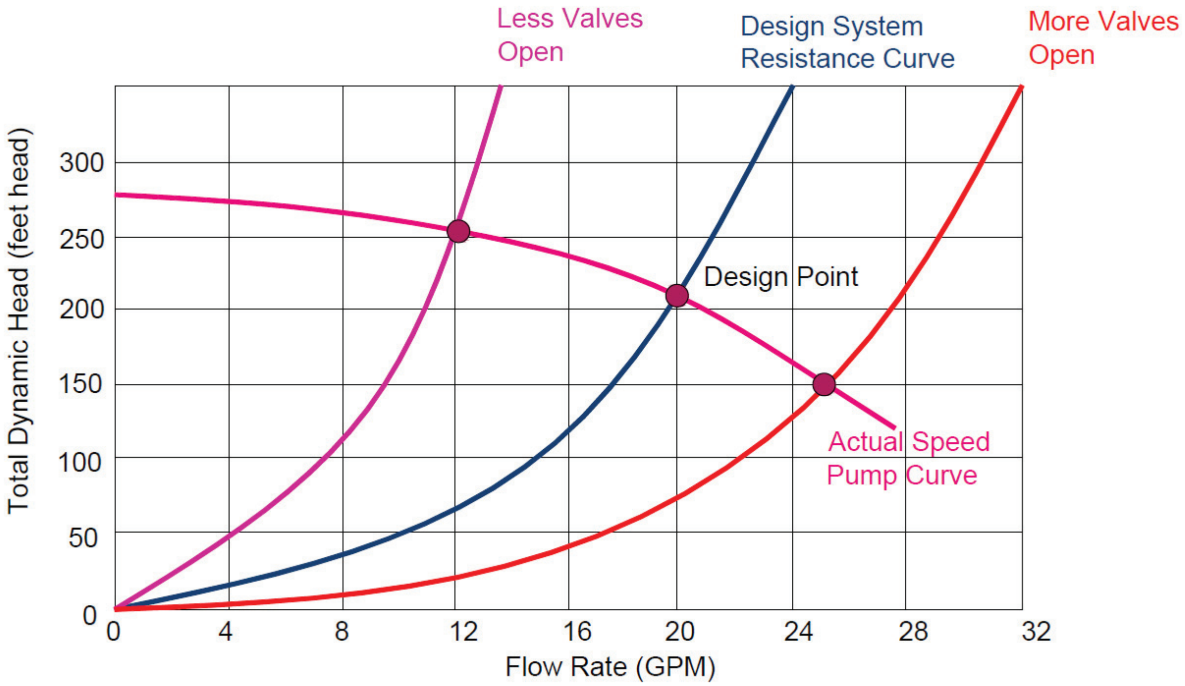


Figure 30: This figure shows that in order to change the flow rate and pressure produced from a constant flow pump, you must change the system resistance curve.

12.0 SCALING LAWS FOR FANS, PUMPS & COMPRESSORS

This section is in reference to the affinity and similarity laws for centrifugal fans, pumps and compressors. These laws do not apply to other types of fans, pumps and compressors. For example, they do not apply to positive displacement pumps or compressors, nor do they apply to propeller fans.

12.1 AFFINITY LAWS FOR FANS, PUMPS & COMPRESSORS

First, if the impeller diameter is held constant and the speed of the fan is changed, then flow rate varies directly with the speed, available pressure varies with the square of the speed and the power use varies with the cube of the speed.

$$\boldsymbol{CASE\ 1:\ N_{old} = N_{new}}$$

$$CFM_{new} = \left(\frac{RPM_{new}}{RPM_{old}}\right)^{1} CFM_{old}$$

$$P_{new} = \left(\frac{RPM_{new}}{RPM_{old}}\right)^{2} P_{old}$$

$$BHP_{new} = \left(\frac{RPM_{new}}{RPM_{old}}\right)^{3} BHP_{old}$$

Second, if the speed is held constant and the impeller diameter of the fan is changed, then flow rate varies directly with the diameter, available pressure varies with the square of the diameter and the power use varies with the diameter raised to the cube power.

$$\boldsymbol{CASE\ 2:\ RPM_{old} = RPM_{new}}$$

$$CFM_{new} = \left(\frac{D_{new}}{D_{old}}\right)^{1} CFM_{old}$$

$$P_{new} = \left(\frac{D_{new}}{D_{old}}\right)^{2} P_{old}$$

$$BHP_{new} = \left(\frac{D_{new}}{D_{old}}\right)^{3} BHP_{old}$$

12.2 SIMILARITY LAWS FOR FANS, PUMPS & COMPRESSORS

You may come across these formulas, if you encounter a question that compares two similar fans. These formulas are called the similarity laws. These laws compare similar fans within the same series of fans. The previous formulas compared the original condition and new condition of the same fan. These formulas compare two similar fans, with different diameters. In order to best understand what is meant by same series of fans, visit a manufacturer's website and you will see various fan series that have varying sizes within the same series. Within a series of fans, a fan with "x" diameter wheel and "y" diameter casing can be compared to another fan in the same series of fans, but with "2x" diameter wheel and "2y" diameter casing. The second pump is similar but has twice the diameter of the first fan.

$\frac{Q_1}{Q_2}$	=	$\left(\frac{D_1}{D_2}\right)^{3}\left(\frac{N_1}{N_2}\right)$
$\frac{P_1}{P_2}$	=	$\left(\frac{D_1}{D_2}\right)^{2}\left(\frac{N_1}{N_2}\right)^{2}\left(\frac{\rho_1}{\rho_2}\right)$
$\frac{W_1}{W_2}$	=	$\left(\frac{D_1}{D_2}\right)^{5}\left(\frac{N_1}{N_2}\right)^{3}\left(\frac{\rho_1}{\rho_2}\right)$
$\frac{Q_1}{Q_2}$	=	$\left(\frac{D_1}{D_2}\right)^{2}\left(\frac{P_1}{P_2}\right)^{.5}\left(\frac{\rho_2}{\rho_1}\right)^{.5}$
$\frac{N_1}{N_2}$	=	$\left(\frac{D_2}{D_1}\right)\left(\frac{P_1}{P_2}\right)^{.5}\left(\frac{\rho_2}{\rho_1}\right)^{.5}$

$\frac{W_1}{W_2}$	=	$\left(\frac{D_1}{D_2}\right)^2 \left(\frac{P_1}{P_2}\right)^{1.5} \left(\frac{\rho_2}{\rho_1}\right)^{.5}$
$\frac{N_1}{N_2}$	=	$\left(\frac{D_2}{D_1}\right)^3 \left(\frac{Q_1}{Q_2}\right)$
$\frac{P_1}{P_2}$	=	$\left(\frac{D_2}{D_1}\right)^4 \left(\frac{Q_1}{Q_2}\right)^2 \left(\frac{\rho_1}{\rho_2}\right)$
$\frac{W_1}{W_2}$	=	$\left(\frac{D_2}{D_1}\right)^4 \left(\frac{Q_1}{Q_2}\right)^3 \left(\frac{\rho_1}{\rho_2}\right)$

$$where\ Q = flow\ rate, D = diameter, N = speed, P = pressure, W = power, \rho = density$$

12.3 MULTIPLE FANS OR PUMPS

There will be times when fans are run in conjunction with each other. It is important for the engineer to understand how the performance is affected depending on the different arrangements of multiple fans. Please refer to *Section 15.0 Mechanical Design and Analysis* for this discussion.

13.0 PRACTICE EXAM PROBLEMS

13.1 PRACTICE PROBLEM 1 – VISCOSITY

Honey has a dynamic viscosity of 1,000 poise, a specific heat capacity of 0.6 cal/g-^{o}C, and a density of 0.05 oz/mL. The kinematic viscosity of honey, in ft^2/sec, is most nearly? Reminder: There are 16 ounces in a pound.

(a) 0.76

(b) 7.1

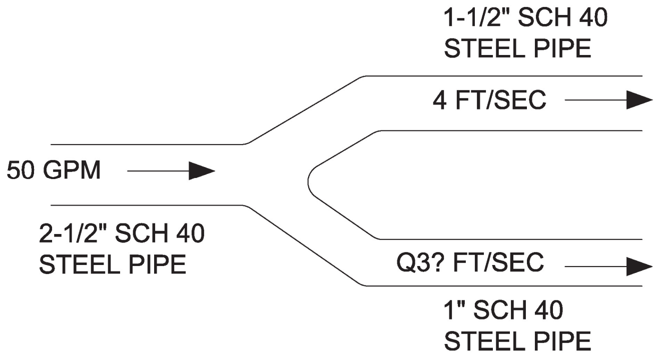
(c) 25

(d) 30

13.2 PRACTICE PROBLEM 2 – PIPE FLOW

50 GPM of water flows through a 2-1/2" pipe and then branches into (2) pipes shown in the figure below. If the velocity in the 1-1/2" pipe is measured at 4 ft/sec, then what is the velocity through the 1" pipe?

Inner Area of Schedule 40 pipes:
    1-1/2” Nominal Diameter: 0.01417 ft^2;
    1” Nominal Diameter: 0.00597 ft^2


1-1/2" SCH 40
STEEL PIPE
4 FT/SEC
50 GPM
2-1/2" SCH 40
STEEL PIPE
Q3? FT/SEC
1" SCH 40
STEEL PIPE

(a) 3.9 ft/sec

(b) 5.6 ft/sec

(c) 7.2 ft/sec

(d) 9.1 ft/sec

13.3 PRACTICE PROBLEM 3 – REYNOLDS NUMBER

Water at 50 °F flows through a 6" schedule 40 steel pipe (Inner diameter = 0.506 ft.) at a volumetric flow rate of 200 gallons per minute. What is the Reynolds number?

(a) 80,000

(b) 100,000

(c) 150,000

(d) 220,000

13.4 PRACTICE PROBLEM 4 – PRESSURE DROP

500 GPM of an unknown fluid flows through a 6" schedule 40 pipe. If the pipe has a relative roughness value of .0004 and a Reynolds number of 250,000, then what is the pressure drop through 100 ft of pipe?

(a) 1.7 ft of head

(b) 3.3 ft of head

(c) 4.2 ft of head

(d) 4.9 ft of head

13.5 PRACTICE PROBLEM 5 – REYNOLDS NUMBER

A fluid has is traveling at a velocity of 8 ft/sec through a 6” pipe. The fluid has a density of 60 lb/ft^3 and has a kinematic viscosity of 1.5 x 10^{-5} ft^2/sec. What is the Reynolds number?

The Reynolds number is most nearly,

(a) 270,000

(b) 540,000

(c) 1,115,000

(d) 3,200,000

13.6 PRACTICE PROBLEM 6 – FRICTION LOSS

A fluid is traveling at 8 feet per second through a 2” pipe (assume an inner diameter of 1.98 inches). If the friction factor is .018, then what is the pressure drop per 100 feet of pipe? Assume a density of 60 lb/ft^3.

The answer is most nearly,

(a) 1.7 psi

(b) 2.3 psi

(c) 4.5 psi

(d) 7.0 psi

13.7 PRACTICE PROBLEM 7 - BUOYANCY

A 100 foot long, 4” pipe is anchored to the ocean bottom below a pier at a depth of 150 feet from the surface. Assume the density of seawater as 60 lb/ft^3. What upward force must the anchors resist, when the pipe is completely empty? The inner diameter of the pipe is 3.75”. Assume the weight of the pipe is negligible.

The answer is most nearly?

(a) 350 lbf

(b) 460 lbf

(c) 610 lbf

(d) 780 lbf

13.8 PRACTICE PROBLEM 8 – PRESSURE DROP

20 GPM of water at 50 °F is flown through a 2” plastic pipe (2.067” ID). The Moody friction factor is .025. What is the pressure drop (psi) through 50’ of pipe?

(a) 0.18 psi

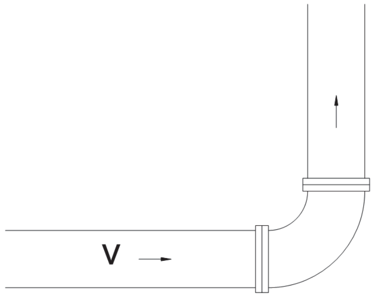
(b) 0.24 psi

(c) 0.41 psi

(d) 0.52 psi

13.9 PRACTICE PROBLEM 9 – IMPULSE MOMENTUM

Water flows through a 50 cm diameter pipe at a flow rate of 190 kg/s and gauge pressure of 4 kPa. As the water flows, it hits a 90 degree elbow. The elbow discharges to atmosphere. What is the force needed to restrain the elbow?



(a) 35 N

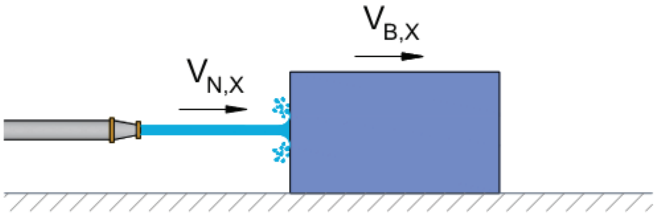
(b) 70 N

(c) 706 N

(d) 986 N

13.10 PRACTICE PROBLEM 10 – IMPULSE MOMENTUM

Water flows out of a nozzle in a 60 m/s horizontal jet stream towards the vertical surface of a box on the ground, as shown below. The jet stream has a cross sectional area of 0.01 m^2 and water disperses after hitting the box. If the nozzle pushes the box horizontally at velocity of 5 m/s, what is the force of the water on the box?



(a) 30 kN

(b) 33 kN

(c) 36 kN

(d) 40 kN

13.11 PRACTICE PROBLEM 11 – POWER AND EFFICIENCY

A hydraulic pump must pump 50 GPM of a fluid with a specific gravity 1.75 from one reservoir to the next reservoir. The pump is located at the same elevation as the first reservoir and the second reservoir is located 150 feet above the first reservoir. Assume that the friction loss in the piping is 4 psi. What is the minimum motor horsepower required of the pump. Assume the pump has an efficiency of 69% and the motor has an efficiency of 90%.

The answer is most nearly,

(a) 3 HP

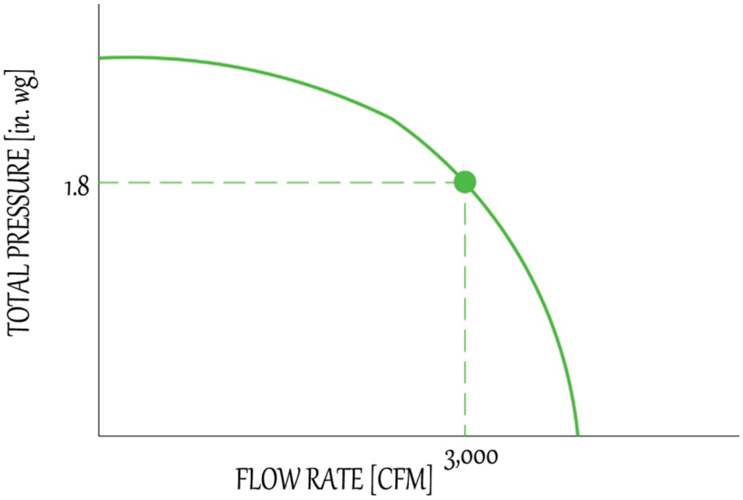
(b) 5 HP

(c) 7.5 HP

(d) 10 HP

13.12 PRACTICE PROBLEM 12 – PERFORMANCE CURVE

Background: A fan has been selected at the design point shown on the below fan curve. What is the minimum number of fans required to be placed in series, to achieve a flow rate of 3,000 CFM at a pressure of 4.0 in. wg?



(a) 1

(b) 2

(c) 3

(d) 4

13.13 PRACTICE PROBLEM 13 – SCALING LAWS

A pump is sized for 200 GPM at 150 ft of head. If the pump speed is decreased by 25%, what is the new flow of the pump? Assume the diameter remains the same.

(a) 100 GPM

(b) 115 GPM

(c) 150 GPM

(d) 200 GPM

13.14 PRACTICE PROBLEM 14 – EXTERNAL FLOW

Water flows around a 4cm diameter pipe that is placed across a large channel. Water flows in the 20 m long, 10 m deep channel at a rate of 10 m/s and a temperature of 20 ˚C. What is the drag force on the pipe? Assume a drag coefficient of 0.4.

(a) 8 kN

(b) 16 kN

(c) 20 kN

(d) 25 kN

13.15 PRACTICE PROBLEM 15 – COMPRESSIBLE FLOW

Air flows through a duct at 21 psig, 110 °F, and a Mach number of 0.4. What is the stagnation pressure of the air?

(a) 23 psig

(b) 25 psig

(c) 36 psig

(d) 38 psig

14.0 PRACTICE EXAM SOLUTIONS

14.1 SOLUTION 1 - VISCOSITY

Honey has a dynamic viscosity of 1,000 poise, a specific heat capacity of 0.6 cal/g-°C, and a density of 0.05 oz/mL. The kinematic viscosity of honey, in ft^2/sec, is most nearly?

Kinematic viscosity is defined as:

$$\nu = \frac{\mu}{\rho}$$

The specific heat capacity is irrelevant.

The dynamic (or absolute) viscosity is irrelevant:

$$\mu = (1000\, poise) * \frac{1\ \mathrm{Pa \cdot s}}{10\ \mathrm{poise}} * \frac{0.020885 \frac{\mathrm{lbf}}{\mathrm{ft}^2}}{Pa} = 2.09 \frac{lbf \cdot s}{ft^2}$$

The density is:

$$\rho = 0.05 \frac{oz}{mL} * \frac{0.0625\, lbm}{1\, oz} * \frac{28317\, mL}{1\, ft^3} = 88.4 \frac{lbm}{ft^3}$$

Therefore the kinematic viscosity is:

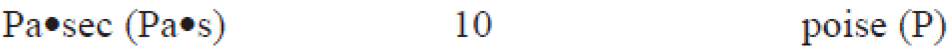
$$\nu = 2.09 \frac{lbf \cdot s}{ft^2} * \frac{ft^3}{88.4\, lbm} * 32.2 \frac{lbm * \frac{ft}{s^2}}{lbf} = 0.76 \frac{ft^2}{s}$$

The correct answer is most nearly, **(a) 0.76**

For more detail on this solution, please see in depth SI and USCS solutions.

First convert poise to Pa-S

$$\mu = (1000\, poise) * \frac{1\ \mathrm{Pa \cdot s}}{10\ \mathrm{poise}} = 100\ \mathrm{Pa - s}$$



Pa•sec (Pa•s)	10	poise (P)

Next convert Pa to psi

$$100\ \mathrm{Pa - s} * \left(\frac{1\ \mathrm{psi}}{6895\ \mathrm{Pa}}\right) = 0.014503\ \mathrm{psi - s}$$

psi	6,895	Pa

Next convert square inches to square feet

$$0.014503\,\frac{\mathrm{lbf}}{\mathrm{in}^2}-\mathrm{s}*\frac{144\mathrm{in}^2}{1ft^2}=\boldsymbol{2.09\,\frac{lbf\cdot s}{ft^2}}$$

You need to have remembered that 16 oz. equals 1 pound. This conversion is not included in the handbook.

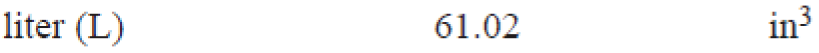
$$\rho=0.05\,\frac{oz}{mL}*\frac{1\ lbm}{16\ oz}=0.003125\,\frac{lbm}{mL}$$

Next convert mL to Liters

$$\rho=0.003125\,\frac{lbm}{mL}*\frac{1000ml}{1L}=3.125\,\frac{lbm}{L}$$

Next convert L to cubic inches.

$$\rho=3.125\,\frac{lbm}{L}*\frac{1\ L}{61.02\ in^3}=0.0512\,\frac{lbm}{in^3}$$



liter (L)	61.02	in^3

$$\rho=0.0512\,\frac{lbm}{in^3}*\frac{12*12*12\ in^3}{ft^3}=\boldsymbol{88.495\,\frac{lbm}{ft^3}}$$

You then need to convert the mass that is currently in pound-mass into slugs. You complete this by dividing by 32.2. Slugs are a unit of mass. Pound-mass is the mass (slugs) x gravity.

$$\boldsymbol{88.495\,\frac{lbm}{ft^3}\div 32.2=2.75\,\frac{slugs}{ft^3}}$$

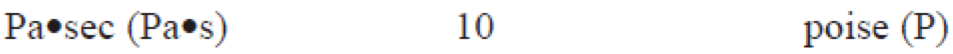
Therefore the kinematic viscosity is shown below. Units of LBF = slugs –ft/s^2.

$$\boldsymbol{\nu}=\frac{\mu}{\rho/g_c}=\frac{\boldsymbol{2.09\,\frac{lbf\cdot s}{ft^2}}}{\boldsymbol{\frac{2.75\ slugs}{ft^3}}}=\frac{\boldsymbol{2.09\,\frac{slugs-ft\cdot s}{s^2-ft^2}}}{\boldsymbol{\frac{2.75\ slugs}{ft^3}}}=\frac{\boldsymbol{2.09\,\frac{\cancel{slugs}}{s-ft}}}{\boldsymbol{\frac{2.75\ \cancel{slugs}}{ft^3}}}=\boldsymbol{0.76\,\frac{ft^2}{s}}$$

SI UNITS VERSION

First convert poise to Pa-S

$$\mu=(1000\ poise)*\frac{1\ \mathrm{Pa}\cdot\mathrm{s}}{10\ \mathrm{poise}}=100\ \mathrm{Pa}-\mathrm{s}$$



Pa•sec (Pa•s)	10	poise (P)

Next convert Pa to N/m^2

$$100\ \mathrm{Pa}-\mathrm{s}*\left(\frac{1\,\frac{\mathrm{N}}{\mathrm{m}^2}}{1\ \mathrm{Pa}}\right)=\mathbf{100\,\frac{N}{m^2}-s}$$

You need to have remembered that 16 oz. equals 1 pound. This conversion is not included in the handbook.

$$\rho=0.05\,\frac{oz}{mL}*\frac{1\ lbm}{16\ oz}=0.003125\,\frac{lbm}{mL}$$

Next convert milliliters to liters

$$\rho=0.003125\,\frac{lbm}{mL}*\frac{1000ml}{1L}=3.125\,\frac{lbm}{L}$$

Next convert pounds to kilograms

$$\rho=3.125\,\frac{lbm}{L}*\frac{1\ kg}{2.205\ lbm}=1.417\,\frac{kg}{L}$$

Next convert L to cubic meters

$$\rho=1.417\,\frac{kg}{L}*\frac{10^3\ L}{1\ m^3}=\boldsymbol{1417.2\,\frac{kg}{m^3}}$$

Therefore the kinematic viscosity is:

$$\nu=\frac{\mathbf{100\,\frac{N}{m^2}-s}}{\boldsymbol{1417.2\,\frac{kg}{m^3}}}=\frac{\mathbf{100\,\frac{kg-m/s^2}{m^2}-s}}{\boldsymbol{1417.2\,\frac{kg}{m^3}}}=\frac{\mathbf{100\,\frac{kg}{m-s}}}{\boldsymbol{1417.2\,\frac{kg}{m^3}}}=\boldsymbol{0.07056\,\frac{m^2}{s}}$$

Lastly, convert meters to feet.

$$\boldsymbol{0.07056\,\frac{m^2}{s}*\frac{ft^2}{0.3048^2\ m^2}=0.76\,\frac{ft^2}{s}}$$

14.2 SOLUTION 2 – PIPE FLOW

This is a flow rate balance type problem.

$$Q_{2.5"}=Q_{1.5"}+Q_{1"}$$

$$50\,\frac{gallons}{minute}*\frac{1\ ft^3}{7.48\ gallons}=4\,\frac{ft}{sec}*60\,\frac{sec}{min}*0.01417\ ft^2+X\,\frac{ft}{sec}*60\,\frac{sec}{min}*0.00597\ ft^2$$

$$50\,\frac{\cancel{gallons}}{minute}*\frac{1\ ft^3}{7.48\ \cancel{gallons}}=4\,\frac{ft}{\cancel{sec}}*60\,\frac{\cancel{sec}}{min}*0.01417\ ft^2+X\,\frac{ft}{sec}*60\,\frac{sec}{min}*0.00597\ ft^2$$

$$6.69\,\frac{ft^3}{minute} = 3.40\,\frac{ft^3}{minute} + X\,\frac{ft}{\cancel{sec}} * 0.36\,\frac{ft^2 * \cancel{sec}}{minute}$$

$$\boldsymbol{X = 9.17\ \frac{ft}{sec}}$$

The correct answer is most nearly **(d), 9.1 ft/sec.**

14.3 Solution 3 – Reynolds Number

First, find the kinematic viscosity of water at 50 F in the *NCEES FE Reference Handbook.*

$$v = 1.410\ x\ 10^{-5}\ ft^2/sec$$

Next find the velocity of water through the 6" schedule 40 steel pipe.

$$Internal\ Area_{6"\ Sch\ 40\ pipe} = 0.201\ ft^2$$

$$Inner\ Diameter_{6"\ Sch\ 40\ pipe} = 0.506\ ft$$

$$Convert\ 200\,\frac{gallons}{minute}\ to\,\frac{ft^3}{sec}$$

$$200\,\frac{gallons}{minute} * \left(\frac{1}{448.83}\right) = 0.4456\ \frac{ft^3}{sec}$$

$$Velocity = \ 0.4456\ \frac{ft^3}{sec} \div 0.201\ ft^2$$

$$Velocity = \ 2.22\ ft/sec$$

Finally, find Reynolds number.

$$Re = \frac{velocity * inner\ diameter}{v}$$

$$Re = \frac{2.22\,\frac{ft}{sec} * 0.5054ft}{1.41\ x\ 10^{-5}\ ft^2/\,\mathrm{sec}}$$

$$Re = 79{,}573$$

The correct answer is most nearly, **(a) 80,000.**

14.4 Solution 4 – Pressure Drop

500 GPM of an unknown fluid flows through a 6" schedule 40 pipe. If the pipe has a relative roughness value of .0004 and a Reynolds number of 250,000, then what is the pressure drop through 100 ft of pipe?

(a) 1.7 ft of head

(b) 3.3 ft of head

(c) 4.2 ft of head

(d) 4.9 ft of head

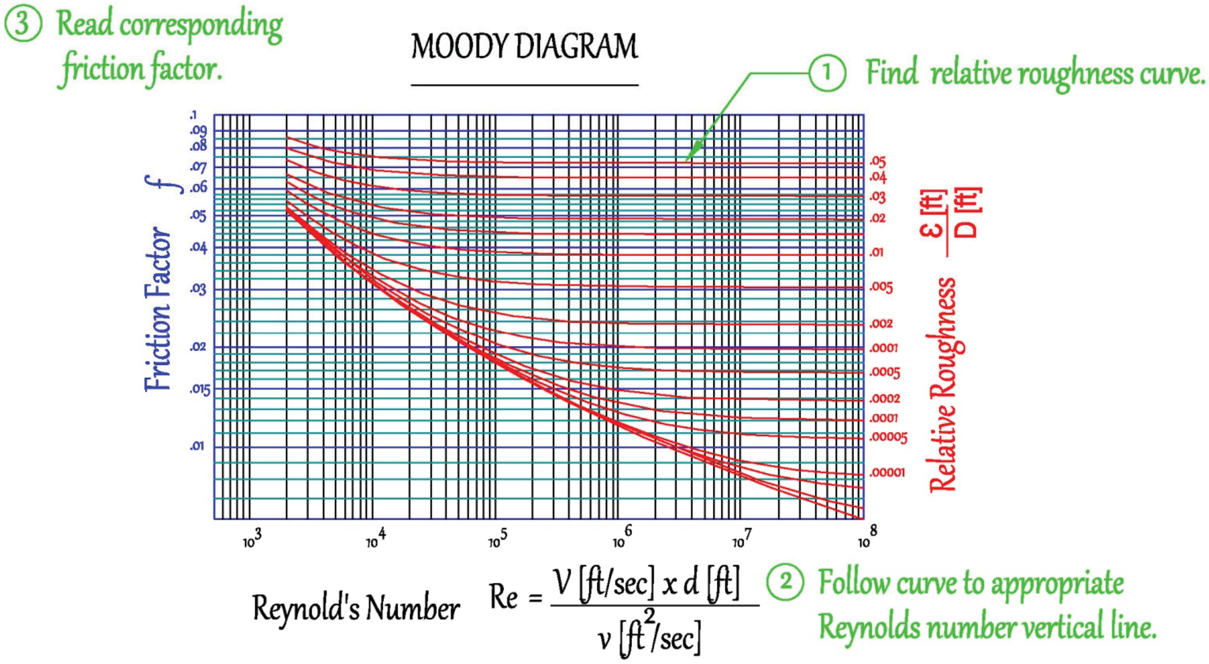
Use the Darcy Weisbach equation to find the pressure drop.

$$h = \frac{fLv^2}{2Dg}\ [Darcy\ Weisbach\ Equation]$$

$$where\ h = ft\ of\ head; f = Darcy\ friction\ factor; v = velocity\ \left[\frac{ft}{sec}\right],$$

$$D = inner\ diameter\ [ft], g = gravity\ [32.2\frac{ft}{sec^2}]$$

First, you need to use the Moody Diagram in the FE Handbook to find the friction factor.



$$Relative\ rougness = 0.0004\ and\ Reynold's\ Number = 250{,}000$$

$$f = .018$$

$$V = 500\ gpm * \left(\frac{1\frac{ft^3}{sec}}{448.83\ gpm}\right) * \left(\frac{1}{0.20063\ ft^2}\right) = 5.55\frac{ft}{s}$$

Use the Darcy Weisbach equation

$$h = \frac{.018 * 100\ ft * 5.5^2}{2 * 0.506\ ft * 32.2\frac{ft}{sec^2}}$$

$$h = 1.7\ ft\ of\ head$$

The correct answer is most nearly, **(a) 1.7 ft of head.**

14.5 SOLUTION 5 – REYNOLDS NUMBER

A fluid has is traveling at a velocity of 8 ft/sec through a 6" pipe. The fluid has a density of 60 lb/ft^3 and has a kinematic viscosity of 1.5 x 10^{-5} ft^2/sec. What is the Reynolds number?

$$Re = \frac{V * D}{v}$$

$$V = \frac{ft}{sec}; D = diameter\ (ft); v = kinematic\ viscosity\ (\frac{ft^2}{\text{sec}})$$

$$Re = \frac{8\frac{ft}{sec} * \frac{6\ in}{12\frac{in}{ft}}}{1.5 * 10^{-5}\frac{ft^2}{sec}}$$

$$Re = 266{,}667$$

The correct answer is most nearly, **(a) 270,000.**

14.6 SOLUTION 6 – FRICTION LOSS

A fluid is traveling at 8 feet per second through a 2" pipe (assume an inner diameter of 1.98 inches). If the friction factor is .018, then what is the pressure drop per 100 feet of pipe? Assume a density of 60 lb/ft^3.

$$Pressure\ drop\ equation \rightarrow\ \Delta P = f * \frac{L}{D} * \rho * \frac{1}{2} * \frac{V^2}{g_c};$$

$$f = 0.018; L = 100\ ft; D = \frac{1.98}{12} = 0.165\ ft;$$

$$\rho = 60\frac{lb}{ft^3}; V = 8\frac{ft}{sec}; g_c = \frac{32.2ft}{sec^2};$$

$$\Delta P = 0.018 * \frac{100\ ft}{0.165\ ft} * \rho * \frac{1}{2} * \frac{8^2}{32.2};$$

$$\Delta P = 0.018 * \frac{100\ ft}{0.165\ ft} * \rho * \frac{1}{2} * \frac{8^2}{32.2};$$

$$\Delta P = 650\frac{lb}{ft^2};$$

$$Then\ convert\ to\ feet\ of\ head\ with\ your\ Engineering\ unit\ conversions\ section$$

$$\Delta P = 650\frac{lb}{ft^2} * \frac{1}{144} = 4.51\ psi$$

The answer is most nearly, **(c) 4.5 psi.**

14.7 SOLUTION 7 - BUOYANCY

A 100 foot long, 4” pipe is anchored to the ocean bottom below a pier at a depth of 150 feet from the surface. Assume the density of seawater as 60 lb/ft^3. What upward force must the anchors resist, when the pipe is completely empty? The inner diameter of the pipe is 3.75”. Assume the weight of the pipe is negligible.

$$The\ upward\ force\ is\ equal\ to\ the\ displaced\ volume, using\ the\ Buoyancy\ equation$$

$$B = \rho_{fluid}\frac{lb}{ft^3} * V_{fluid\ displaced}$$

$$V_{fluid\ displaced} = (\pi * \frac{D^2}{4}) * L;$$

$$V_{fluid\ displaced} = \left(\pi * \frac{\left(\frac{3.75}{12}ft\right)^2}{4}\right) * 100ft;$$

$$V_{fluid\ displaced} = \left(\pi * \frac{\left(\frac{3.75}{12}ft\right)^2}{4}\right) * 100ft;$$

$$V_{fluid\ displaced} = 7.67\ ft^3;$$

$$B = 60\frac{lb}{ft^3} * 7.67\ ft^3$$

$$B = 460.2\ lbf$$

The answer is most nearly, **(b) 460 lbf.**

14.8 SOLUTION 8 – PRESSURE DROP

For this equation use the Darcy Weisbach Equation below:

$$h = \frac{fLv^2}{2Dg}\ [Darcy\ Weisbach\ Equation]$$

$$where\ h = ft\ of\ head; f = Darcy\ friction\ factor; v = velocity\ \left[\frac{ft}{sec}\right],$$

$$D = inner\ diameter\ [ft], g = gravity\ [32.2\frac{ft}{sec^2}]$$

Prepare each term to be plugged into the equation. First, find velocity. Convert 20 GPM to velocity [ft/sec].

$$\text{Multiply GPM by } \frac{1}{448.83} \text{ to get} \frac{\text{FT}^3}{\text{sec}}.$$

$$20\ \text{GPM} * \frac{1}{448.83} = 0.04456\frac{\text{FT}^3}{\text{sec}}.$$

$$\text{Area} = \text{PI} * \frac{\left(\frac{2.067}{12}\right)^2}{4} = 0.0233\ \text{ft}^2$$

$$\text{Velocity} = .\frac{04456\ \text{ft}^3/\text{sec}}{0.0233\ \text{ft}^2} = 1.91\frac{ft}{sec}$$

Plug in the variables to the equation.

$$h = \frac{.025 * 50 * 1.91^2}{2 * (\frac{2.067}{12})32.2}\ [ft\ of\ head]$$

$$h = \frac{.025 * 50 * 1.91^2}{2 * (\frac{2.067}{12})32.2}\ [ft\ of\ head]$$

$$h = 0.41\ [ft\ of\ head]$$

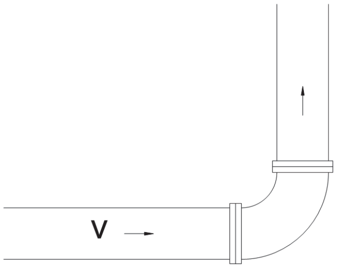
Next convert to PSI

$$h = 0.41\, ft\, of\, head * \frac{0.433\, psi}{1\, ft\, of\, head} = 0.18\, PSI$$

The answer is most nearly, **(a), 0.18 psi.**

14.9 SOLUTION 9 – IMPULSE MOMENTUM

Water flows through a 50 cm pipe at a flow rate of 190 kg/s and gauge pressure of 4kPa. As the water flows, it hits a 90 degree elbow. The elbow discharges to atmosphere. What is the force needed to restrain the elbow?



(a) 35 N

(b) 70 N

(c) 706 N

(d) 986 N

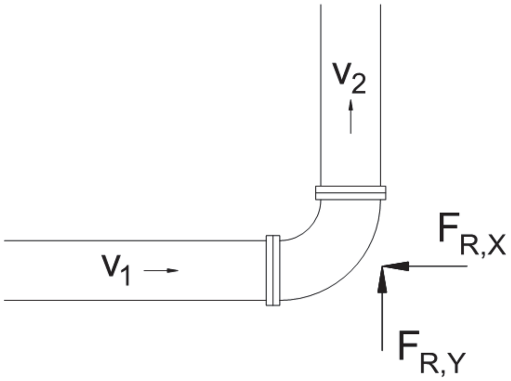
Use the momentum equation for steady flow, where the subscript 1 designates fluid entering the control volume and 2 designates fluid leaving the control volume.

$$\sum F = \sum Q_2 \rho_2 v_2 - \sum Q_1 \rho_1 v_1$$

The velocity entering and leaving the elbow is the same and is calculated as:

$$v_1 = v_2 = 190 \frac{kg}{s} * \frac{1m^3}{1000kg} * \frac{1}{\pi * \left(\frac{0.5\, m}{2}\right)^2} = 0.97 \frac{m}{s}$$

Separate the forces into the x and y components.



$$\sum F_x = -F_{R,x} + F_1 = F_{R,X} + P_1 A_1$$

$$-F_{R,X} + P_1 A_1 = 0 - Q_1 \rho_1 v_1$$

Since,

$$Q_1 \rho_1 = \dot{m}_1$$

then,

$$F_{R,X} = \dot{m}_1 v_1 + P_1 A_1 = 190 \frac{kg}{s} * 0.97 \frac{m}{s} * \frac{1N}{1\ kg * 1 \frac{m}{s^2}} + 4000 \frac{N}{m^2} * \pi * \left(\frac{0.5m}{2}\right)^2 = 969\ N$$

$$\sum F_Y = F_{R,Y} = Q_2 \rho_2 v_2$$

$$F_{R,Y} = \dot{m}_2 v_2 = 190 \frac{kg}{s} * 0.97 \frac{m}{s} * \frac{1N}{1\ kg * 1 \frac{m}{s^2}} = 184\ N$$

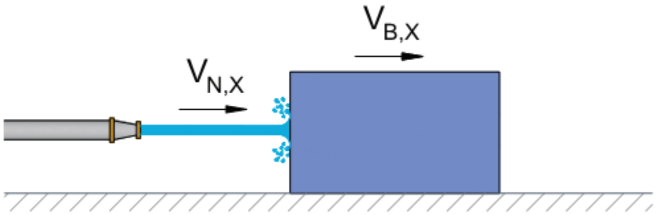
Then, find the total reactive force.

$$F = \sqrt{F_{R,X}^2 + F_{R,Y}^2} = \sqrt{(969\ N)^2 + (184\ N)^2} = 986\ N$$

The correct answer is most nearly, **(d) 986 N.**

14.10 SOLUTION 10 – IMPULSE MOMENTUM

Water flows out of a nozzle in a 60 m/s horizontal jet stream towards the vertical surface of a box on the ground, as shown below. The jet stream has a cross sectional area of 0.01 sq m and water disperses after hitting the box. If the nozzle pushes the box horizontally at velocity of 5 m/s, what is the force of the water on the box?



(a) 30 kN

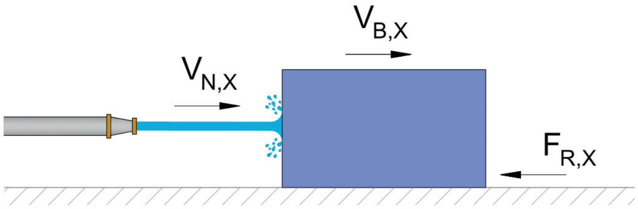
(b) 33 kN

(c) 36 kN

(d) 40 kN

Use the momentum equation for steady flow, where the subscript 1 designates fluid entering the control volume and 2 designates fluid leaving the control volume.

$$\sum F = \sum Q_2\rho_2 v_2 - \sum Q_1\rho_1 v_1$$



To simplify the equation, find the velocity of the jet stream relative to the movement of the box and take the vertical face of the box as the control surface.

$$v_{water,relative} = v_{N,X} - v_{B,X} = 60\ m/s - 5\ m/s = 55\ m/s$$

This means that water enters the control volume at 55 m/s and leaves at 0 m/s, while the box is still and the reactive force on the box is $F_{R,X}$. Use 1000 kg/m^3 as the density of water.

$$-F_{R,X} = \left[Q_2\rho_2 * \left(\frac{0m}{s}\right)\right] - \left[\left(\frac{55m}{s} * 0.01\ m^2\right) * \frac{1000kg}{m^3} * \frac{55m}{s}\right]$$

$$-F_{R,X} = -30{,}250\frac{kg * m}{s^2} * \frac{1kN}{1000\ kg * m/s^2}$$

$$F_{R,X} = 30kN \leftarrow$$

The water is hitting the surface of the box at the same force as the reactive force.

The correct answer is most nearly, **(a) 30 kN.**

14.11 Solution 11 – Power and Efficiency

A hydraulic pump must pump 50 GPM of a fluid of specific gravity 1.75 from one reservoir to the next reservoir. The pump is located at the same elevation as the first reservoir and the second reservoir is located 150 feet above the first reservoir. Assume that the friction loss in the piping is 4 psi. What is the minimum motor horsepower required of the pump. Assume the pump has an efficiency of 69% and the motor has an efficiency of 90%.

$$Refer\ to\ pump\ power\ equation\ in\ FE\ Handbook, page\ 192$$

$$Mechanical\ work = \frac{\Delta H * Q * Density}{Pump\ Efficiency * Motor\ Efficiency}$$

$$\Delta H = pressure\ drop\ in\ feet\ of\ fluid; Q = flow\ rate\ in\ ft^3/s$$

$$We\ need\ to\ convert\ 150\ feet\ of\ elevation\ head\ to\ pressure$$

$$(mass) * gravity * (150\ feet)$$

Convert the density in specific gravity to pound mass and then to slugs per cubic feet. The standard density is 62.4. However, you have a specific gravity of 1.75 which means the density of this fluid is 1.75 x the standard density. You divide by 32.2 in order to get mass in units of slugs.

$$\frac{mass}{volume} = 62.4\frac{lbm}{ft^3} * 1.75 * \left(\frac{1}{32.2}\right) = 3.39\frac{slugs}{ft^3}$$

With the mass per volume, you can find the pressure due to elevation, which is mass x gravity x height.

$$P_{elevation} = 3.39\frac{slugs}{ft^3}\ x\ 32.2\frac{ft}{s^2} x 150\ ft = 16{,}380\frac{slugs}{ft^2} - \frac{ft}{s^2}$$

The slugs x gravity (slugs –ft/s^2) is the same as lbf.

$$P_{elevation} = 16{,}380\frac{}{ft^2} - slugs - \frac{ft}{s^2} = 16{,}380\frac{lbf}{ft^2}$$

Now, convert the friction loss from lbf per square inches to square feet.

$$P_{friction} = 4\frac{lbf}{in^2} * \frac{144\ in^2}{ft^2} = 576\frac{lbf}{ft^2}$$

Next, convert the gallons per minute to a cubic feet per second.

$$Q = 50\frac{gallons}{minute} * \left(\frac{.134\ ft^3}{gallon}\right) * \frac{1\ min}{60\ s} = 0.112\frac{ft^3}{s}$$

Next, use the power equation to find the work.

$$Mechanical\ work = \left(576\frac{lbf}{ft^2} + 16{,}380\frac{lbf}{ft^2}\right) * 0.122\frac{ft^3}{s}$$

$$Mechanical\ work = 2068\ \ ft - \frac{lbf}{s}$$

This is the amount of mechanical work required at the outlet of the pump. But the pump is not 100% efficiency so the pump needs more input work to produce this amount of work. This is why you divide by the efficiency. The input work to the pump is the same output work of the motor. In order to find the input work to the motor, you divide the motor output by the efficiency of the motor.

$$Pump\ Input\ Power = \frac{2068\ \ ft - \frac{lbf}{s}}{0.69} = 2998\ \ ft - \frac{lbf}{s}$$

$$Motor\ Input\ Power = \frac{2998\, ft - \frac{lbf}{s}}{0.9} = 3331\ \ ft - \frac{lbf}{s}$$

Finally use the conversion that 1 HP is equal to 550 ft-lbf/sec

$$Convert\ to\ HP \rightarrow 3331\ \ ft - \frac{lbf}{s} * \frac{1\ HP}{550\, ft - \frac{lbf}{s}} = 6.0\ HP$$

The correct answer is most nearly, **(C) 7.5 HP**. The motor must be at least able to provide 6 HP. The 5 HP is not acceptable. The question also asks for the minimum motor, so the 10 HP can meet the load, but it isn't the smallest motor that can meet the load.

14.12 SOLUTION 12 – PERFORMANCE CURVE

Background: A fan has been selected at the design point shown on the below fan curve. What is the minimum number of fans required to be placed in series, to achieve a flow rate of 3,000 CFM at a pressure of 4.0 in. wg?

If fans are placed in series, then their pressures are added. (3) Fans are required to achieve a pressure of 5.4 in. wg.

The correct answer is most nearly, **(c) 3.**

14.13 SOLUTION 13 – SCALING LAWS

A pump is sized for 200 GPM at 150 ft of head. If the pump speed is decreased by 25%, what is the new flow of the pump? Assume the diameter remains the same.

Use the affinity laws.

$$\frac{Q_1}{Q_2}=\frac{N_1}{N_2}; if\ diameter\ is\ held\ constant$$

$$\frac{200}{Q_2}=\frac{x}{0.75x}$$

$$200*0.75*x=Q_2*x$$

$$Q_2=150\ gpm$$

The correct answer is most nearly, **(c) 150 GPM.**

14.14 SOLUTION 14 – EXTERNAL FLOW

The find the drag force on the pipe, use the following equation for immersed objects in a large body of water.

$$F_D=\frac{C_D\rho v^2A}{2}$$

From the NCEES FE Reference Handbook, the density of water at 20^{o}C is 998.2 kg/m^3 (997 at 25). Solve for the front face area of the pipe that the fluid flow will see.

$$A=D*Channel\ Width=4cm*\frac{1m}{100cm}*20m=0.8m^2$$

Solve for the drag force.

$$F_D=\frac{0.4*(998.2kg/m^3)(10m/s)^2(0.8m^2)}{2}*\frac{1N}{1kg*m/s^2}$$

$$F_D=15{,}970N$$

The correct answer is most nearly, **(b) 16kN.**

14.15 SOLUTION 15 – COMPRESSIBLE FLOW

The stagnation pressure is the static pressure of the air, when all the kinetic energy is isentropically converted to pressure energy. It is calculated from the pressure in the duct using the following isentropic relationship.

$$\frac{P_0}{P}=\left(1+\frac{k-1}{2}*Ma^2\right)^{\frac{k}{k-1}}$$

Where k is the specific heat ratio, c_p/c_v and P_o is the stagnation pressure. The specific heat ration of air is 1.4. The pressure in the above equation is given in absolute values.

$$P = 21psig + 14.7 = 35.7psia$$

Solve for the stagnation pressure.

$$P_0 = P\left(1 + \frac{k-1}{2} * Ma^2\right)^{\frac{k}{k-1}} = 35.7psia * \left(1 + \frac{1.4-1}{2} * 0.4^2\right)^{\frac{1.4}{1.4-1}}$$

$$P_0 = 39.9psia$$

$$P_0 = 39.9psia - 14.7 = 25.2psig$$

The correct answer is most nearly, **(b) 25 psig.**

11 - Thermodynamics

MECHANICAL FE TEXTBOOK

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Section 11.0 –Thermodynamics

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1.0 INTRODUCTION

Thermodynamics accounts for approximately 10 to 15 questions on the FE Mechanical exam. Thermodynamics is the largest topic on the FE Mechanical Exam. It contains so much information that most Mechanical Engineering college courses do not cover all of this information within their entire Mechanical Engineering degree course curriculum. 11G, 11I, 11J are most often left out of Thermodynamic college courses. In addition, many college courses cover a lot of theory and do not focus on the key concepts and skills that are necessary for practical application of the theory. The following list is the outline of the Thermodynamics topics that can appear on the exam. This section will go into detail on the key concepts and skills that are necessary under each topic. Following, the key concepts and skills there will be problems that you can use to test your understanding of the concepts and skills.

Section 11.0 Thermodynamics (10 to 15 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Topic 1.0	Introduction
11A	Topic 2.0	Properties of Ideal Gases and Pure Substances
11B	Topic 3.0	Energy Transfers
11C	Topic 4.0	Laws of Thermodynamics
11D	Topic 5.0	Processes
11E	Topic 6.0	Performance of Components
11F	Topic 7.0	Power Cycles
11G	Topic 8.0	Refrigeration and Heat Pump Cycles
11H	Topic 9.0	Non-reacting Mixtures of Gases
11I	Topic 10.0	Psychrometrics
11J	Topic 11.0	Heating, Ventilating and Air-Conditioning (HVAC) Processes
11K	Topic 12.0	Combustion and Combustion Products
	Topic 13.0	Practice Exam Problems

2.0 PROPERTIES OF IDEAL GASES & PURE SUBSTANCES

This topic, Properties of Ideal Gases & Pure Substances, describes the typical properties for both gases and liquids. On the exam, you should be able to find thermodynamic properties very easily through the use of your *NCEES FE Reference Handbook* for given gases and liquids. The most common fluids in Thermodynamics are air, moist air, liquid water, steam, combustion fuels (butane, ethane, methane and propane) and refrigerants like R-134a. The properties for these fluids are the building blocks for solving the problems on the exam. You should also have a concept of what these properties mean in the real world. These concepts will help to reality check your answers, instead of blindly following the results of your equations. Hopefully, this helps you to catch any math errors and speeds up your elimination of incorrect answers.

2.1 PRESSURE

Pressure is one of the two most likely properties that you will start off with in a real world situation, because pressure is a thermodynamic property that is easily measured.

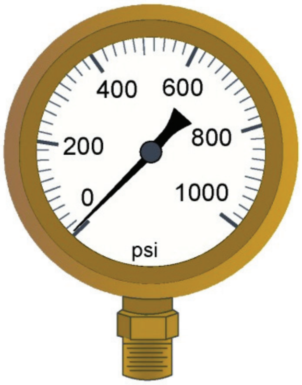


Figure 1: Pressure gauge

The pressure of a fluid indicates the amount of force per unit area that the fluid imparts on the system around it. Pressure is measured in units of pounds per square inch $(psi = \frac{lbf}{square\ inch})$ or kilopascals (KPa). There are two different types of pressure scales, (1) absolute pressure and (2) gauge pressure. These two pressure scales differ by their 0 reference point. Gauge pressures have a 0-reference point as 1 atm. Thus 0 psig, where the g indicates gauge pressure, is equal to 1 atmospheric or 14.7 psia or 101.3529 KPa, where the “a” in psia indicates absolute pressure. Most real world applications encountered by practicing engineers will have pressures indicated in gauge pressure. These include pressures measured at the discharge and intake of pumps and fans and the pressures measured at other pieces of equipment like heat exchangers, chillers and cooling towers. The relationship between gauge and atmospheric pressure is shown with the following equation and figure.

$$P_{abs}[psi] = P_{gauge}[psi] + 14.7\ psi$$

$$P_{abs}[KPa] = P_{gauge}[KPa] + 101.3529\ KPa$$

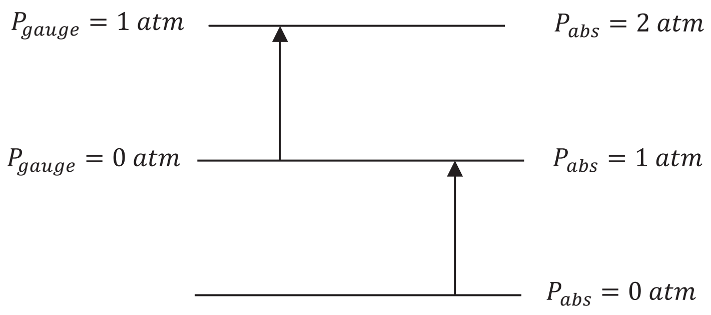


Figure 2: The relationship between gauge and absolute pressures

2.2 TEMPERATURE

Temperature is the second of the two most likely properties that you will start off with in a real world situation, because temperature is easy to measure.

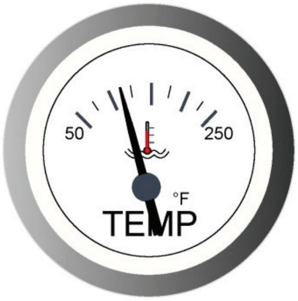


Figure 3: A temperature gauge.

This property is the one most people are familiar with, because it is shown on thermostats and thermometers. Temperature is a direct indication of the amount of heat in the fluid. The USCS units used for temperature are Fahrenheit and Rankine. Typical temperatures for chilled water (medium used for water-cooled Air Conditioning) range from 45°F (7.22 °C) to 55°F (12.78°C) and hot water temperatures range from 120°F (48.89 °C) to 140°F (60 °C). The temperature at which water boils is 212°F (100 °C) and water freezes at 32°F (0 °C).

Rankine (Kelvin) temperatures are used when it is necessary to define an absolute temperature scale having only positive values, relative to absolute zero. The conversion between Fahrenheit (Celsius) and Rankine (Kelvin) is shown below. When using equations during the exam, ensure that the correct temperature units are used. Always double check the required units for your equation.

$$°R = °\mathrm{F} + 460$$

$$°K = °\mathrm{C} + 273.15$$

2.3 ENTHALPY

Enthalpy describes the amount of energy in a system. This property is used extensively in HVAC and Thermodynamics. It defines the entering and leaving energy of a fluid through a system. On the exam, enthalpy refers to total enthalpy. Total enthalpy is equal to the internal energy of the fluid plus the energy due to pressure-volume.

$$h = u + pv * (\frac{0.18505 \frac{BTU}{lbm}}{1\, psi * ft^3})$$

$$where\ h = enthalpy \left(\frac{Btu}{lbm}\ or\ \frac{KJ}{kg}\right); u = internal\ energy\ \left(\frac{Btu}{lbm} or \frac{KJ}{kg}\right);$$

$$p = pressure\ (psi\ or\ KPa); v = specific\ volume(\frac{ft^3}{lbm}\ or\ or\ \frac{m^3}{kg})$$

Enthalpy is commonly found in steam tables as shown in the below example.

Temperature	Pressure	Specific volume		Internal energy		Enthalpy		
Temp (°F)	Press. $(\frac{lbf}{in^2})$	sat. liquid v_f $\left(\frac{ft^3}{lbm}\right)$	sat. vapor v_g $\left(\frac{ft^3}{lbm}\right)$	sat. liquid u_f $\left(\frac{Btu}{lbm}\right)$	sat. vapor u_g $\left(\frac{Btu}{lbm}\right)$	sat. liquid h_f $\left(\frac{Btu}{lbm}\right)$	evap h_{fg} $\left(\frac{Btu}{lbm}\right)$	sat. vapor h_g $\left(\frac{Btu}{lbm}\right)$
650	2208	.026	0.16	685	1054	696	424	1120

If you take the saturated liquid values and compare them with the above equation, you should find that the equation holds true.

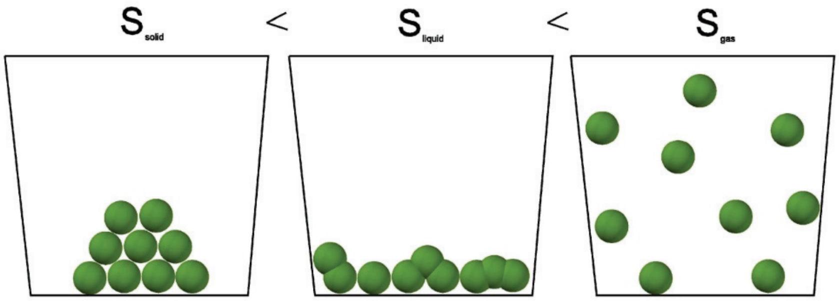
$$h_f = u_f + pv_f * (\frac{0.18505 \frac{BTU}{lbm}}{1\, psi * ft^3})$$

$$696 = 685 + 2{,}208 * .026 * (\frac{0.18505 \frac{BTU}{lbm}}{1\, psi * ft^3})$$

$$696 = 696$$

2.4 ENTROPY

Entropy is the measure of disorder in a fluid. For example, a solid has low entropy, because it is orderly and the molecules have less possible configurations at any given time. A liquid has a higher entropy than a solid, because of an increased amount of disorder. Finally, a gas has the highest entropy in this example. At any given time, the configuration of the gas can be one of many different configurations.



Entropy is mostly known for its use in the 2nd law, which states that a system's entropy never decreases. Also entropy is used to describe thermodynamic transitions. If there is no change in entropy then the process is determined to be isentropic. Also a process is reversible if the entropy is not increased and the process is irreversible if the entropy increases.

2.5 SPECIFIC HEAT

The specific heat describes the ease of a fluid or solid to increase in temperature, when heat is applied. Specific heat is also known as heat capacitance and can be thought of as an objects ability to hold and gain heat. For solids and liquids, specific heat is shown as the variable.

$$heat\ capacity = c_p\ (\frac{Btu}{lbm\ °F}\ or\ \frac{kJ}{kg\ °C})$$

Water has a specific heat of $1.0\frac{Btu}{lbm\ °F}$, while aluminum has a specific heat of $0.23\ \frac{Btu}{lbm\ °F}$. As heat is added to water, water will increase in temperature at a slow rate. Since aluminum has a lower specific heat, it needs less energy to raise its temperature.

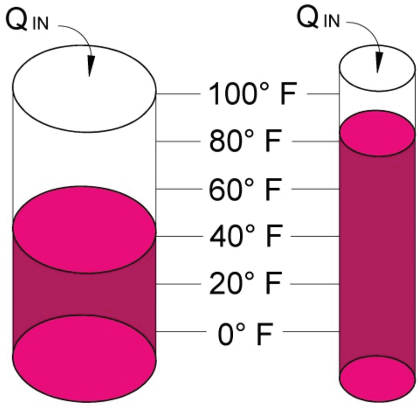


Figure 4: The heat capacity on the left is much larger, thus the same input of heat only raises the temperature by a small amount. The heat capacity on the right is smaller, thus the same amount of heat, greatly raises the temperature.

For gases, the heat capacity value is given in two different forms, one for a gas undergoing a constant-volume process and one for a gas undergoing a constant-pressure process.

$$constant\ volume = c_v;\ constant\ pressure = c_p$$

The most common specific heats are air and water. You should have these values memorized or be able to quickly find these values in the FE Reference Handbook.

$$Gas \rightarrow c_{p,air} = 0.240 \frac{Btu}{lbm\ °R}\ or\ 1.0\ \frac{kJ}{kg\ °K};\ c_{v,air} = 0.171 \frac{Btu}{lbm\ °R}\ or\ 0.718\ \frac{kJ}{kg\ °K}$$

$$Liquid \rightarrow c_{p,water} = 1.0 \frac{Btu}{lbm\ °F}\ or\ 4.18\ \frac{kJ}{kg\ °K}$$

If you encounter a different gas and need the specific heat value, then you should look back at the *NCEES FE Reference Handbook – Thermodynamics Chapter*.

3.0 ENERGY TRANSFERS (ENERGY BALANCES)

Conservation of energy is a law in nature and states that energy cannot be created or destroyed. Therefore, when transfers of states or changes in phases occur in a system, an energy balance equation can be created to solve for the unknown properties in a system or cycle.

Essentially, the equations are derived from the law that

$$\sum \text{energy}_{\text{in}} = \sum \text{energy}_{\text{out}}$$

This equation is used throughout the exam just in different forms. The following shows examples of energy balances on various pieces of equipment.

3.1 TURBINES, PUMPS & COMPRESSORS

Turbines, pumps and compressors change the amount of energy of the incoming fluid by changing the pressure of the fluid. Pumps and compressors increase the pressure of the fluid, while turbines reduce the pressure of the fluid. In a pump and compressor, energy is transferred from a power source to the fluid. In a turbine, energy is transferred from a fluid to provide a power source. In both cases, this power source is called Work. Essentially work is transferred to or from these pieces of equipment.

$$Work_{pump,compressor} = \dot{m}_{fluid} * \left(h_{leaving} - h_{entering}\right)$$

$$\dot{m}_{fluid} * \left(h_{leaving} - h_{entering}\right) = Work_{turbine}$$

3.2 BOILERS, CONDENSERS, EVAPORATORS

Boilers, condensers and evaporators are pieces of equipment where a phase change in a fluid occurs. In boilers and evaporators, liquid is changed to gas and in condensers, gas is changed to liquid. In boilers and evaporators, energy is added into the system by the boiler or evaporator in order to heat the liquid to gas. In condensers, heat is removed from the system by the condenser in order to change the gas to a liquid. The energy balance equations governing these pieces of equipment take into account the phase change of the fluids.

$$Q_{boiler/evap} + \dot{m}_{liquid} * h_{liquid} = \dot{m}_{gas} * h_{gas}$$

$$Q_{condenser} + \dot{m}_{liquid} * h_{liquid} = \dot{m}_{gas} * h_{gas}$$

3.3 HEAT EXCHANGERS

Heat exchangers are used to transfer heat from one fluid the other. Heat exchangers can be used to transfer heat from one hot liquid to a cold liquid, a hot air stream to a cold air stream, from air to liquid or from liquid to air. Since there is no phase change, the energy balance equation is only based on the mass flow rate of both fluids, the temperatures of these fluids and the heat capacity of both fluids.

$$\dot{m}_{cold} * c_{p,cold} * (T_{cold,out} - T_{cold,in}) = \dot{m}_{hot} * c_{p,hot} * (T_{hot,in} - T_{hot,out})$$

3.4 MIXING

In feedwater heaters or in tanks, two fluids may be mixed together. The energy balance on these types of systems involves calculating the total energy of the fluids entering the system, which will equal the energy of the mixed fluid. A few equations shown below highlight this relationship. This mixing energy balance can be applied to both air and liquid. Similarly to boilers, condensers and evaporators, if a phase change occurs, then the energy change due to the phase change must be taken into account.

$$\dot{m}_{cold} * c_{p,cold} * T_{cold} + \dot{m}_{hot} * c_{p,hot} * T_{hot} = \dot{m}_{mix} * c_{p,mix} * T_{mix}$$

$$\dot{m}_{cold} * h_{cold} + \dot{m}_{hot} * h_{hot} = \dot{m}_{mix} * h_{mix}$$

4.0 LAWS OF THERMODYNAMICS

There are three main laws of thermodynamics as shown below. But you don't need to understand the theory behind these thermodynamic laws, you just need to be able to apply these laws to the thermodynamic problems as shown throughout the remaining topics in this section.

1st Law of Thermodynamics: The first law of thermodynamics is the conservation of energy, which was briefly discussed in the previous topic.

2nd Law of Thermodynamics: The second law is that entropy always increases.

3rd Law of Thermodynamics: The third law is that the entropy of a system approaches zero as the temperature approaches absolute zero.

5.0 THERMODYNAMICS PROCESSES

The thermodynamic process that can be tested include, Isentropic, Isobaric, Adiabatic, Isothermal and Isochoric. These processes are all different and have their specific uses in Power Cycles and Refrigeration Cycles. This section will cover everything you need to know about these processes to prepare you for these cycles. But before you get to the processes, you need to understand four key skills which are (1) How to use a pressure-enthalpy diagram and (2) How to use thermodynamic tables, (3) How to use enthalpy-entropy diagrams and (4) how to find thermodynamic properties.

5.1 PRESSURE-ENTHALPY DIAGRAMS (REFRIGERANT)

For fluids undergoing phase changes, especially in steam and refrigerant cycles, the pressure-enthalpy diagram is a very useful tool. The pressure-enthalpy diagram for steam is also an excellent way to gain an understanding of the steam tables. The pressure-enthalpy (P-h) diagram describes the liquid, vapor and mixed region of water. As shown in the following figure, the P-h diagram consists of pressure "P" (psia) on the y-axis and enthalpy "h" (Btu/lbm) on the x-axis. It is important to note that pressure is shown on a logarithmic scale while enthalpy is shown in a normal scale. In the middle of the diagram is the vapor dome. This dome separates the sub-cooled liquid (aka water) on the left side, super-heated vapor (aka steam) on the right side and the liquid-vapor mix region (aka mixed region or wet region) in the middle.

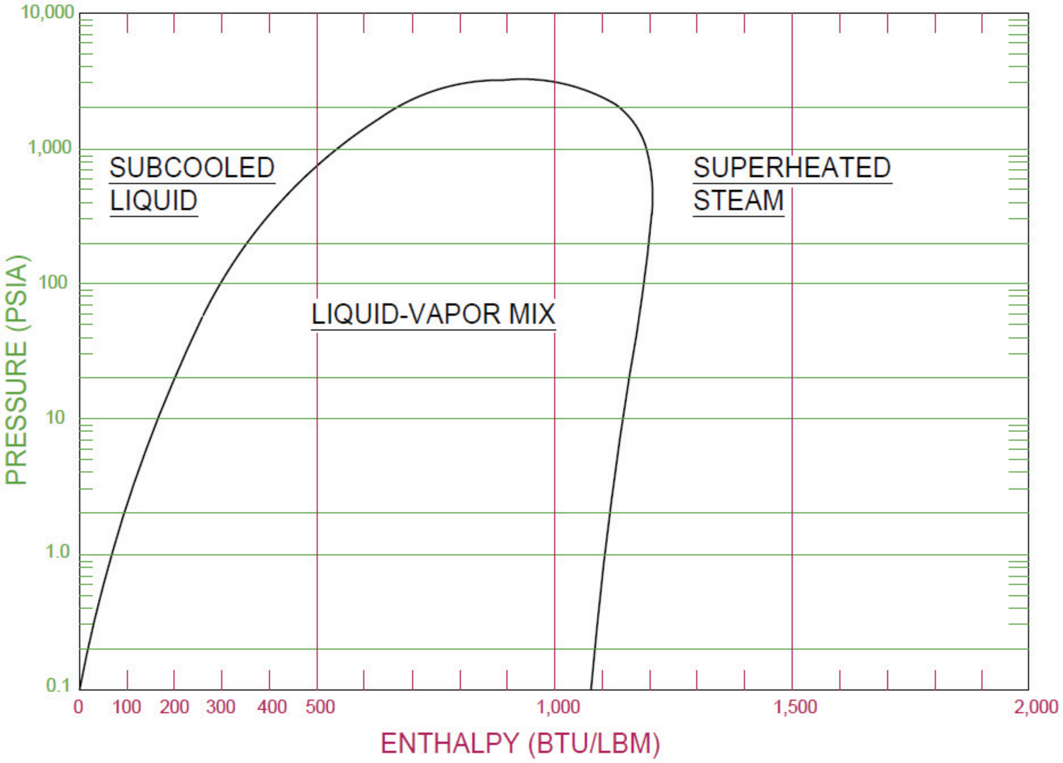


Figure 5: Pressure-enthalpy (P-h) diagram: phases of water. A pressure-enthalpy diagram is not provided for steam, only steam tables are provided. However, if you are able to visualize the steam table, then you will be better equipped to use the steam tables.

The mixed region is cut by upward sloping lines that represent the percentage of vapor, as shown in the following figure. The figure shows that as you move from left to right on a constant pressure line, the percentage of vapor increases from 0% at the saturated liquid to 100% at the saturated vapor line. The percentage of vapor is also known in other terms as steam quality and dryness fraction, where saturated vapor has a steam quality or dryness fraction of 1.

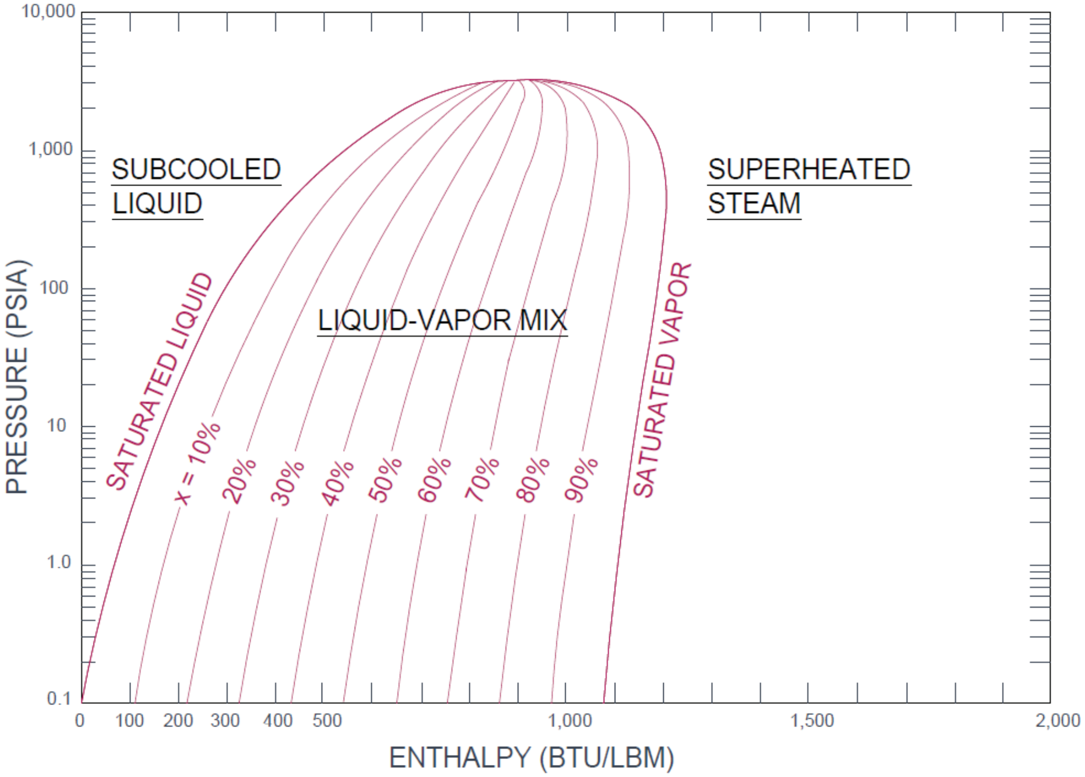


Figure 6: P-h diagram for steam - dryness fraction/steam quality

The P-h diagram is also helpful in illustrating the relationship between the enthalpy of the saturated liquid, saturated vapor and the enthalpy of vaporization.

First, take a horizontal line (constant pressure) from the saturated liquid curve to the saturated vapor curve. These horizontal lines are shown on the next figure. Then read the corresponding enthalpies of the saturated liquid and saturated vapor, shown as h_f and h_g, where h_f is the enthalpy of saturated liquid (fluid) and h_g is the enthalpy of saturated vapor (gas). The difference between these two enthalpies at constant pressure is the enthalpy of vaporization, shown as h_{fg}. The enthalpy of vaporization is the amount of enthalpy required to evaporate liquid at a certain pressure. This relationship between the enthalpy of saturated liquid, vapor and enthalpy of vaporization is shown by the following equation.

$$h_g = h_f + h_{fg}$$

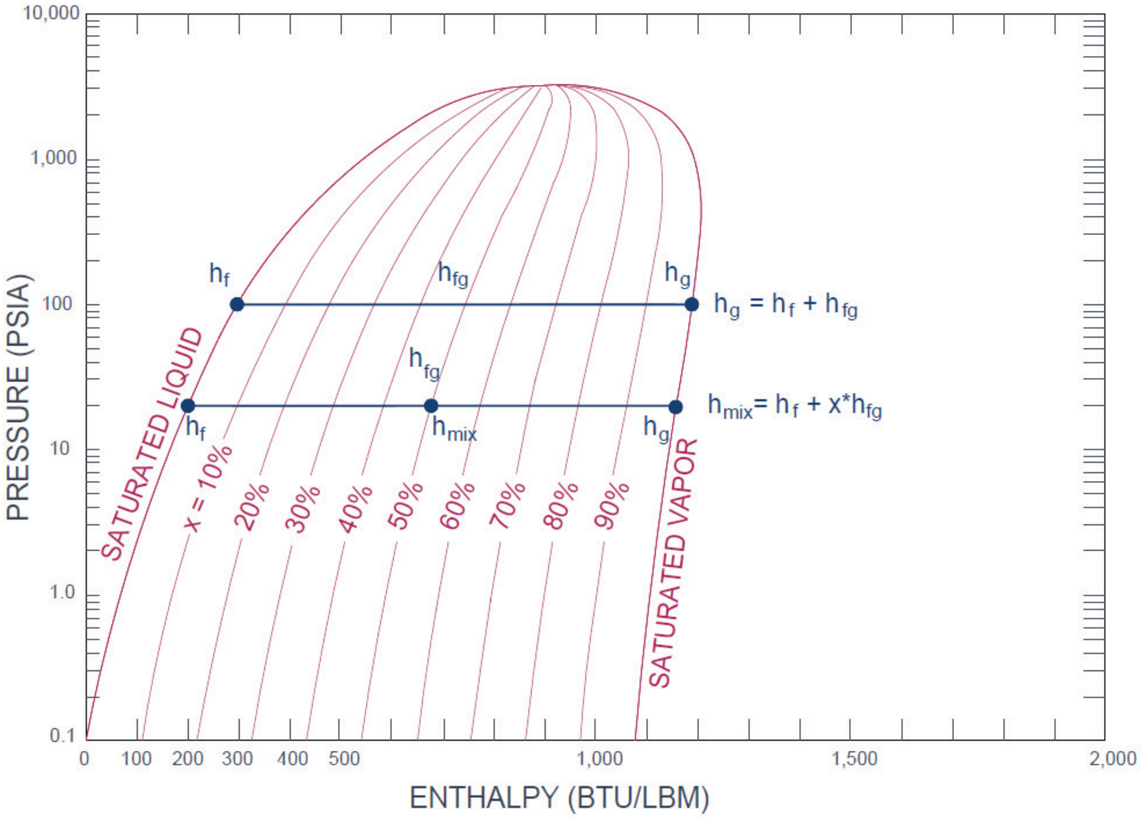


Figure 7: P-h diagram for steam: Enthalpy of vaporization is equal to the different between the enthalpy of the saturated vapor and the enthalpy of the saturated liquid at a certain pressure/temperature.

If a point in the mix region is selected, then the relationship between the enthalpy of the mixed steam and the enthalpy of saturated liquid, enthalpy of vaporization and steam quality is as shown below. "x" is the steam quality or dryness fraction.

$$h_{mix} = h_f + x * h_{fg}$$

In the figure above, the point is shown on the 50% steam quality. Therefore only 50% of the enthalpy of vaporization has been added to the enthalpy of saturated liquid. Substituting 50% for "x" results in the following equation.

$$h_{mix} = h_f + 0.5 * h_{fg}$$

The next important part of the P-h diagram is the constant temperature lines. These lines are characterized by nearly vertical lines in the sub-cooled liquid and super-heated steam region. This means that any increase in enthalpy during these phases, causes the temperature of the liquid or steam to increase and vice versa for decreases in enthalpy. In the mixed region, temperatures are shown to remain constant with increasing enthalpy and are identified as horizontal lines. As enthalpy is added to a saturated liquid, the temperature does not change, because the enthalpy is used to evaporate the liquid to a vapor.

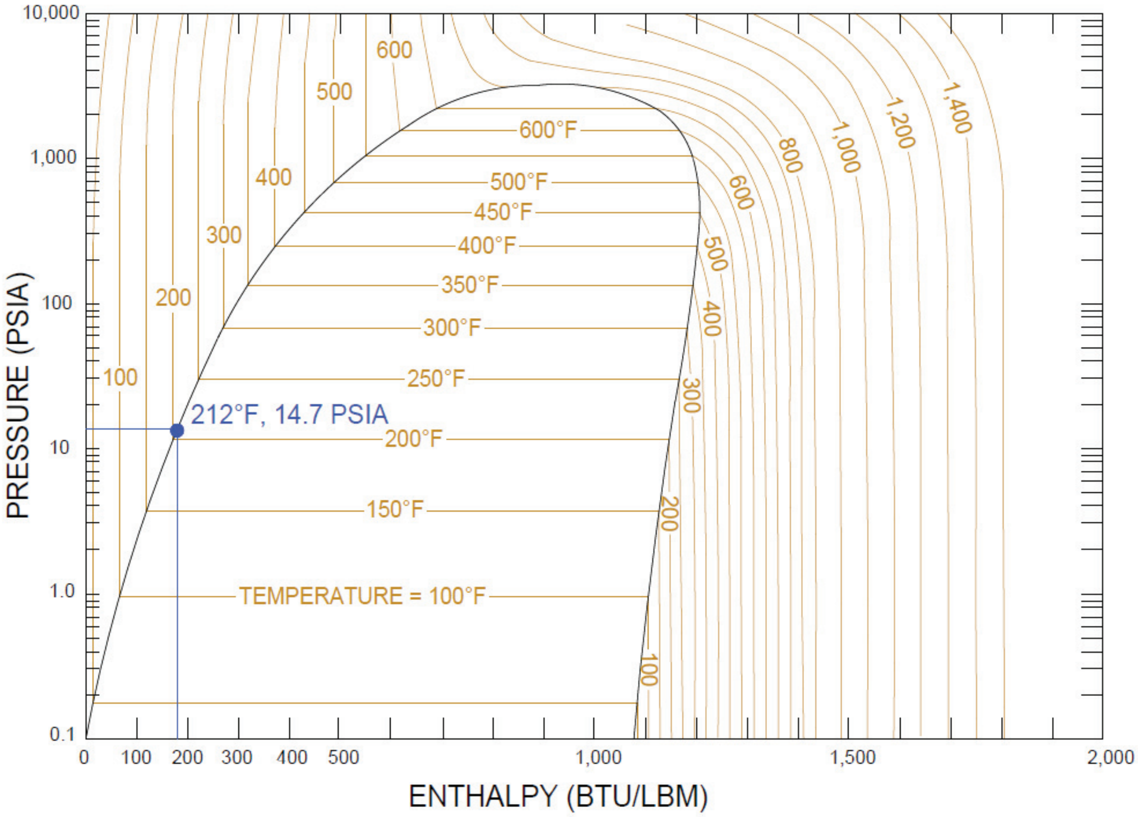


Figure 8: P-h diagram for steam: Constant temperature lines in the mixed region are shown as horizontal lines. This is because any addition/subtraction of enthalpy in this region will not change the temperature. The constant temperature lines are shown vertically in the sub-cooled and super-heated region. In these regions, any subtraction/addition of enthalpy will only affect the temperature.

A common point on the P-h diagram that the engineer should memorize is the location of the boiling point of water at 1 atmosphere (14.7 PSIA), which is 212 °F. It is important to note that if the temperature of a saturated liquid/vapor mixture is known then the pressure can be determined. This is because in the phase change region, pressure and temperature are dependent on each other. In the mixed region, the engineer is unable to determine the location on the P-h diagram with only temperature and pressure. Another value must also be known, like entropy, specific volume or steam quality. For example, if the engineer was asked to determine the enthalpy of water at 212 F, 14.7 PSIA, it would be impossible, because the point could be located anywhere in between the saturated vapor and liquid lines, along the constant pressure/temperature lines in the dome.

Entropy lines on the P-h diagram are shown as upward sloping curves, refer to the figure below.

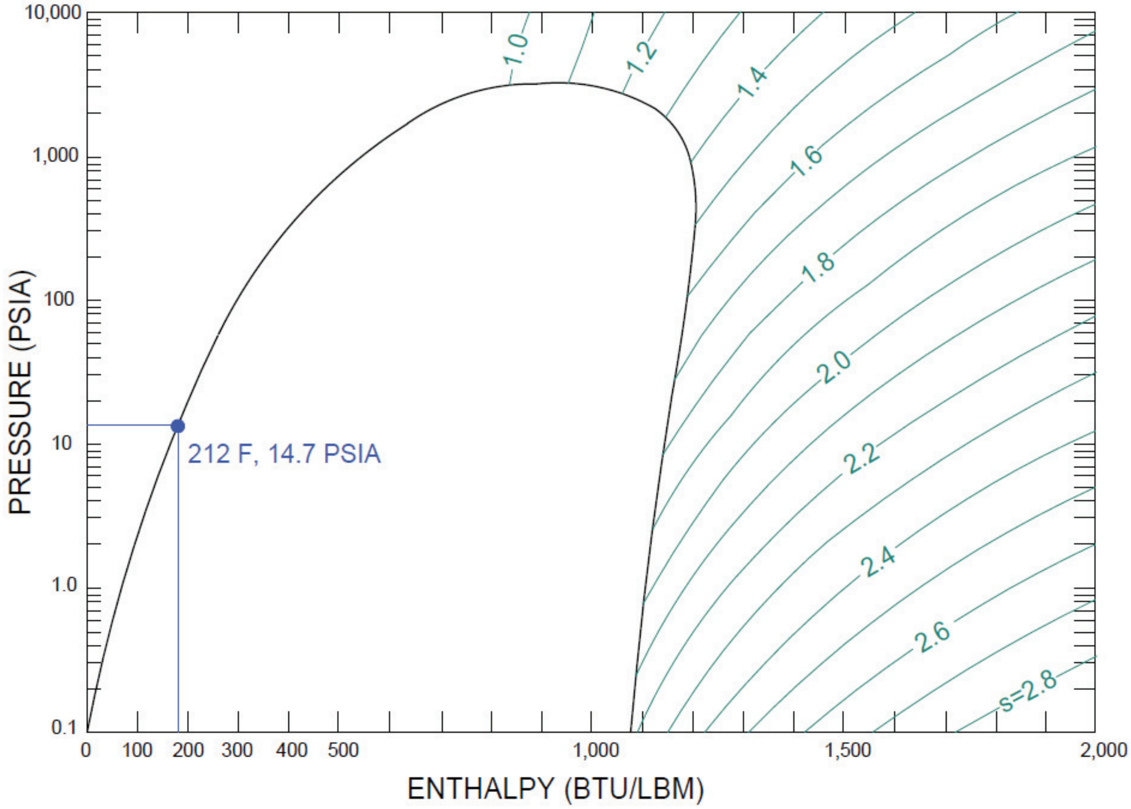


Figure 9: P-h diagram for steam: constant entropy lines

Entropy increases as enthalpy is increased in all three regions. Entropy is shown to decrease in the super-heated steam region when pressure is increased. Constant entropy lines are used during an isentropic process, which means a conversion in which entropy is held constant. One common process is the flow of steam through an ideal steam turbine. Steam enters the turbine at a high pressure and leaves at a lower pressure, transmitting the thermal energy to mechanical work.

The final set of lines on the P-h diagram is the constant specific volume lines, shown below. Specific volume lines are nearly horizontal in the vapor region and nearly vertical in the liquid region. It can be seen that in the liquid region, there is very little change in specific volume. However, in the superheated vapor region, there is a wide range of specific volumes. Specific volume is shown to increase as pressure is lowered and enthalpy is constant.

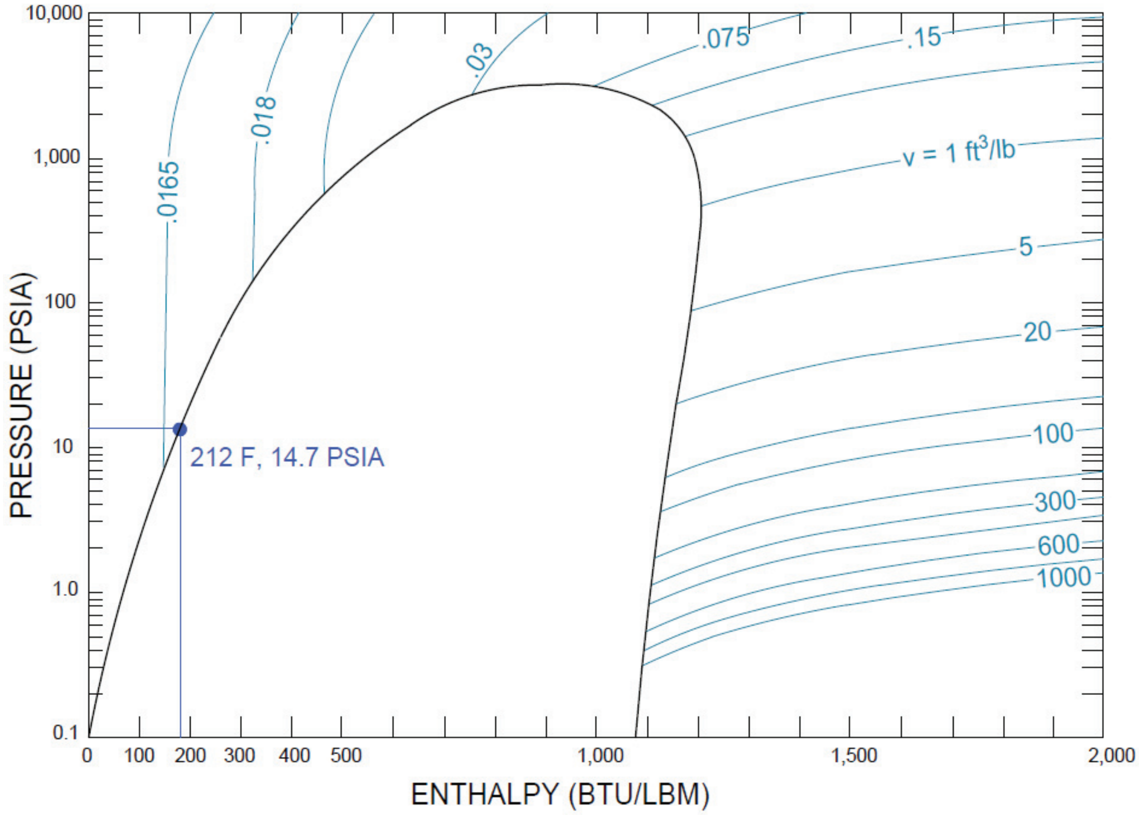
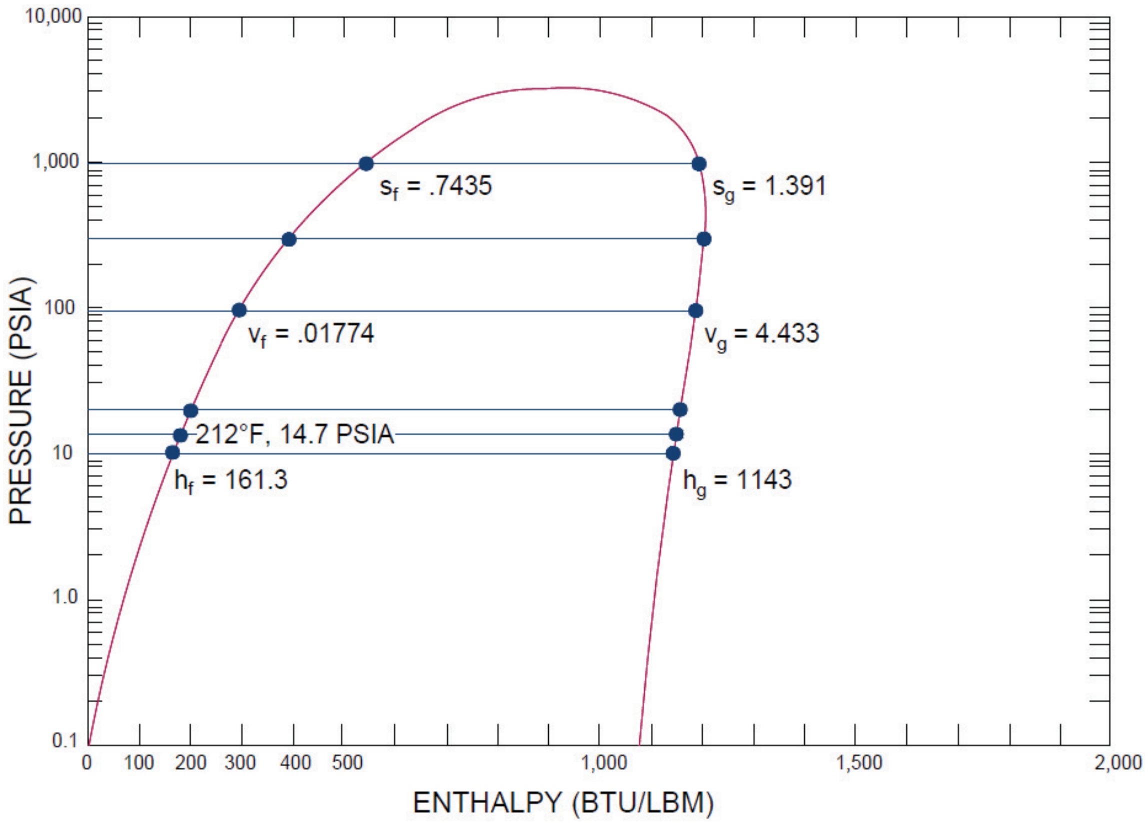


Figure 10: P-h diagram: constant specific volume lines

Although the P-h diagram is a very powerful tool, typically the tables are used to solve steam problems. Steam tables are simply a listing of the values of specific volume, enthalpy and entropy as a function of pressure and temperature at the saturated liquid and vapor curves.

5.2 THERMODYNAMIC TABLES (STEAM & REFRIGERANT)

There are three main types of steam tables that the engineer must be able to use the, (1) Saturation Tables as a function of pressure; (2) Saturation Tables as a function of temperature and (3) Superheated Steam Tables. Graphically the steam tables show the values of the outer dome on the pressure-enthalpy diagram. The following figure shows the points that are selected for the steam tables. This figure shows the values as a function of pressure.

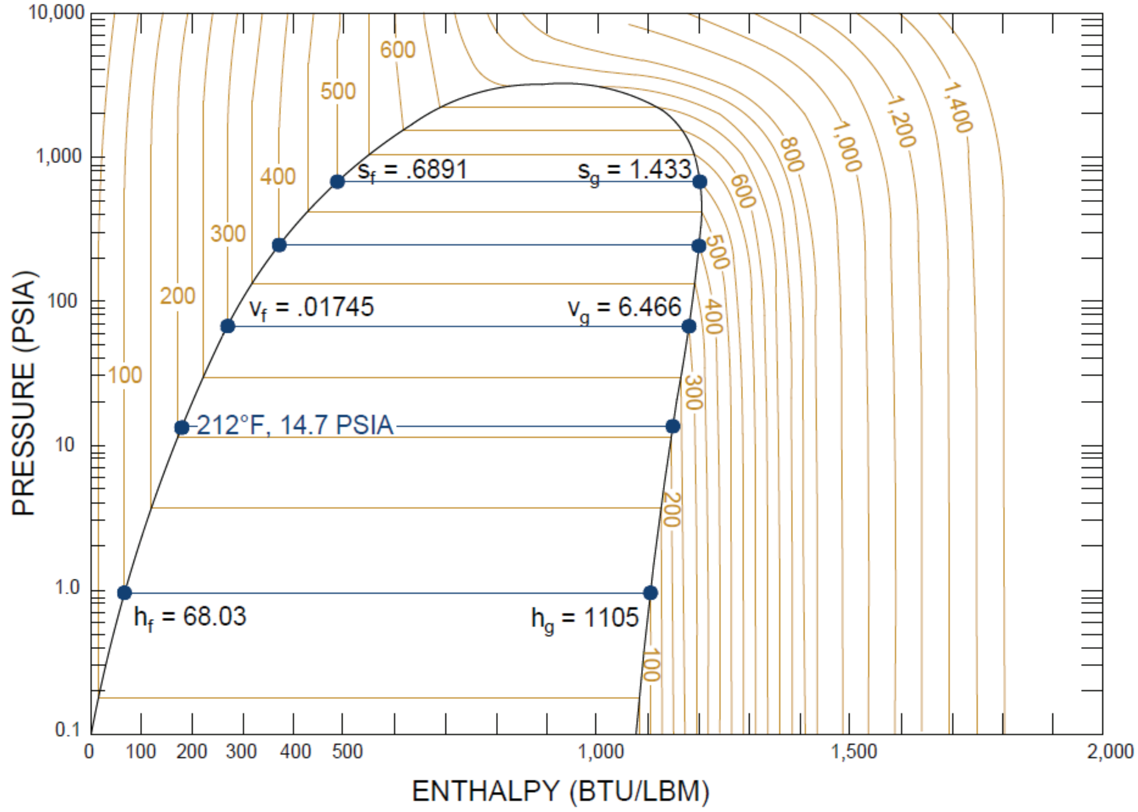


STEAM TABLES BY PRESSURE

P (psia)	T (°F)	v=specific volume (ft^3/lbm)		h=enthalpy (Btu/lb)			s = entropy (Btu/lb-F)	
		v_f	v_g	h_f	h_g	h_{fg}	s_f	s_g
10	193	.01659	38.42	161.3	1143	981.9	.2836	1.788
14.7	212	.01671	26.80	180.2	1150	970.1	.3122	1.757
20	228	.01683	20.09	196.3	1156	959.9	.3359	1.732
100	328	.01774	4.433	298.5	1188	889.0	.4694	1.607
300	417	.01890	1.544	394	1203	809.4	.5882	1.511
1000	545	.02160	0.446	542.7	1193	649.9	.7435	1.391

Figure 11: Steam tables as a function of pressure

The following figure shows the points that are selected for the steam tables. This figure shows the values as a function of temperature.

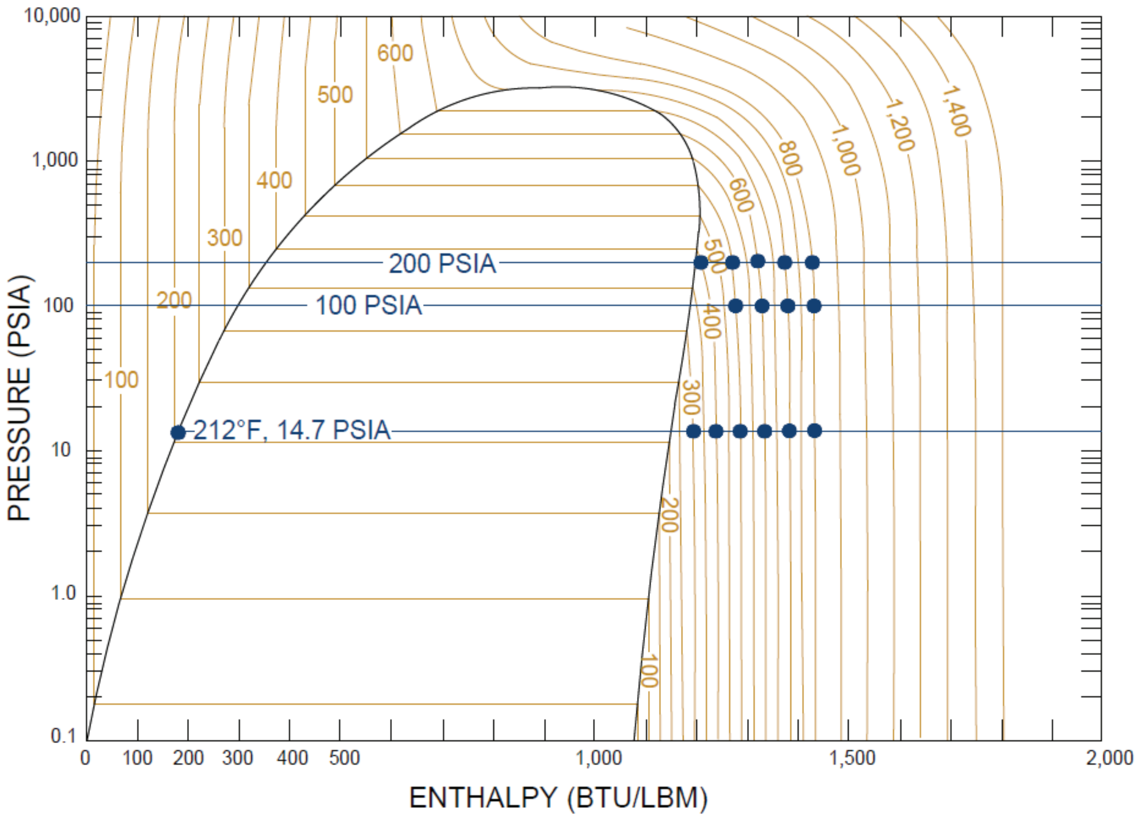


STEAM TABLES BY TEMPERATURE

T (°F)	P (psia)	v=specific volume (ft^3/lbm)		h=enthalpy (Btu/lb)			s = entropy (Btu/lb-F)	
		v_f	v_g	h_f	h_g	h_{fg}	s_f	s_g
100	.951	.01613	349.8	68.03	1105	1037	.1296	1.982
212	14.7	.01671	26.80	180.2	1150	970.1	.3122	1.757
300	67.03	.01745	6.466	269.7	1180	910.3	.4372	1.635
400	247	.01864	1.864	375.1	1201	826.4	.5667	1.528
500	681	.02044	0.6756	487.9	1202	714.4	.6891	1.433

Figure 12: Steam tables as a function of temperature

There are also steam tables for steam in the super-heated region. These steam tables are shown as a function of pressure and temperature. These values are selected because they are the easiest to measure in practice.



SUPERHEATED STEAM TABLES

P (psia)		300 F	400 F	500 F	600 F	700 F
14.7	v_g	30.53	34.67	38.77	42.86	46.93
	h_g	1193	1240	1287	1335	1384
	s_g	1.816	1.874	1.927	1.974	2.018
100	v_g	-	4.936	5.588	6.217	6.834
	h_g	-	1228	1279	1329	1380
	s_g	-	1.652	1.709	1.759	1.804
200	v_g	-	2.362	2.725	3.059	3.380
	h_g	-	1211	1269	1322	1374
	s_g	-	1.560	1.624	1.677	1.724

Figure 13: Superheated steam tables

The steam tables show that as temperature increases the specific volume, the enthalpy and the entropy increase. As pressure increases there is a decrease in specific volume, enthalpy and entropy.

5.3 ENTHALPY-ENTROPY DIAGRAMS (MOLLIER DIAGRAM)

The Mollier diagram also known as the enthalpy-entropy diagram shows graphically the various properties of steam ranging from superheated steam to the mixed region. The diagram does not provide water (liquid) properties. A sample of the diagram is shown below in order to illustrate the main points of the diagram and how to use the diagram. Refer to the actual tables located in the *NCEES FE Reference Handbook*.

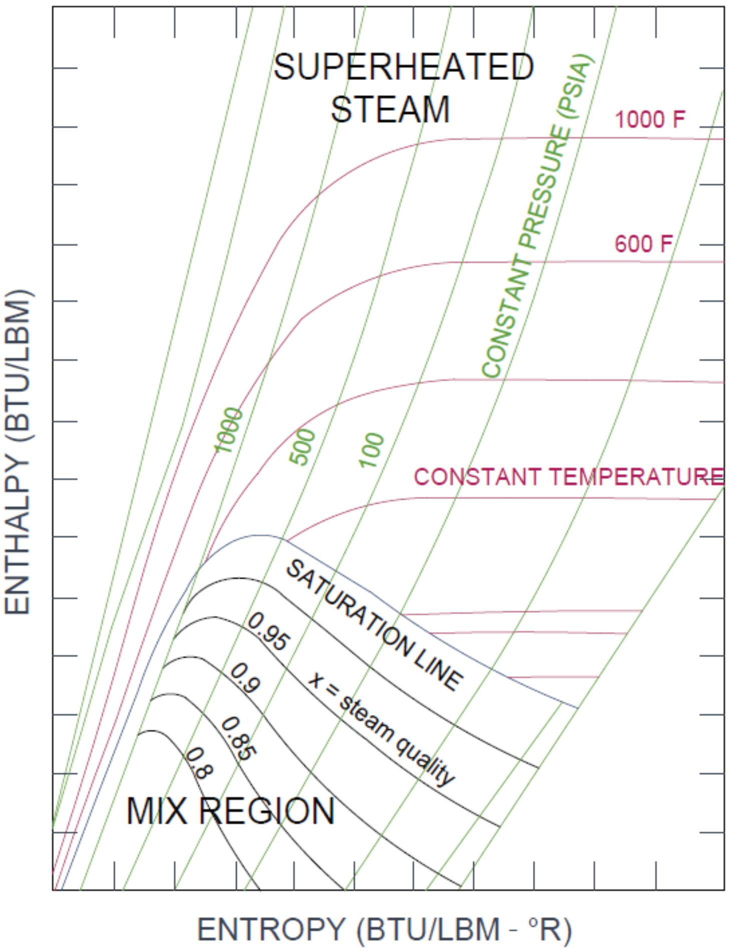

SUPERHEATED STEAM
CONSTANT PRESSURE (PSIA)
1000 F
600 F
1000
500
100
CONSTANT TEMPERATURE
SATURATION LINE
0.95
0.9
0.85
0.8
x = steam quality
MIX REGION
ENTHALPY (BTU/LBM)
ENTROPY (BTU/LBM - °R)

Figure 14: Simplified Mollier diagram

First, inspect the axes, note that the y-axis indicates enthalpy and the x-axis indicates entropy. The Mollier diagram shows two regions, the mixed region of vapor and liquid and the super-heated vapor steam region. The two regions are separated by the downward sloping saturation line, where steam quality is equal to 1. Secondly, notice the upward sloping (left to right) constant pressure lines. Constant dryness fraction or steam quality lines are shown as downward sloping in the mix region. Finally, the diagram has slightly downward sloping constant temperature lines, which is only applicable in the super heat region.

5.4 FINDING THERMODYNAMIC PROPERTIES (STEAM)

In practice, the pressure and temperature of steam can easily be found via measurement. The other useful properties of steam, such as entropy, enthalpy and specific volume, must be found through the use of the (1) P-h Diagram, (2) Mollier Diagram and (3) Steam Tables.

A simple way to find the properties of steam given the temperature and pressure is to draw a simple P-h diagram. For example, assume water is at 14.7 psia and 60 F. Next draw a simple P-h diagram. It is known that at 14.7 PSIA (1 ATM), the boiling point is 212 F, thus the constant temperature line in blue can be drawn. Since the temperature of the water is 60 F, then the point must be located to the left along the horizontal constant pressure line. Note that since constant temperature lines are vertical in the sub-cooled liquid region, that the enthalpy of water at 14.7 PSIA, 60 F is equal to the enthalpy of saturated liquid water at 60 F (take vertical line down from 14.7 PSIA, 60 F to the intersection of the saturated liquid curve.

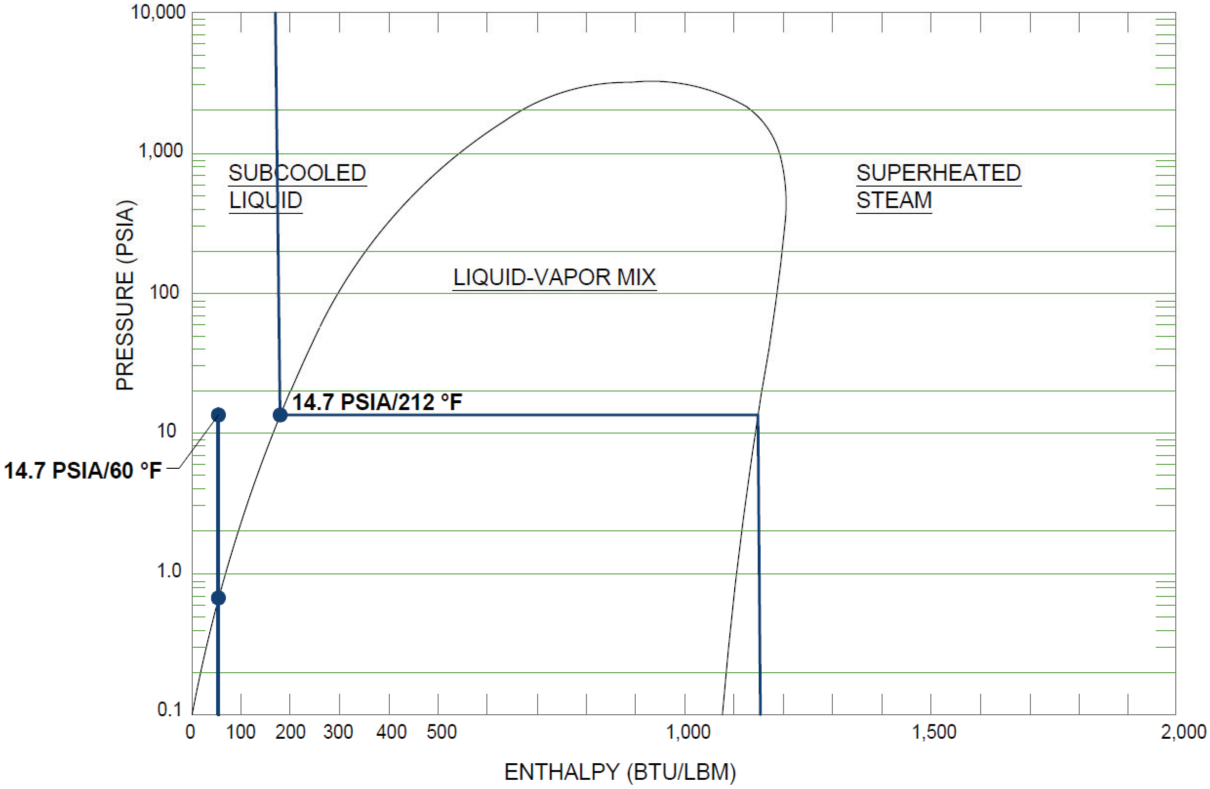


Figure 15: Determining location on p-h diagram

If water was given at a temperature of 212 F and a pressure of 14.7 PSIA, then it would be impossible to find the location. The point could be located anywhere at the intersection of the constant temperature line in blue and the horizontal constant pressure line, which is anywhere on and in between the saturated liquid and vapor curves.

The previous example is simple because it is near the standard boiling point. However, the same method can be used for varying temperatures and pressures. For example, water at 600 °F and 300 PSIA. Start the sketch by drawing in the constant temperature line for 600 °F. Then look up the steam tables as a function of temperature and find 600 °F. In the table, saturated steam at 600 °F corresponds to a pressure of 1542.5 PSIA. It is known that the point in question (600 °F and 300 PSIA) must lie on the constant temperature line and it must lie below the horizontal portion corresponding to a pressure of 1542.5 because the pressure is only 300 PSIA. Thus 600 °F and 300 PSIA steam is located in the super-heated steam region and the super-heated tables should be used.

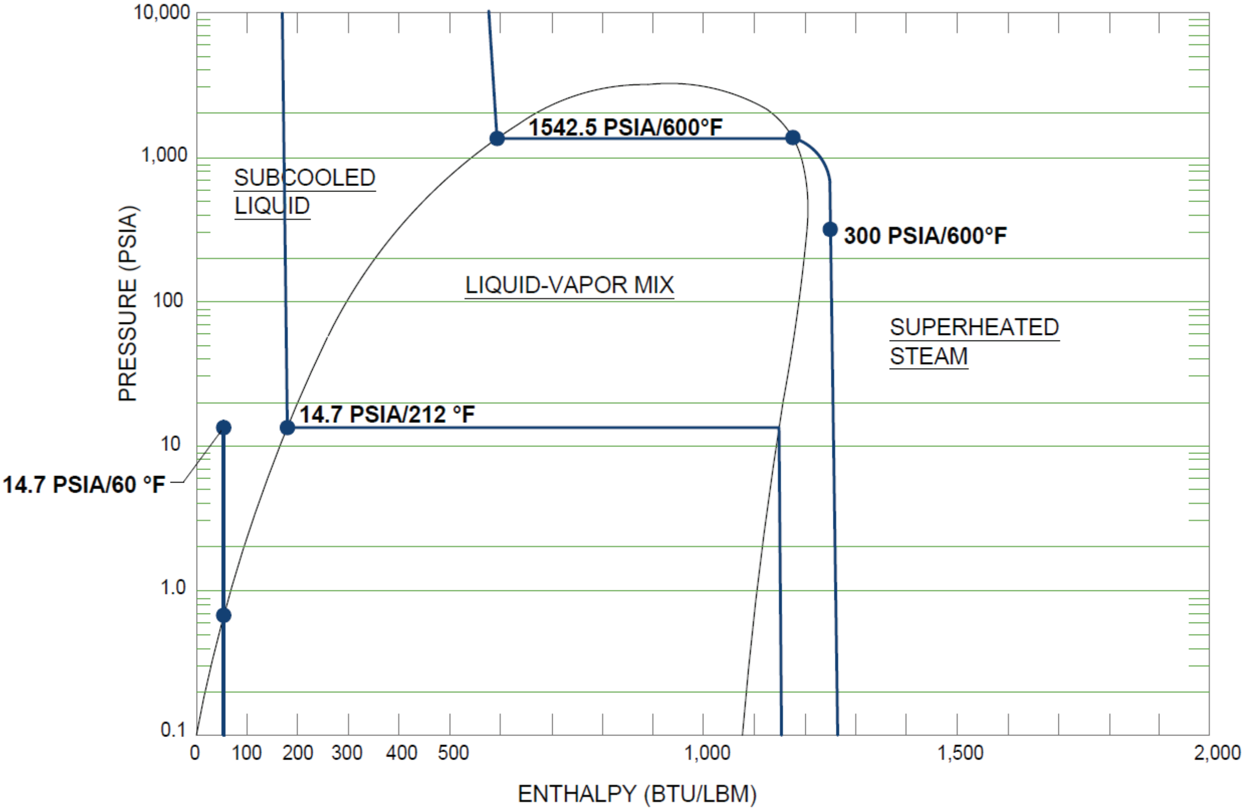


Figure 16: Finding a point on the p-h diagram

5.5 THERMODYNAMIC TRANSITIONS

Thermodynamic transitions describe various movements a fluid may take in a thermodynamic cycle. These are the individual steps that will make a complete thermodynamic cycle. For example, a thermodynamic transition may be a fluid compressed from pressure state 1 to pressure state 2 or a fluid expanded from pressure state 3 to pressure state 4. The way the fluid moves from one state to the next will also provide more information on the final thermodynamic properties of the fluid. For example, one of these transitions may occur isentropically (no change in entropy), adiabatically (no change in heat), or isothermally (no change in temperature).

The following equations are used to find the changes in a fluid's properties after it undergoes a particular thermodynamic transition. The fluid in these equations are assumed to be *ideal gases*. As a reminder, an ideal gas is a theoretical gas that follows the relationship, *PV=nRT*. This was discussed in the Section 12.0 Fluid Mechanics. For the following equations, ideal gases are assumed to be air, nitrogen, oxygen, carbon dioxide, hydrogen, etc. Gas properties are available in the "Gases" table at the end of the Thermodynamics chapter of the *NCEES FE Reference Handbook.* Superheated steam approximates as an ideal gas when pressures are low, below 1.5 psia (10kPa), or when temperatures are very high and pressures are moderate. Therefore, the following equations *should not be used for steam cycles*.

5.5.1 ISENTROPIC TRANSITION FROM PRESSURE STATE 1 TO PRESSURE STATE 2

An isentropic process assumes the following ideal conditions: (1) the process is reversible (e.g. no friction losses) and (2) there is no external heat transfer (adiabatic). This process is often assumed to occur in ideal compressors, nozzles, pumps and turbines, where the entering entropy, s_1, is equal to the leaving entropy, s_2. When you encounter an isentropic transition where the fluid heat capacities also do not change, you can use the following equations to find your final thermodynamic properties. The fluid will most likely be air.

Solving for Pressure change from state 1 to state 2

$$P_2 = P_1 * \left(\frac{v_1}{v_2}\right)^k \quad [ideal\ gas, isentropic, constant\ c_p\ \&\ c_v]$$

The resulting pressure can be found based on the change in volume from state 1 to state 2 and the specific heat ratio of the fluid. k = 1.4 for air.

$$P_2 = P_1 * \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \quad [ideal\ gas, isentropic, constant\ c_p\ \&\ c_v]$$

The resulting pressure can also be found based on the change in temperature from state 1 to state 2 and the specific heat ratio of the fluid. k = 1.4 for air.

Solving for Temperature change from state 1 to state 2

$$T_2 = T_1 * \left(\frac{v_1}{v_2}\right)^{k-1} \quad [ideal\ gas, isentropic, constant\ c_p\ \&\ c_v]$$

The resulting temperature can be found based on the change in volume from state 1 to state 2 and the specific heat ratio of the fluid. k = 1.4 for air.

$$T_2 = T_1 * \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} \quad [ideal\ gas, isentropic, constant\ c_p\ \&\ c_v]$$

The resulting temperature can also be found based on the change in pressure from state 1 to state 2 and the specific heat ratio of the fluid. k = 1.4 for air.

Solving for Work from state 1 to state 2

You can use the entering and leaving enthalpies, along with the mass flow rate to find the work.

$$W = \dot{m} * (h_2 - h_1)$$

Or you can use the input and output pressure, volume, or temperatures conditions.

$$W = \frac{P_2 v_2 - P_1 v_1}{1-k} = \frac{(T_2 - T_1)R}{1-k} \quad [ideal\ gas, isentropic, constant\ c_p \&\ c_v]$$

5.5.2 ISOBARIC TRANSITION WITH HEAT GAIN OR HEAT REJECTION

An isobaric process assumes the following ideal conditions; (1) there are no friction losses and (2) no change in pressure. This process is often assumed to occur at heat transfer equipment like cooling/heating coils, but also at combustion chambers. At these heat exchangers, the fluid loses or gains heat, but the pressure remains the same.

Solving for Temperature change from state 1 to state 2

$$T_2 = T_1 * \left(\frac{v_2}{v_1}\right) \quad [ideal\ gas\ law, isobaric]$$

Solving for Volume change from state 1 to state 2

$$v_2 = v_1 * \left(\frac{T_2}{T_1}\right) \quad [ideal\ gas\ law, isobaric]$$

Solving for heat gain from state 1 to state 2

You can use the entering and leaving enthalpies, along with the mass flow rate to find the heat gain.

$$Q = \dot{m} * (h_2 - h_1)$$

Or you can use the input and output temperatures, along with the mass flow rate to find the heat gain.

$$Q = \dot{m} * c_p * (T_2 - T_1) \quad [ideal\ gas, constant\ c_p]$$

The *NCEES FE Reference Handbook* shows this process in the Thermodynamics section, but it shows it with work as opposed to heat.

$$W = P\ (v_2 - v_1)$$

As the volume increases, the temperature will also increase. When the volume decreases, the temperature will decrease.

5.5.3 ISOCHORIC TRANSITION WITH HEAT GAIN OR REDUCTION

An isochoric transition is a constant volume process. An example of this process is a container with a fixed volume of a gas. The gas is then heated or cooled. The change in heat will result in a change in temperature and pressure of the gas.

Solving for Temperature change from state 1 to state 2

$$T_2 = T_1 * \left(\frac{P_2}{P_1}\right) \quad [ideal\ gas\ law, isochoric]$$

Solving for Volume change from state 1 to state 2

$$P_2 = P_1 * \left(\frac{T_2}{T_1}\right) \quad [ideal\ gas\ law, isochoric]$$

Solving for heat gain from state 1 to state 2

Solving for the heat gain will be the same equation.

$$Q = \dot{m} * (h_2 - h_1) = \dot{m} * c_p * (T_2 - T_1)$$

Work for a closed isochoric system, however, will equal zero.

5.5.4 ADIABATIC TRANSITIONS

An adiabatic transition occurs when heat does not enter or leave a system. This is characterized by no change in enthalpy. An example of an adiabatic transition is adiabatic expansion through a turbine or expansion valve.

$$\Delta Q = 0 \quad [Adiabatic]$$

Work for an adiabatic system will equal the change in enthalpy.

$$W = \dot{m} * (h_2 - h_1)$$

5.5.5 ISOTHERMAL TRANSITIONS

An isothermal transition will have no change in temperature. An example of an isothermal transition is isothermal compression. This next example will take you through how to use the isothermal equations in the Thermodynamics section of the *NCEES FE Reference Handbook*.

Example 1 – Isothermal Compression

A compressor is initially at a volume 0.7 m^3 of air at 112 kPa and 0°C. The air is compressed to 0.2 m^3. The compressor is cooled to keep the temperature the same at both volumes. What is the work done?

This problem can be done with either the initial pressure and volume or the final pressure and volume. You should get the same answer.

$$Work = P_{initial}V_{initial} \ln\left(\frac{V_{final}}{V_{initial}}\right) = (112\ kPa)(0.7\ m^3)\ln\left(\frac{0.2}{0.7}\right) = -98.2\ kJ$$

$$Work = -98.2\ kJ$$

$$Conversion\ 1\ kJ = 1\frac{kPa}{m^3}$$

Also remember that for an ideal gas, the pressure-volume relationship will be maintained.

$$P_{initial}V_{initial} = P_{final}V_{final}$$

$$112\ kPa * 0.7\ m^3 = P_{final} * 0.2\ m^3$$

$$P_{final} = 392\ kPa$$

Another equation that can be used for isothermal transitions is shown below.

$$Work = RT\ln\left(\frac{V_{final}}{V_{initial}}\right)$$

5.5.6 POLYTROPIC TRANSITIONS

A polytropic transition will have a slight change in temperature. It is a transition that is somewhere in between isentropic and isothermal. An example of a polytropic transition is compression with cooling. This next example will take you through how to use the isothermal equations in the Thermodynamics section of the *NCEES FE Reference Handbook*.

$$P(v)^n = constant \quad [ideal\ gas, polytropic]$$
$$where\ 1 < n < k;\ \ k = specific\ heat\ ratio$$

$$when\ n = k \Rightarrow isentropic; n = 1 \Rightarrow isothermal$$

Example 1 – Polytropic Compression

A compressor is initially at a volume 0.7 m^3 of air at 112 kPa and 0°C. The air is compressed to 0.2 m^3. The compressor is slightly cooled, but the process is polytropic, not isothermal. What is the work done? n=1.2

This problem can be done with the polytropic equation in the *NCEES FE Reference Handbook*. First, you need to find the final pressure. The ideal gas pressure-volume relationship changes to the equation shown below.

$$P_{initial}(V_{initial})^n = P_{final}(V_{final})^n$$

$$112\ kPa(0.7\ m^3)^{1.2} = P_{final}(0.2\ m^3)^{1.2}$$

$$P_{final} = 503.6\ kPa$$

Now, you can find the work done, with the below equation.

$$Work = \frac{(P_{final}V_{final} - P_{initial}V_{initial})}{1-n}$$

$$Work = \frac{\left(503.6\ kPa * (0.2\ m^3) - 112\ kPa\ (0.7\ m^3)\right)}{1-1.2}$$

$$Work = -111.6\ kJ$$

5.5.7 ISENTROPIC TRANSITIONS

An isentropic transition will have the maximum change in temperature. An example of an isentropic transition is isentropic compression. This next example will take you through how to use the isothermal equations in the Thermodynamics section of the *NCEES FE Reference Handbook*.

Example 1 – Isentropic Compression

A compressor is initially at a volume 0.7 m^3 of air at 112 kPa and 0 °C. The air is compressed to 0.2 m^3. The compressor is not cooled. What is the work done? k=1.4

This problem can be done with the isentropic equation in the FE Reference Handbook. First, you need to find the final pressure. The ideal gas pressure-volume relationship changes to the equation shown below. The k-value is the ratio of the c_p to c_v and for air this becomes 1.4.

$$k = \frac{c_p}{c_v} = 1.4$$

$$P_{initial}(V_{initial})^k = P_{final}\left(V_{final}\right)^k$$

$$112\ kPa(0.7\ m^3)^{1.4} = P_{final}(0.2\ m^3)^{1.4}$$

$$P_{final} = 647\ kPa$$

Now, you can find the work done, with the below equation.

$$Work = \frac{\left(P_{final}V_{final} - P_{initial}V_{initial}\right)}{1-k}$$

$$Work = \frac{\left(647\ kPa * (0.2\ m^3) - 112\ kPa\ (0.7\ m^3)\right)}{1-1.4}$$

$$Work = -127.5\ kJ$$

As you can see, the work increases from isothermal to polytropic to isentropic compression.

6.0 PERFORMANCE OF COMPONENTS

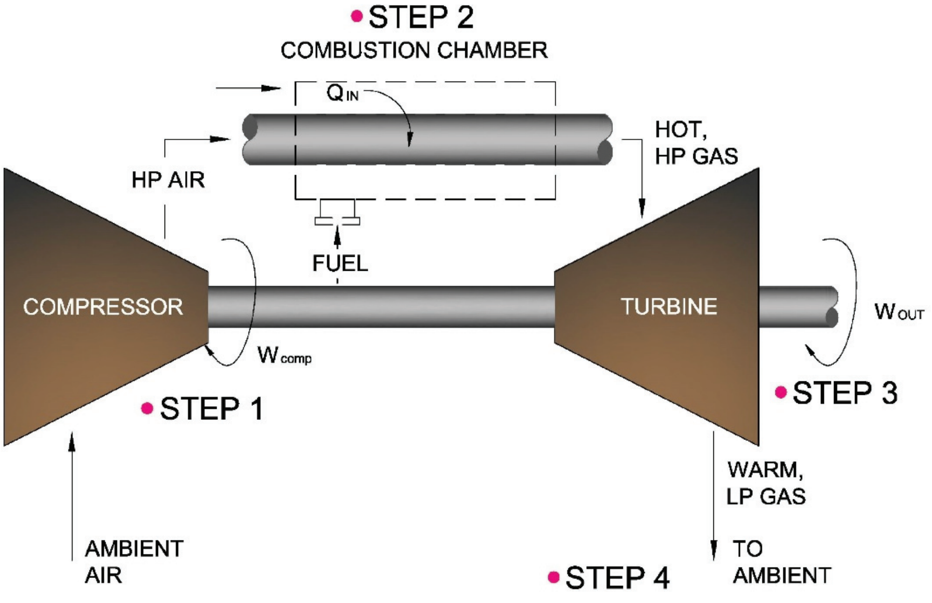
The performance of components topic is covered in the other topics throughout this section. But as a brief overview, performance of components describes the operational efficiency of the various components found in thermodynamics. A list of these components is shown below.

- Boilers
- Condensers
- Evaporators
- Turbines
- Heat Exchangers
- Expansion Valves
- Combustion Chambers

7.0 POWER CYCLES

7.1 IDEAL OPEN GAS TURBINE CYCLE

The two major application areas of the open gas turbine cycle are for aircrafts and electric power generation. In an open cycle the working fluid (air) only passes through the cycle once and is then exhausted to the atmosphere. In a closed cycle, the working fluid (air) is recycled through the cycle. One assumption that you should be aware of is that the mass flow rate through both the open and closed cycles are assumed to be constant. Although fuel does enter the cycle, it is assumed that the only mass flow rate to be considered in doing problems is the mass flow rate of the air. This is typically a safe assumption because the ratio of air to fuel is quite large, typically fuel can be around 2% of the total mass flow rate.



STEP 2
COMBUSTION CHAMBER
Q_{IN}
HOT, HP GAS
HP AIR
FUEL
COMPRESSOR
TURBINE
W_{OUT}
W_{comp}
STEP 1
STEP 3
WARM, LP GAS
AMBIENT AIR
STEP 4
TO AMBIENT

Figure 17: The above figure shows the components of an open Brayton cycle. Step 1 to 2 is the isentropic compressor. Step 2 to 3 is a constant pressure combustion chamber. Step 3 to 4 is the isentropic turbine.

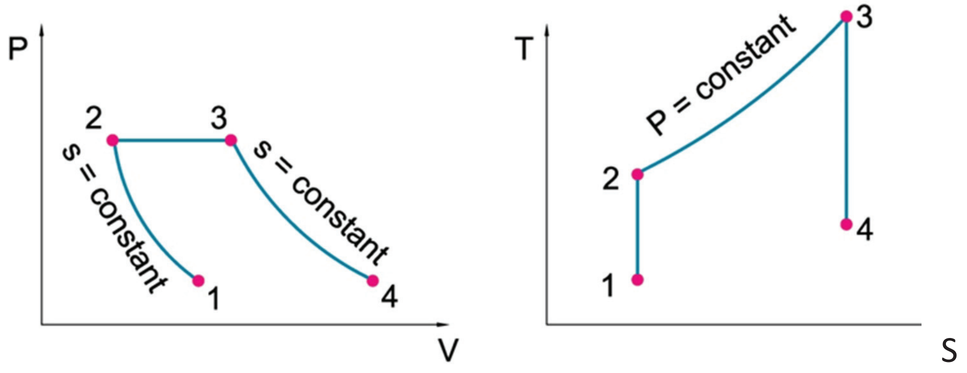


Figure 18: These figures show the constant entropy and constant pressure paths in the Brayton cycle. Use these figures with the previous figure to match each point to each piece of equipment.

7.1.1 STEP 1: COMPRESSOR

The first stage of the cycle is the compressor. Fresh air enters the compressor and is compressed to a higher temperature and pressure. Typically is assumed that this compression occurs isentropically, meaning that there is no change in entropy.

$$W_{comp} = \dot{m} * (h_2 - h_1)$$

7.1.2 STEP 2: COMBUSTION CHAMBER

Next, the high pressure/temperature air enters a combustion chamber. Here fuel goes through the combustion process, which will be discussed later in this section. The result of the combustion process is added heat to the air and combustion gas mixture.

$$Q_{in} = \dot{m} * (h_3 - h_2)$$

7.1.3 STEP 3: TURBINE

This hot gas mixture then enters the turbine and as the gas expands (loses pressure and energy). The energy is converted to work to turn the turbine. Following the turbine, an electric generator is turned to produce electricity. Also part of the work produced by the turbine is used to operate the compressor. Notice how the turbine shaft is connected to the shaft that drives the compressor. The back work ratio is used to describe the ratio of the turbine work that is used to drive the compressor.

$$W_{turb} = \dot{m} * (h_3 - h_4)$$

$$Back\ work\ ratio = \frac{W_{comp}}{W_{turbine}} = \frac{h_2 - h_1}{h_3 - h_4}$$

7.1.4 STEP 4: EXHAUST

Finally the hot gas mixture is exhausted to the atmosphere.

7.1.5 BRAYTON CYCLE EFFICIENCY

The cycle efficiency can be found by taking the amount of useful work produced by the cycle and dividing it by the input energy.

$$Efficiency = \frac{W_{out}}{W_{in}} = \frac{W_{turbine} - W_{comp}}{Q_{in}}$$

7.2 IDEAL BRAYTON CYCLE - CLOSED GAS TURBINE CYCLE

The Brayton cycle or closed gas turbine cycle is similar to the open gas turbine cycle, except the air is not exhausted to the atmosphere. In order to accomplish this, heat exchangers are placed at the combustion chamber section and at the exhaust section.

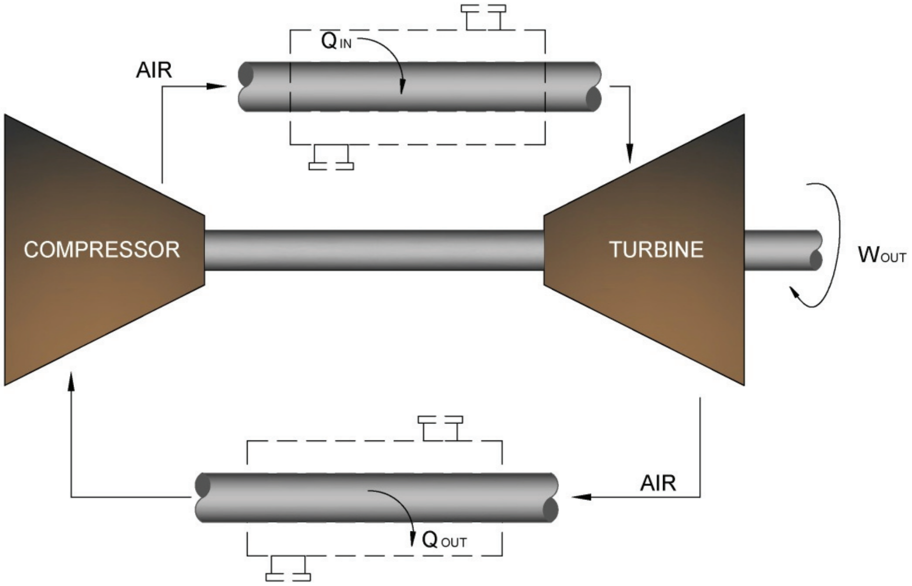


Figure 19: The above figure shows the components of a Brayton cycle. Step 1 to 2 is the isentropic compressor. Step 2 to 3 is a constant pressure heat exchanger. Step 3 to 4 is the isentropic turbine and Step 4 to 1 is a constant pressure heat exchanger.

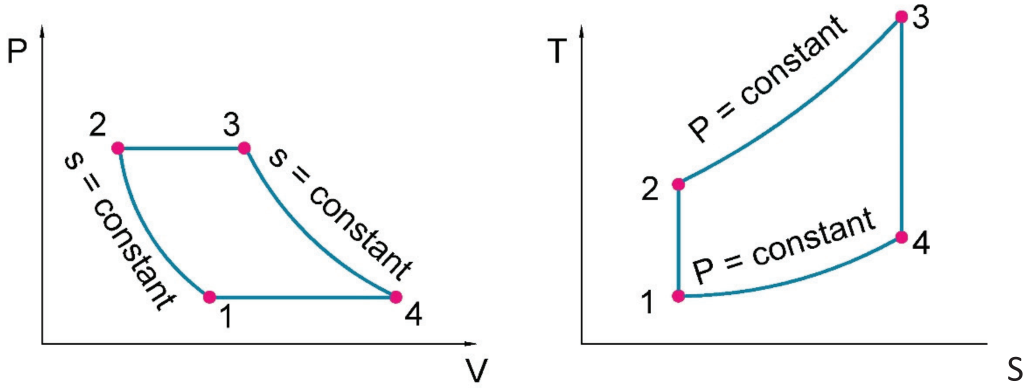


Figure 20: This figures show the constant entropy and constant pressure paths in the Brayton cycle. Use this figure with the previous figure to match each point to each piece of equipment.

7.2.1 STEP 1 TO 2: COMPRESSOR

The compressor state of the cycle still remains first. But instead of fresh air, recycled air enters the compressor and is compressed to a higher temperature and pressure. The compression process is isentropic, meaning that there is no change in entropy.

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}$$

7.2.2 STEP 2 TO 3: COMBUSTION CHAMBER AND HEAT EXCHANGER

Next, the high pressure/temperature air enters a heat exchanger. Here the air is heated up from an outside heat source. This heat source could be a combustion chamber, where fuel goes through the combustion process and adds heat to one side of the heat exchanger. The recycled air then picks up this heat. The combustion gases are kept on the other side of the heat exchanger and are not a part of this cycle.

$$q_{in} = \dot{m} * (h_3 - h_2) = \dot{m} * c_p * (T_3 - T_2)$$

7.2.3 STEP 3 TO 4: TURBINE

This hot pressurized air then enters the turbine and as the gas expands (loses pressure and energy), the energy is converted to work to turn the turbine. Following the turbine, an electric generator is turned to produce electricity.

$$\frac{T_3}{T_4} = \left(\frac{P_3}{P_4}\right)^{\frac{k-1}{k}}$$

7.2.4 STEP 4 TO 1: HEAT EXCHANGER

Finally the hot, low pressure air releases its heat through another heat exchanger and the cool, low pressure air repeats the cycle at step 1.

$$q_{out} = \dot{m} * (h_4 - h_1) = \dot{m} * c_p * (T_4 - T_1)$$

7.2.5 EFFICIENCY

The cycle efficiency can be found by taking the amount of useful work produced by the cycle and dividing it by the input energy.

$$Efficiency = \frac{W_{out}}{W_{in}} = \frac{W_{turbine} - W_{comp}}{Q_{in}}$$

The cycle efficiency can also be found with the below equation.

$$Efficiency = 1 - \left(\frac{P_2}{P_1}\right)^{\frac{1-k}{k}}$$

$$\left(\frac{P_2}{P_1}\right) = (\frac{P_3}{P_4})$$

7.3 IDEAL BRAYTON CYCLE - CLOSED GAS TURBINE CYCLE WITH HEAT RECOVERY

The Brayton cycle or closed gas turbine cycle is similar to the open gas turbine cycle, except the air is not exhausted to the atmosphere. In order to accomplish this, heat exchangers are placed at the combustion chamber section and at the exhaust section.

7.4 ACTUAL BRAYTON CYCLE - CLOSED GAS TURBINE CYCLE

The actual Brayton cycle is a little different from the ideal Brayton cycle, because processes do not occur isentropically in reality, meaning that entropy is added. Also excess heat or less heat can be added during the steps.

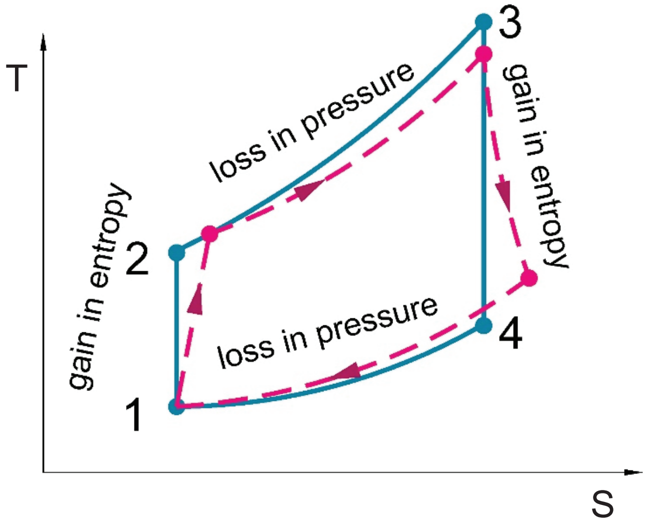


Figure 21: These figures show the constant entropy and constant pressure paths in the Brayton cycle. Use these figures with the previous figure to match each point to each piece of equipment.

7.4.1 STEP 1: COMPRESSOR

First, the compressor will consume excess work than ideally. The excess work will cause an increase in temperature and an increase in entropy.

7.4.2 STEP 2: COMBUSTION CHAMBER AND HEAT EXCHANGER

Secondly, the heat exchange will not occur without a pressure drop. There will be some pressure loss as the fluid passes through the heat exchanger. These losses are due to friction.

7.4.3 STEP 3: TURBINE

Similar to step 1, the turbine will cause an increase in entropy when compared to the ideal case. However, different from step 1, less work will be extracted by the turbine, this will cause a higher temperature as compared to the ideal case.

7.4.4 STEP 4: HEAT EXCHANGER

Similar to step 2, the heat exchange will not occur without a pressure drop. There will be some pressure loss as the fluid passes through the heat exchanger.

The actual gas-turbine cycle is different from the ideal Brayton cycle since there are irreversibilities. In an actual gas-turbine cycle, the compressor consumes more work and the turbine produces less work than that of the ideal Brayton cycle. The irreversibilities in an actual compressor and an actual turbine can be quantified with the following efficiency equations.

The efficiency of the compressor is described through the following equation. The actual change in energy from the incoming to outgoing pressure states is greater than the ideal situation of the compressor because of the heat gains due to the compressor work.

$$n_c = \frac{h_1 - h_{2,isentropic}}{h_1 - h_{2,actual}} = \frac{ideal}{actual} = \frac{no\ compressor\ heat\ gain}{with\ compressor\ heat\ gain}$$

$$where\ n_c = adiabatic\ efficiency\ of\ the\ compressor$$

The efficiency of the turbine is described through the following equation. The actual change is less than the ideal situation of the turbine because of the friction losses in the turbine. The actual turbine cannot convert as much energy as an ideal turbine.

$$n_T = \frac{h_3 - h_{4,actual}}{h_3 - h_{4,isentropic}} = \frac{actual}{ideal} = \frac{no\ pressure\ loss}{with\ pressure\ loss}$$

$$where\ n_T = adiabatic\ efficiency\ of\ the\ turbine$$

7.5 RANKINE CYCLE

Steam power plants run vapor power cycles with water as the working fluid. This section introduces the ideal cycle for vapor power cycle, the Rankine cycle.

These steam powered plants generate electrical power by taking the heat from combusting coal, oil, biofuel or natural gas. The power plant consists of multiple systems working together. An overall view of all systems together is shown below.

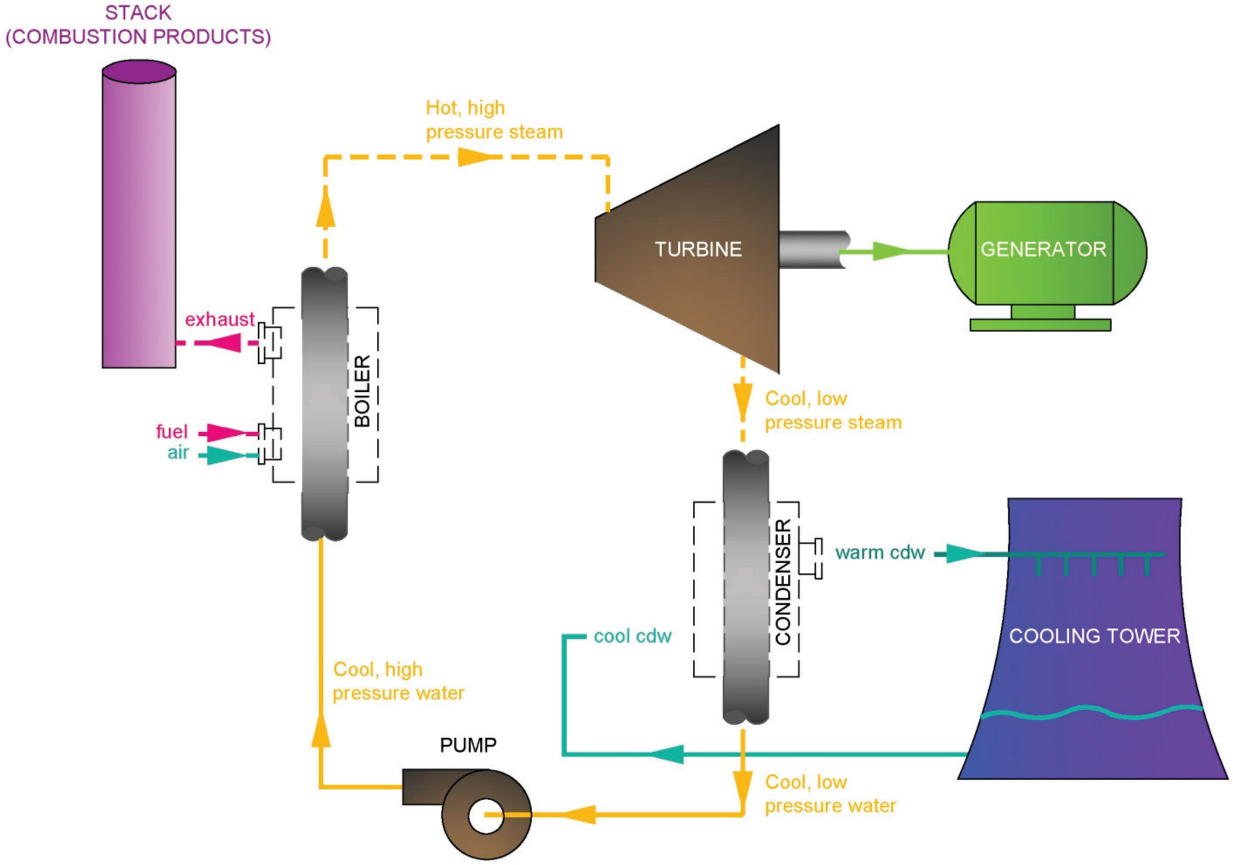

STACK
(COMBUSTION PRODUCTS)
Hot, high
pressure steam
TURBINE
GENERATOR
exhaust
BOILER
fuel
air
Cool, low
pressure steam
CONDENSER
warm cdw
COOLING TOWER
cool cdw
Cool, high
pressure water
PUMP
Cool, low
pressure water

Figure 22: A complete Rankine cycle is shown in yellow.

7.5.1 RANKINE CYCLE (YELLOW)

The thermodynamic cycle described by the yellow lines is called the Rankine cycle. This system consists of a boiler, turbine, condenser and a pump. Fuel, burned in the boiler, heats the water to generate superheated steam. The Rankine system is discussed in this section.

7.5.2 BOILER-COMBUSTION SYSTEM (RED-BLUE)

The combustion process at the boiler and stack is the second system. This system provides the heat to vaporize the water in the Rankine cycle. Fuel is burned in the boiler to heat the water to generate superheated steam. The combustion system is discussed later in this Thermodynamics section.

7.5.3 ELECTRIC GENERATOR (GREEN)

The electric generator uses the rotation of the shaft from the turbine to create electricity.

7.5.4 COOLING TOWER-CONDENSER WATER SYSTEM (BLUE)

The cooling tower subsystem is used to provide cool condenser water to the condenser, such that the Rankine cycle is able to condense the steam from the outlet of the turbine.

The focus of this chapter is subsystem *A*, the Rankine cycle. This steam is used to run the turbine which powers the generator. Electrical energy is generated when the generator windings rotate in a strong magnetic field (subsystem C). After the steam leaves the turbine, it is cooled to its liquid state in the condenser by transferring heat to the cooling water system (subsystem D). The liquid is pressurized by the pump prior to going back to the boiler.

There are four components associated with the ideal Rankine cycle are steady-flow devices, and thus all four processes that make up the Rankine cycle can be analyzed as steady-flow processes. The kinetic and potential energy changes of water are small relative to the heat and work terms, are thus neglected.

The Rankine cycle has four steps:

1. A pump adds pressure to water via isentropic compression.
2. Heat is added to the water at a boiler, which causes the water to vaporize to steam.
3. The high pressure steam is expanded at a turbine, which produces useful work.
4. The low pressure steam is condensed back to a liquid at a condenser and the process is repeated.

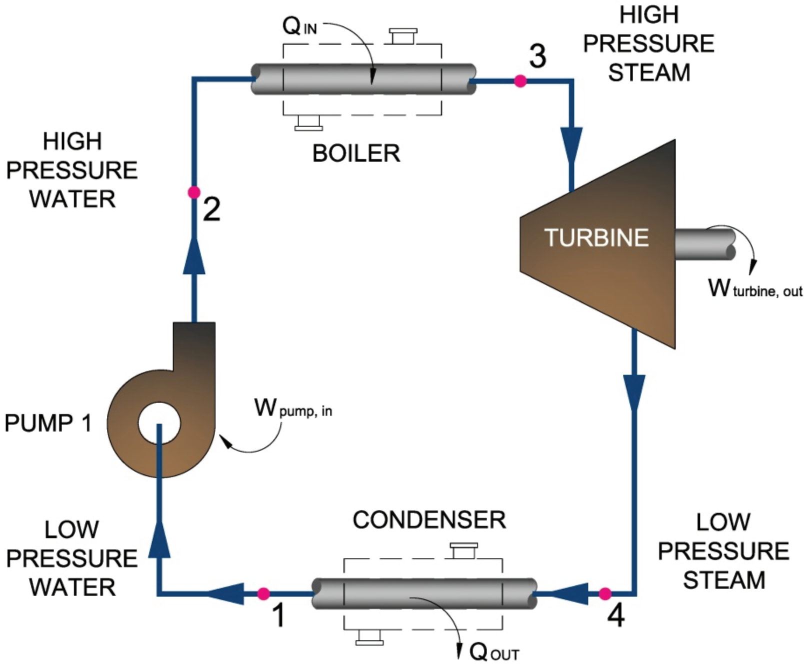


Figure 23: Rankine cycle without reheat

Finding the efficiency and the amount of work produced by the Rankine cycle is heavily dependent on your knowledge of steam tables. Please read through the Thermodynamics section on steam thoroughly. The first set of equations use the enthalpy at each stage to find the work and energy from one stage to the next.

$$Pump \rightarrow Work_{pump} = \dot{m} * (h_2 - h_1)$$

Sometimes you can use this equation, if you are neglecting kinetic and potential energy changes.

$$Pump \rightarrow Work_{pump} = \dot{m} * \frac{P_2 - P_1}{\rho}$$

$$where\ P_2\ and\ P_1 = [psi], \rho = \left[\frac{lbs}{ft^3}\right]; \dot{m} = [\frac{lbs}{hr}]; Work = \left[\frac{Btu}{hr}\right]$$

The heat added at the boiler and the heat removed at the condenser can be found with the following equation and the enthalpies entering and exit each piece of equipment.

$$Boiler \rightarrow Q_{in} = \dot{m} * (h_3 - h_2)$$

$$Condenser \rightarrow Q_{out} = \dot{m} * (h_4 - h_1)$$

The same method can be used to find the work output of the turbine.

$$Turbine \rightarrow W_{turb} = \dot{m} * (h_4 - h_3)$$

The efficiency of the Rankine cycle can be found with the following equation

$$Efficiency = \frac{Work_{turbine} - Work_{pump}}{Q_{in}}$$

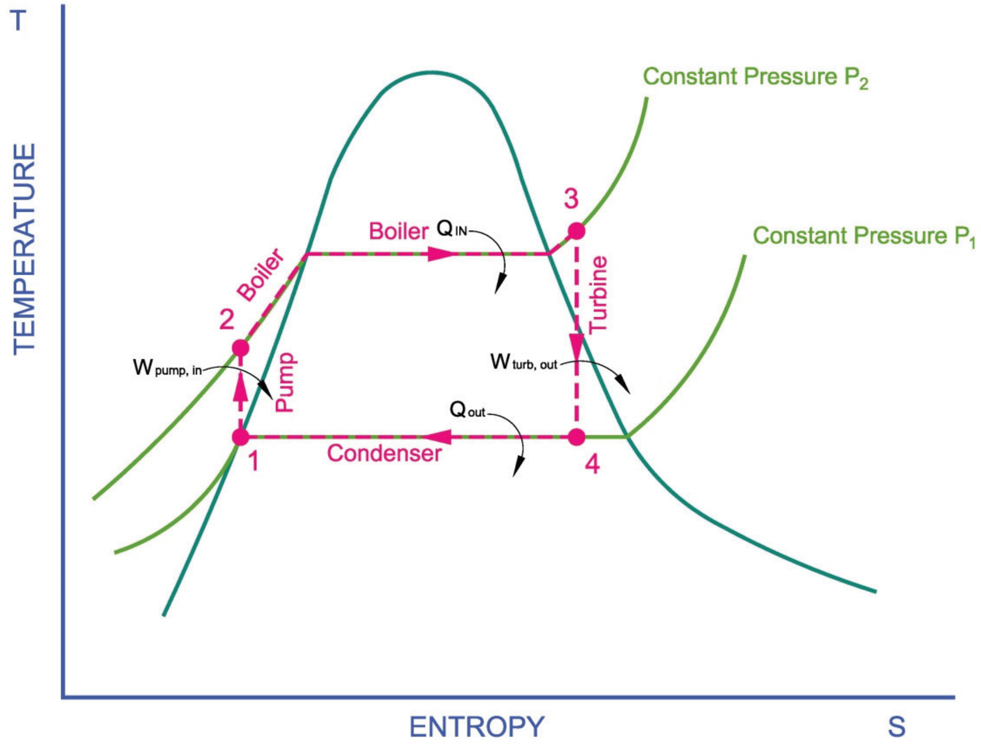


Figure 24: Rankine cycle graph without reheat

The above figure shows the Rankine cycle but in graph format. When you are completing these types of problems, it is helpful to draw this graph to remind yourself of the different stages and the thermodynamic transition between each stage. For example, the turbine and pump transitions are isentropic, meaning the entropy remains the same. The boiler and condenser transitions are isobaric, meaning the pressure remains the same.

The Rankine cycle is covered in the Energy/Power Applications section in more detail. Please see this section for a continuation of the Rankine cycle and the variations that can be applied to this cycle.

7.6 INTERNAL COMBUSTION ENGINES (OTTO CYCLE)

An internal combustion engine is an engine where the combustion of fuel is used to produce hot gases which directly drive a form of mechanical work. An internal combustion system is opposite of an external type of combustion system. In an external type of combustion system, the heat from the combustion process is used to heat up another fluid, typically water to steam, which then drives a turbine for mechanical work. In this external type of system, the combustion gases are kept separate.

There are many different types of combustion engines, but there are a few types that you should be familiar with for the exam.

7.6.1 RECIPROCATING ENGINES

A reciprocating engine is an engine that has pistons which up and down within a cylinder. This engine is also known as a piston engine. These types of engines follow the basic process of in taking fuel and air, then compressing the mixture, followed by combustion. The combustion process releases heat and the combustion products are expanded and then exhausted to the atmosphere.

The next types of engines that you should understand for the exam are the two-stroke, four-stroke and six-stroke engines. This term describes the number of strokes it takes for the combustion engine to complete a cycle.

7.6.2 TWO-STROKE ENGINE

A two-stroke engine takes two strokes to complete a cycle. In the first stroke, the downward movement of the piston, fresh air is added to the cylinder and exhaust air is released from the cylinder. Then on the second stroke, the fresh air is compressed and ignited. The heat released from combustion expands the gases, which drives the mechanical work of the piston.

7.6.3 FOUR-STROKE ENGINE

A four-stroke engine takes four strokes to complete a cycle. The four stroke engine cycle is better known as the Otto cycle. This cycle starts with an intake stroke where gas vapor is drawn into the engine. Then the gas vapor is compressed with the compression stroke in Step 1. Then in Step 2, the gas vapor is combusted at a constant volume, with a spark and heat is released, which increases the temperature of the gas. Next in Step 3, the gas is expanded which forces the piston backwards and finally in step 4, the combustion gases are exhausted to the atmosphere.

7.6.4 COMPRESSION RATIO

The compression ratio describes the ratio of the volume of the combustion chamber at its largest capacity to its smallest capacity. The compression ratio is one indicator on the mechanical efficiency of an internal combustion engine. The higher the ratio, the more heat is extracted from a smaller amount of fluid.

The smallest capacity point is called the top dead center and the largest capacity point is called the bottom dead center. The terms BDC and TDC will be discussed and explained in a following section.

$$r = \frac{v_{BDC}}{v_{tdc}}$$

$$where\ r = compression\ ratio;\ v_{bdc} = bottom\ dead\ center;\ v_{tdc} = top\ dead\ center$$

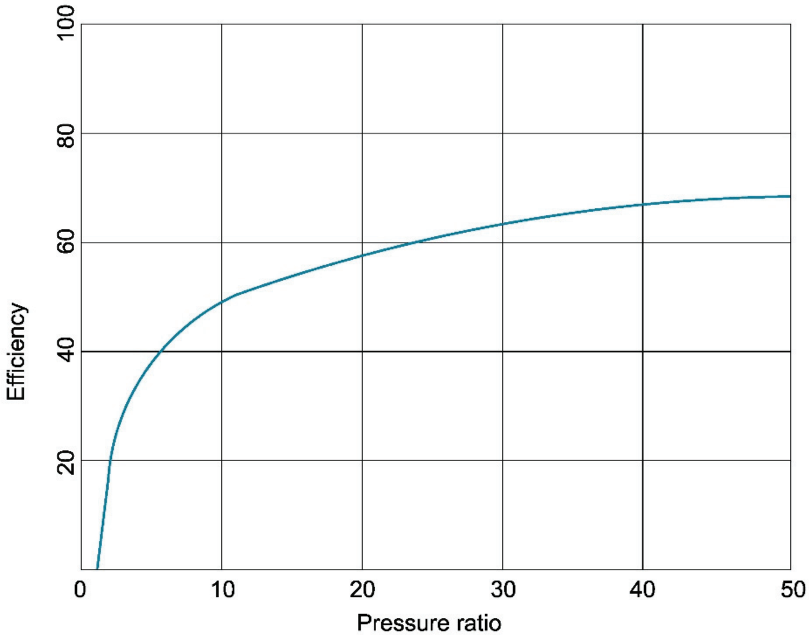


Figure 25: This graph shows how the efficiency changes with the pressure ratio.

7.6.5 BMEP

BMEP stands for brake mean effective pressure. It is a type of effective pressure

$$Two\ strok\ engine \rightarrow\ BMEP\ (psi) = \frac{75.4 * T}{V}$$

$$where\ T = torque\ (lb - ft); V =\ displaced\ volume\ (in^3)$$

The number of strokes is discussed in the Otto Cycle section. A four stroke cycle is shown.

$$Four\ strok\ engine \rightarrow\ BBMEP\ (psi) = \frac{150.8 * T}{V}$$

$$where\ T = torque\ (lb - ft); V =\ displaced\ volume\ (in^3)$$

7.6.6 OTTO CYCLE

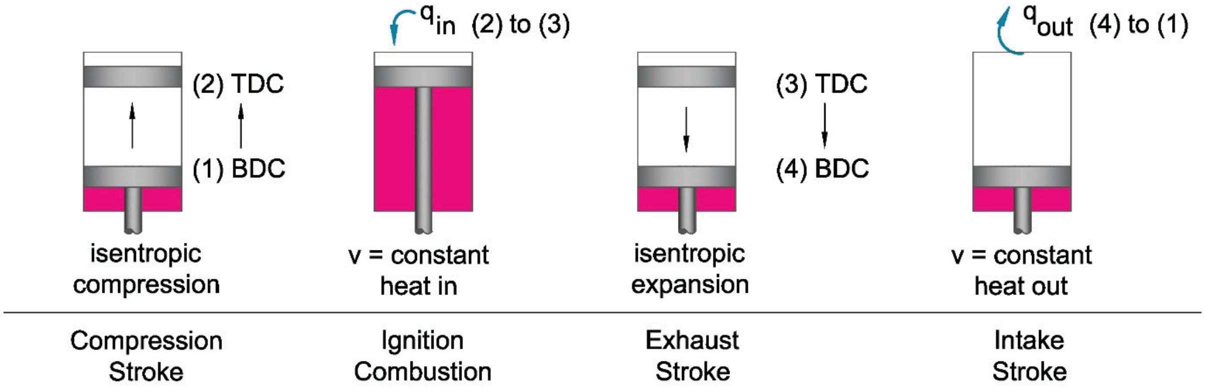
The Otto cycle is the same as the cycle described in the four stroke engine cycle. These steps are shown in the figure below. The figure shows a piston within a cylinder. The diameter of the piston is called the bore and the distance that the piston travels from TDC to BDC is called the stroke.

First, the piston starts at the point called BDC, which stands for bottom dead center. Then the piston compresses the air within the cylinder by decreasing the volume and traveling from BDC to TDC. When the piston is as TDC, the volume within the cylinder is at its minimum and the pressure is at its maximum. This step is called the compression stroke.

Second, the ignition and combustion stage occurs. At this stage, there is no change in volume or pressure. There is only an increase in heat from the products of combustion.

The third step is the exhaust stroke. In this step, the piston travels from TDC to BDC, which is assumed to ideally be an isentropic expansion.

Finally, air is allowed to enter into the cylinder and the process begins again, while the energy leaves the cylinder.



q_{in} (2) to (3)
q_{out} (4) to (1)
(2) TDC
(1) BDC
(3) TDC
(4) BDC
isentropic compression
v = constant heat in
isentropic expansion
v = constant heat out
Compression Stroke
Ignition Combustion
Exhaust Stroke
Intake Stroke

Figure 26: Stages of Otto cycle.

The next figure shows the same stages of the Otto cycle but on a Pressure-Volume graph. The important things to note on this figure are the isentropic expansion and compression at steps 1 to 2 and 3 to 4. Also the change in volume from TDC to BDC and BDC to TDC. The ratio of the volume at BDC to the volume at TDC is called compression ratio. In the figure above, the volume is the white part.

$$Compression\ ratio = \frac{V_{BDC}}{V_{TDC}}$$

Compression ratios are typically in the range of 10:1 and up to 22:1.

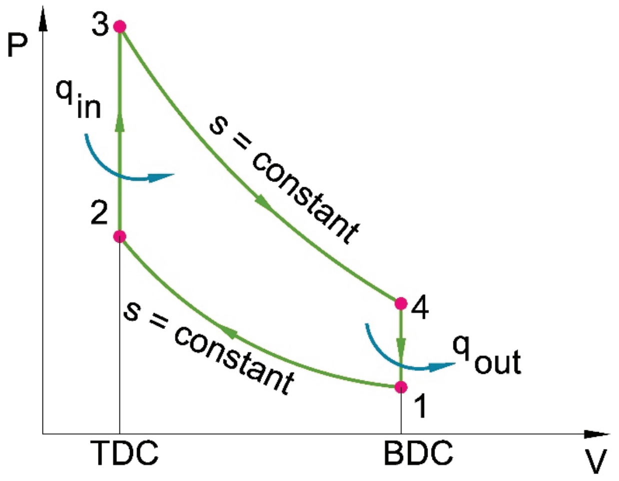


Figure 27: Stages of Otto cycle on a Pressure-Volume graph.

If you have a question on the Otto cycle, you should draw both the Pressure-Volume graph to remember the relationship between pressure and volume at each stage and the Temperature-Entropy graph.

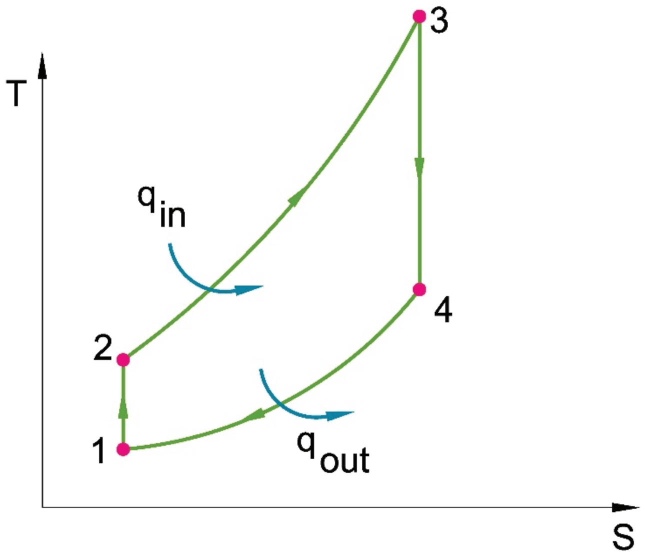


Figure 28: Stages of Otto cycle on a Temperature-Entropy graph

An important question on the Otto cycle is determining the efficiency of the cycle. This can be determined by the simple question, Work out divided by Work in.

$$Efficiency = \frac{Work_{out}}{Work_{in}}$$

The amount of useful work produced is the remaining work after the energy leaves the cylinder minus the energy.

$$Efficiency = \frac{Q_{out} - Q_{in}}{Q_{in}}$$

In practice, the temperature and volume at TDC and BDC can be measured. So it is important to be able to determine the energy in and out given these values. The below equation can be used when all temperatures at each stage are known.

$$Otto\ Cycle\ Thermal\ Efficiency = 1 - \frac{T_4 - T_1}{T_3 - T_2}$$

If you need to find just the energy values of Q_{in} and Q_{out}, then you can use these equations.

$$Q_{in} = \dot{m} * c_v * (T_3 - T_2);\ Q_{out} = \dot{m} * c_v * (T_4 - T_1)$$

$$where\ c_v = 0.1715 \frac{Btu}{lbm\ °R};$$

The final equation for determining efficiency uses the compression ratio and the specific heat ratio.

$$Efficiency = 1 - \frac{1}{r^{k-1}};$$

$$where\ k = 1.4;$$

If you need to find other temperatures or volumes, then you can use the following relationship for isentropic processes.

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{k-1};\ \frac{T_3}{T_4} = \left(\frac{V_4}{V_3}\right)^{k-1}$$

8.0 REFRIGERATION AND HEAT PUMP CYCLES

Refrigerants are fluids used in refrigeration and heat pump cycles to transfer heat from one location to another location. A key concept that you need to understand that allows this heat transfer is boiling pressure and temperature.

8.1 BOILING PRESSURE/TEMPERATURE

One key principle that must be understood for Refrigeration is the relationship between the boiling/condensing point of a fluid, in this case a refrigerant, and the temperature and pressure of the refrigerant. A refrigerant liquid's boiling point is a function of the vapor pressure of the refrigerant vapor that is in equilibrium with the refrigerant liquid. If the pressure is low, then there is a smaller force acting upon the refrigerant liquid, thus it will take a lower temperature to boil the refrigerant liquid. For example, water at a pressure of 1 atmosphere or 14.696 PSI will boil at 212°F. However, if the water was at a pressure of 0.122 PSI, then the water will boil at 40°F. This principle is important to understand: *Low pressure refrigerants boil at a lower temperature, high pressure refrigerants condense at a higher temperature.*

For more detail refer to the following table and figure.

Water Temperature [F] Boiling Point	Pressure [PSI]
212	14.696
190	9.340
160	4.742
130	2.224
100	0.950
70	0.363
40	0.122
10	0.031

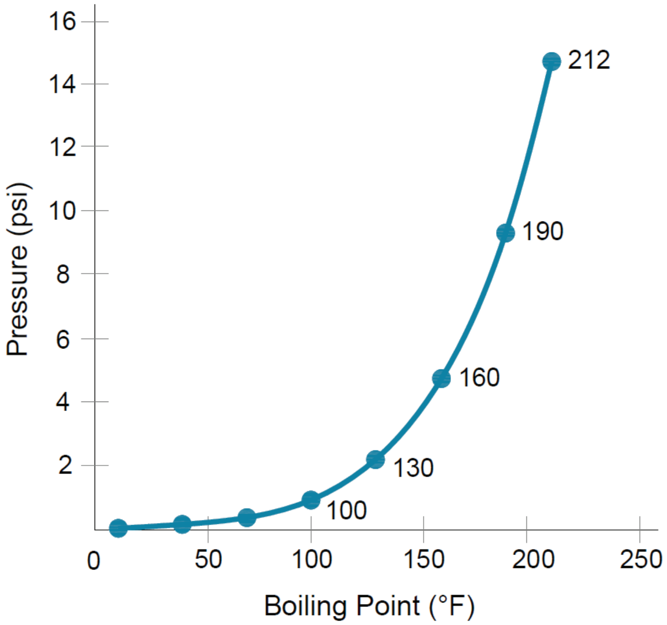


Figure 29: Relationship between boiling point and pressure

When a low pressure refrigerant changes from its liquid phase to a gas phase, it can absorb much more heat than if it were to simply increase in temperature. The same is also true when a high pressure refrigerant changes phase from its gas phase to a liquid phase; it releases much more heat than if it were to decrease in temperature. The energy required to change the phase of a liquid from a liquid to a gas is called the latent heat of evaporation. The energy released to change the phase of a gas to a liquid is called the latent heat of condensation.

8.2 VAPOR COMPRESSION CYCLE

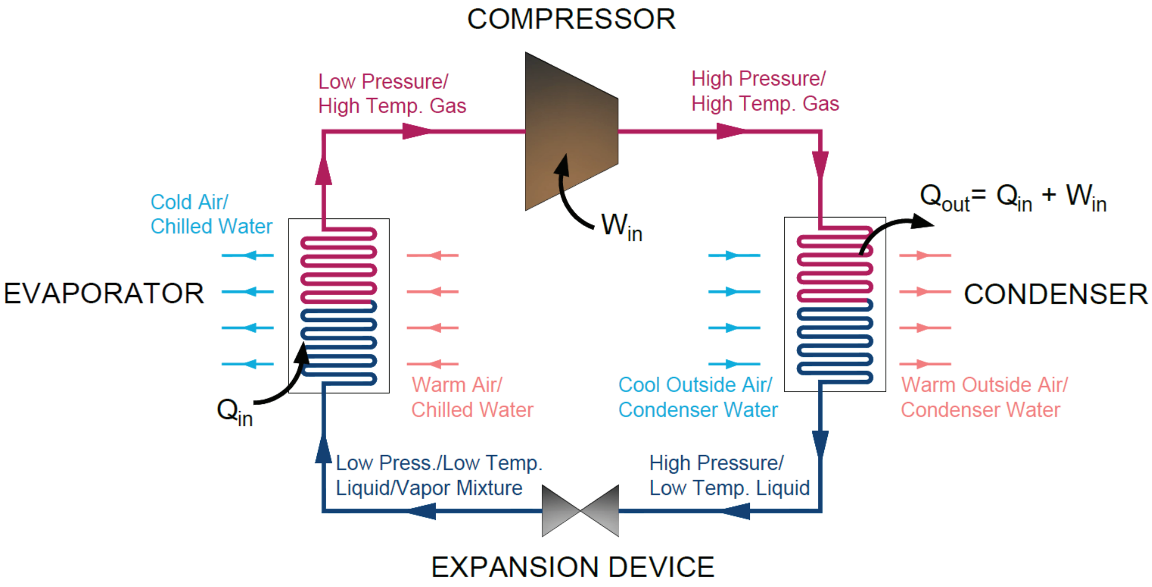

COMPRESSOR
Low Pressure/
High Temp. Gas
High Pressure/
High Temp. Gas
$Q_{out} = Q_{in} + W_{in}$
Cold Air/
Chilled Water
W_{in}
EVAPORATOR
CONDENSER
Q_{in}
Warm Air/
Chilled Water
Cool Outside Air/
Condenser Water
Warm Outside Air/
Condenser Water
Low Press./Low Temp.
Liquid/Vapor Mixture
High Pressure/
Low Temp. Liquid
EXPANSION DEVICE

Figure 30: Vapor compression cycle

The vapor compression cycle is the primary cycle used in commercial refrigeration systems. This cycle is shown in the above figure.

The vapor compression cycle starts at (Step 1) the evaporator, with cold, low-pressure, liquid-vapor refrigerant. It absorbs heat and evaporates to a low-pressure gas. Then the gas is (Step 2) compressed to a high-pressure, high-temperature gas and (Step 3) condensed to a high pressure, low temperature liquid. Finally, the gas is expanded (reduced in pressure) at the (Step 4) expansion device to a cold, low-pressure liquid refrigerant.

8.2.1 EVAPORATOR

Step 1: Evaporator. The first step in the vapor compression cycle is the evaporator, which can also be called a liquid cooler. The evaporator is simply a heat exchanger. Heat is exchanged from the warm medium (air or water) to the cold, liquid-vapor refrigerant mixture. The heat gained by the liquid/vapor refrigerant mixture causes it to change phases to a refrigerant gas.

The refrigerant liquid gains the heat necessary to overcome the latent heat, in order to change to a gas.

There are two types of evaporators, (1) an air cooled evaporator and (2) a water cooled evaporator. The figure below shows the (1) air cooled evaporator which is most commonly referred to as a direct expansion system. In this evaporator, warm air from an air conditioned space is cooled and redistributed to the space. The figure below shows the water cooled system, where chilled water return is cooled and supplied to the chilled water distribution system.

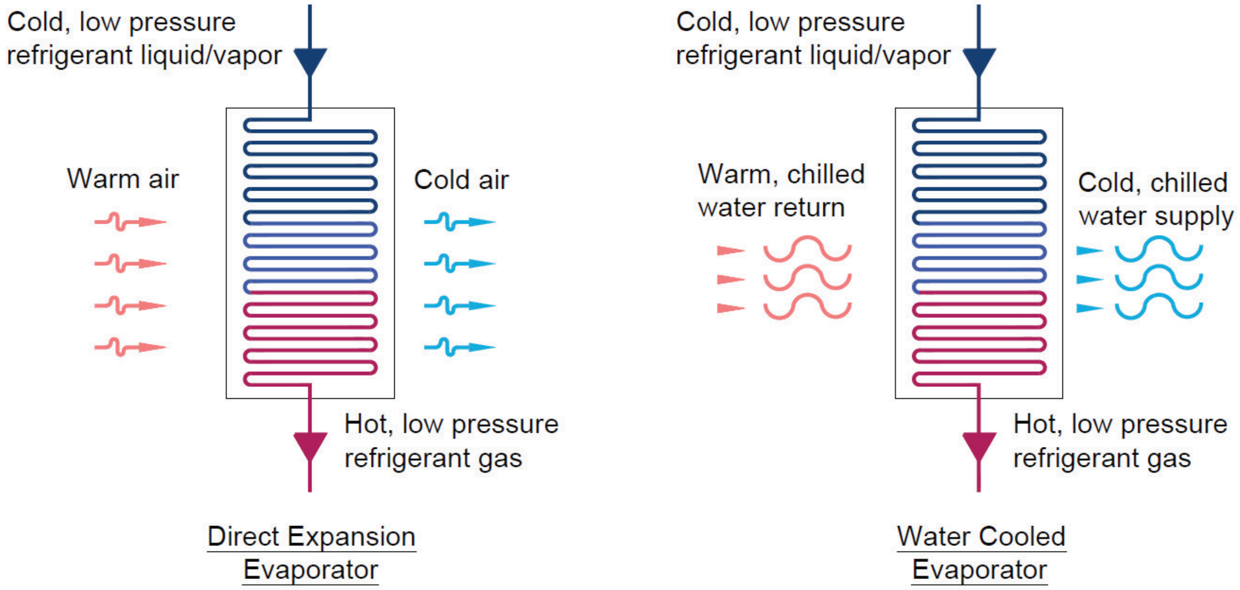

Cold, low pressure refrigerant liquid/vapor
Warm air
Cold air
Hot, low pressure refrigerant gas
Direct Expansion Evaporator
Cold, low pressure refrigerant liquid/vapor
Warm, chilled water return
Cold, chilled water supply
Hot, low pressure refrigerant gas
Water Cooled Evaporator

Figure 31: Direct expansion evaporator on the left and water cooled on the right

The most common system is the direct expansion system. This system is prevalent throughout smaller systems, like those serving residential systems. In this system, the hot air from the space is used to directly evaporate the refrigerant to a hot gas. Note that the hot air from the space is roughly ~75 °F and the refrigerant liquid is typically 40 °F. The 75 °F room air is cooled down to ~55 °F and then distributed back to the space. In a water-cooled system, which is more common for larger commercial systems, chilled water typically at 55 °F is cooled by the evaporator down to ~45 °F. The colder chilled water is then supplied to another heat exchanger, where air is cooled and then distributed to the space.

Besides the two different types of evaporator systems, there are also different types of heat exchangers used in refrigeration. The most common heat exchangers include: (1) Shell and Tube, (2) Tube in Tube and (3) Brazed Plate.

(1) Shell and Tube: This heat exchanger is the most common and consists of copper pipes arranged in a coil that is constructed in a cylindrical shell. One fluid is provided in the shell and contacts the outer surface of the inner tubes. Another fluid is contained inside of the tubes. Heat exchange occurs in the shell at the outer surface of the tubes. Often times aluminum fins

are provided on the copper pipes. These fins provide more surface area for heat exchange to occur.

(2) Tube in Tube: A tube is constructed in a tube, sealed separately to keep the fluids in one tube from contaminating the other. Heat exchange is conducted at the outer surface of the inner tube and the inner surface of the outer tube.

(3) Brazed Plate: This type of heat exchanger consists of multiple thin plates separated by a small distance. Each plate either carries the hot or cold fluid. Heat exchange occurs between the surface areas of each plate.

As previously mentioned the evaporator acts as a heat exchanger with a cold side and a hot side. The cold side consists of a mixture of refrigerant gas and liquid. At this point, the partial liquid-gas refrigerant mixture moves through the evaporator, picking up heat from the hot side. But instead of heating the gas, the heat is used to boil the remaining liquid. It is important for the evaporator to boil all of the liquid, prior to the refrigerant entering the compressor in the following step. Once all the liquid has boiled, the liquid-gas mixture turns into a refrigerant gas (vapor), called a saturated vapor. Any additional heat will now increase the temperature of the refrigerant vapor, into a region called super heat. Any release in heat will cause some of the gas to condense back to a liquid.

It is important for the engineer to understand that the amount of cooling provided through the evaporation of the refrigerant liquid is much more than simply increasing the temperature of the refrigerant liquid. For example, R-134a takes 92.82 Btu of heat to change 1 lb of refrigerant from liquid to gas. While it takes 0.204 Btu of heat to increase 1 lb of refrigerant gas by 1°F.

8.2.2 COMPRESSOR

Step 2: Compressor. The next step is where the refrigerant gas is compressed by the compressor, which raises the temperature and pressure of the gas. The compressor is where the work takes place. The compressor is also the driving force that moves the refrigerant through the vapor compression cycle and prepares the refrigerant before it enters the condenser. It is important that the refrigerant gas is raised to a temperature that is above the temperature of the fluid in the condenser. This will allow heat to be transferred from the refrigerant to the condenser fluid. The compression of the refrigerant gas occurs isentropically, meaning that there is no change in entropy. Since the compressor is not completely efficient there will be an increase in enthalpy as the heat generated by the compressor is transferred to the refrigerant gas.

Entropy - a measure of the amount of disorder in a thermodynamic system.

Enthalpy - a measure of the total energy in a thermodynamic system (sensible and latent energy).

8.2.3 CONDENSER

Step 3: Condenser. The third step in the vapor compression cycle is the condenser. The condenser is the counterpart of the evaporator. Similar to the evaporator, the condenser is simply a heat exchanger. Except in this case, heat is exchanged from the warm refrigerant gas to the cold medium. The heat released by the warm refrigerant gas causes it to change phases. The refrigerant gas condenses to refrigerant liquid.

There are two types of condensers, similar to the two types of evaporators. The figure below shows a sample water cooled condenser, where cool condenser water at ~85 °F is used to remove heat from the refrigerant, causing it to increase in temperature to approximately ~95 °F. The figure below shows the air cooled system, where heat is removed from the refrigerant by blowing outside air over the coil. The location will determine the condenser water and outside air temperatures.

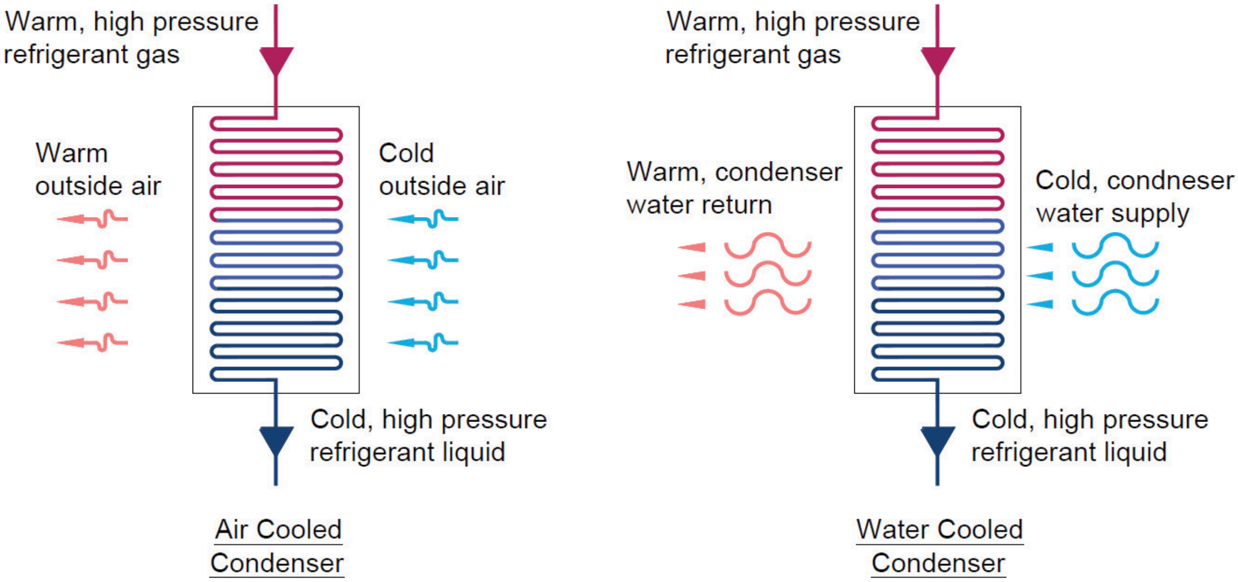

Warm, high pressure refrigerant gas
Warm outside air
Cold outside air
Cold, high pressure refrigerant liquid
Air Cooled Condenser
Warm, high pressure refrigerant gas
Warm, condenser water return
Cold, condneser water supply
Cold, high pressure refrigerant liquid
Water Cooled Condenser

Figure 32: Air cooled condenser on the left and water cooled on the right

The methods of heat exchange are similar to that of the evaporator. Refer to the evaporator section for the different types of heat exchangers.

8.2.4 EXPANSION DEVICE

Step 4: Expansion Device. The final step is the expansion device, which is the counterpart of the compressor. The expansion device reduces the pressure of the liquid, which causes not only the pressure to decrease but also the temperature to decrease. During this process, some of the liquid refrigerant is turned into a gas, this is called flash gas. The resultant of the expansion device is a cold partial liquid-vapor refrigerant mix. The cold refrigerant liquid-vapor mix then repeats the process at the evaporator.

The expansion device that is primarily used in air conditioning systems is called a thermostatic expansion valve (TXV). The TXV as its name describes, opens and closes, based on a thermal device. The adjustment of the opening/closing determines the amount of refrigerant that is passed through and evaporated. The TXV uses the temperature of the evaporator output as a basis for determining the amount of refrigerant.

For example, if the TXV senses that the evaporator is producing an output refrigerant temperature that is too cold, then there is too much refrigerant for the heat load (hot side of the evaporator) and the refrigerant sent to the evaporator needs to be throttled down (decrease cold side of the evaporator). If the TXV senses that the output of the evaporator is too high, then the amount of refrigerant cannot keep up with the heat load (hot side) then the TXV should allow more refrigerant to the evaporator (increase cold side).

8.3 PRESSURE-ENTHALPY DIAGRAM

The pressure-enthalpy diagram or P-H diagram is a tool that is used heavily in the HVAC & Refrigeration field. This diagram describes the relationship of pressure and enthalpy of a select refrigerant. In order to properly understand this diagram, it is best to go through the vapor compression cycle on a P-H diagram.

On the P-H diagram, pressure is indicated on the y-axis and enthalpy is indicated on the x-axis. Typically enthalpy is in units of Btu/lb and pressure is in units of pounds per square inch (psi). The upside down U figure shown on the diagram designates the points at which the refrigerant changes phase. The left vertical curve indicates the saturated liquid curve and the right vertical curve indicates the saturated vapor curve. The region in-between the two curves describes refrigerant states that contain a mixture of both liquid and vapor. The locations to the left of the saturated liquid curve indicate that the refrigerant is in liquid form and locations to the right of the saturated vapor curve indicate that the refrigerant is in vapor form. The point at which the two curves meet is called the critical point. The importance of this point is that at any point above, no additional pressure will change the vapor into a liquid. A simplified pressure-enthalpy diagram is shown in the figure below, describing this information.

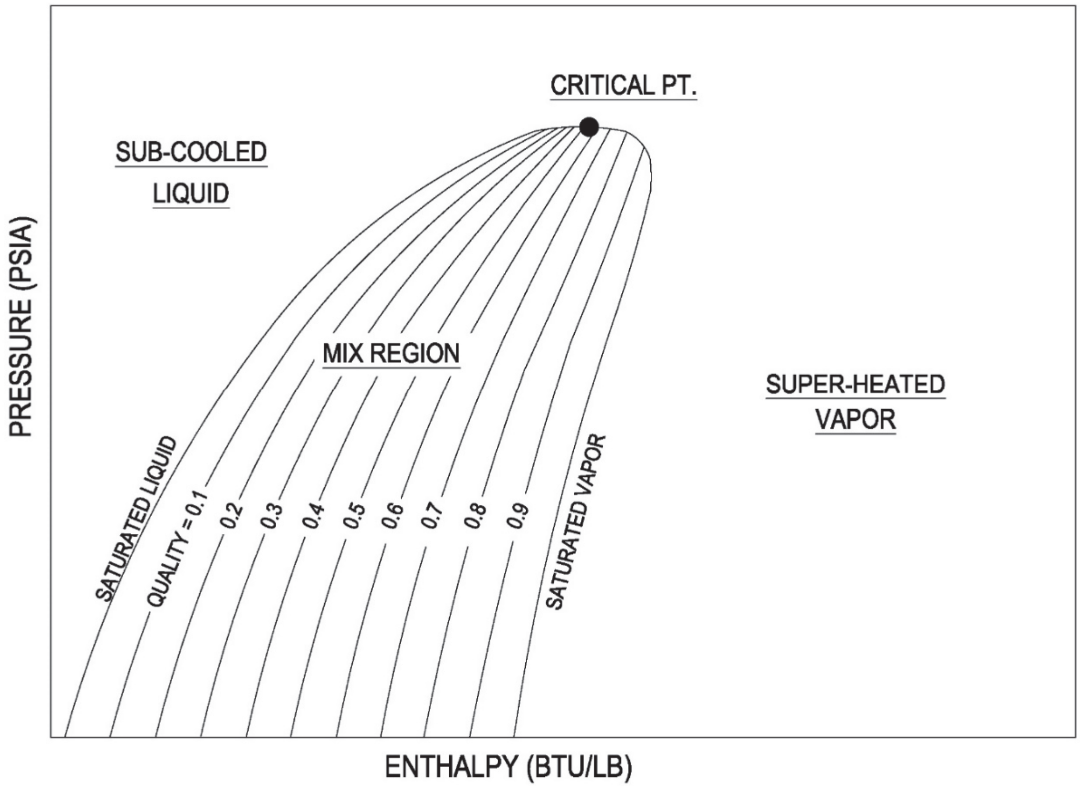


Figure 33: Pressure-enthalpy diagram

The curves break up the diagram into three regions (1) Liquid, (2) Vapor and (3) Mix.

(1) Liquid Region: The liquid region is also known as the sub-cooled region. In this region there are vertical temperature lines, which increase as enthalpy is increased. The figure below is a simplified P-H diagram illustrating the constant temperature lines.

(2) Vapor Region: The vapor region is also known as the super-heated region. In this region there are vertical temperature lines, which increase as enthalpy is increased. Refer to the figure below. There are also lines of constant entropy, which are also important. Entropy is the measure of the amount of disorder in the system.

(3) Liquid-Vapor Mix Region: In this region, the P-H diagram shows horizontal temperature lines, which indicate constant temperature. The mix region is the phase change region, where any addition of enthalpy will cause additional liquid to vaporize instead of raising the temperature. The below figure illustrates the horizontal temperature lines in the mix region. There are also upward sloping curves which indicate quality. Quality is a measure of the ratio of vapor mass to total mass. For example quality of 0.1 or 10%, which is located near the saturated liquid line, describes points that have 10% vapor by mass. The 0.9 or 90% line, which is located near the saturated vapor line, describes points that have 90% vapor by mass. The previous figure indicates the quality lines.

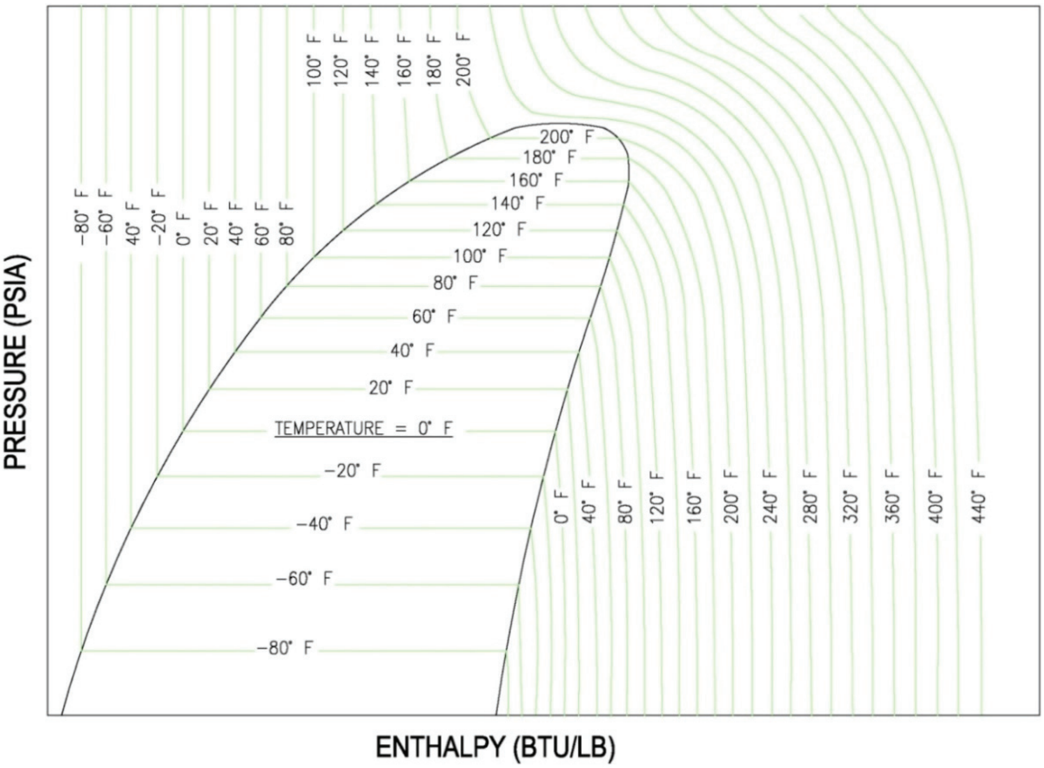

100° F
120° F
140° F
160° F
180° F
200° F
−80° F
−60° F
40° F
−20° F
0° F
20° F
40° F
60° F
80° F
200° F
180° F
160° F
140° F
120° F
100° F
80° F
60° F
40° F
20° F
TEMPERATURE = 0° F
−20° F
−40° F
−60° F
−80° F
0° F
40° F
80° F
120° F
160° F
200° F
240° F
280° F
320° F
360° F
400° F
440° F
PRESSURE (PSIA)
ENTHALPY (BTU/LB)

Figure 34: P-H diagram, constant enthalpy lines

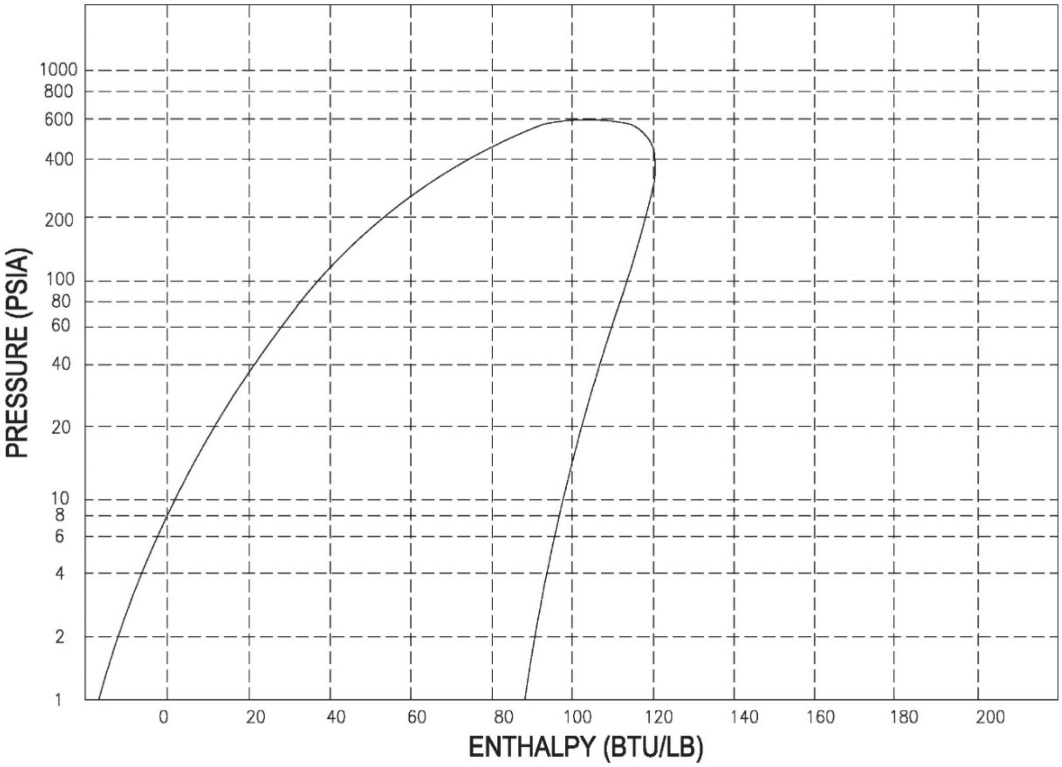


Figure 35: P-H Diagram, Constant Pressure & Enthalpy Lines

The x-y axes of the P-H diagram are the pressure lines running from left to right. The enthalpy lines are the vertical lines. The skeletal graph shown in the figure above shows the pressure-enthalpy lines.

The next important lines on the pressure-enthalpy diagram are those describing lines of constant entropy, which are used and discussed in section 6.1.2 Compressor.

8.3.1 REFRIGERATION CYCLE

One of the most important skills needed for the professional engineer in the HVAC & Refrigeration field is navigating the refrigeration cycle on a pressure-enthalpy diagram. The following sections will show each specific part of the refrigeration cycle on the pressure-enthalpy diagram and it will also highlight the important points and calculations needed.

Throughout this explanation the refrigerant R-134a is used as an example. It is recommended that the engineer be familiar with the P-H diagram for R-134a and the other common refrigerants. These diagrams can be found in the NCEES FE Handbook. A sample R-134a diagram is shown below, with a sample refrigeration cycle, identifying (Step 1) Evaporator, (Step 2) Compressor, (Step 3) Condenser and (Step 4) Expansion Device.

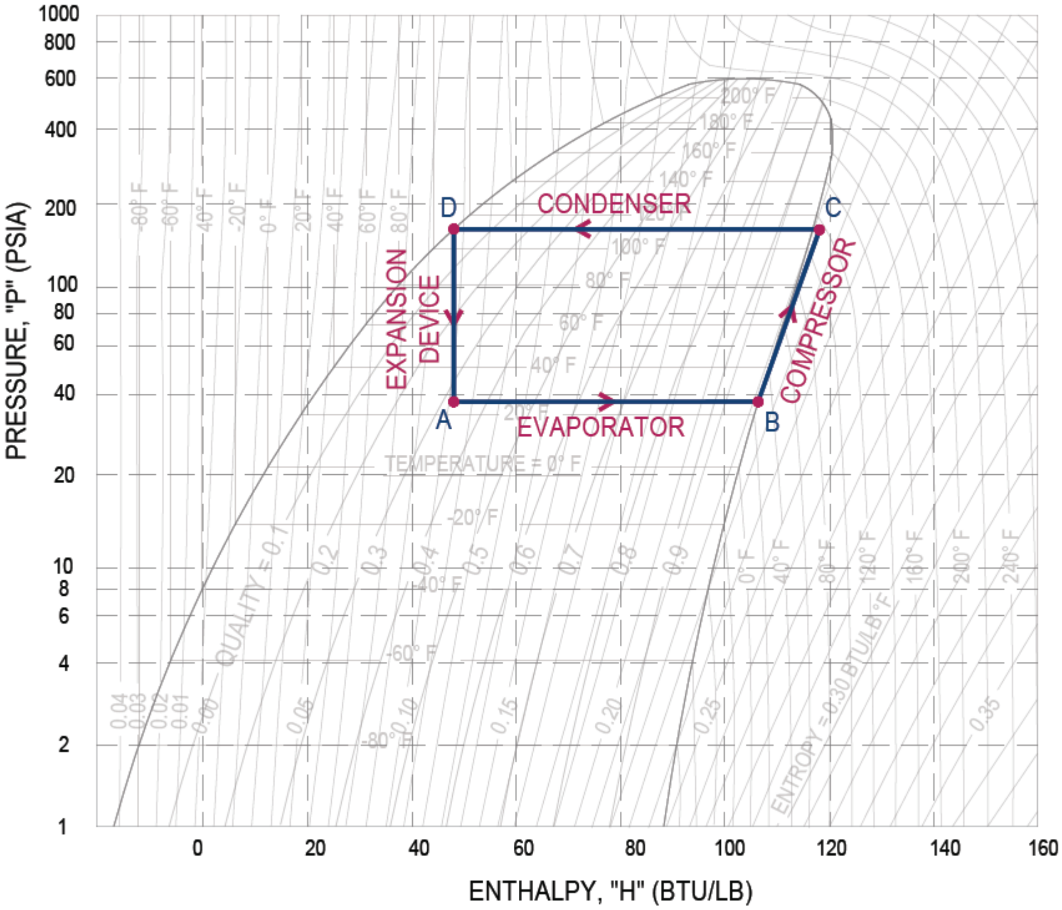


Figure 36: Sample R-134a P-H diagram

8.3.1 Step 1 Evaporator

The refrigerant entering the evaporator is a cold, partial liquid-vapor mixture. The operating pressure and temperature of the evaporator is called the suction pressure and suction temperature. The suction line is the piping that routes refrigerant gas from the evaporator to the compressor. It is important to note that pressure and temperature are constant in the evaporator (the mixed region) and their values are dependent on each other.

For example, if a compressor operates at a suction pressure of 36.8 psia, then the corresponding evaporator pressure is 36.8 psia and the corresponding evaporator temperature is 25 °F, see below figure for points A and B (Values are for Refrigerant R-134a). If the compressor operates at a suction pressure of 49.7 psia, then the corresponding evaporator pressure is also 49.7 psia and the evaporator temperature is 40 °F. See below figure for points A' and B'(Values are for Refrigerant R-134a).

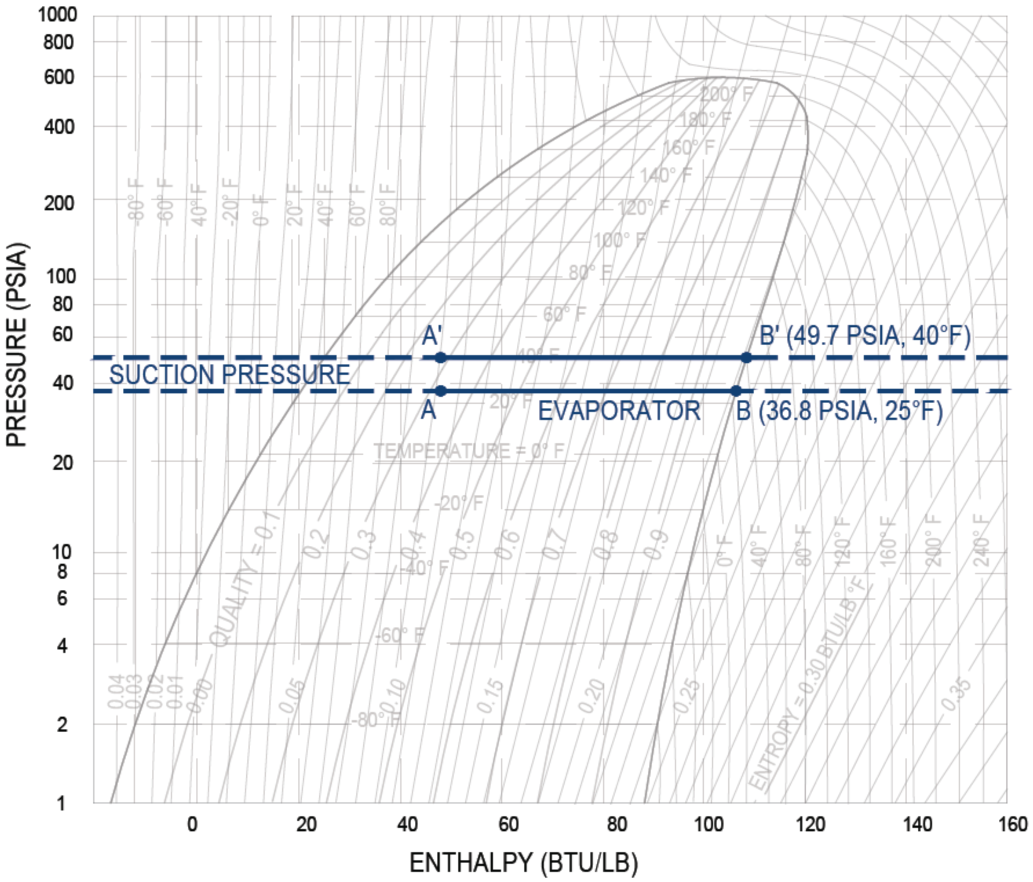


Figure 37: R-134a P-H diagram with evaporator, suction pressure versus evaporator temperature

The evaporator moves the refrigerant from point A (partial liquid-vapor mixture) to point B, a fully saturated vapor refrigerant. As the evaporator transfers heat to the refrigerant, there is no gain in temperature, since all the heat is used to convert the remaining liquid to a gas. In an ideal evaporator, there is just enough heat transfer to convert all the liquid to gas and nothing more. Thus, the output of an ideal evaporator is 100% vapor at the same entering temperature, refer to the figure below. In this figure, we see that as the refrigerant moves through the evaporator, the temperature remains the same and the percentage of vapor increases, until it reaches saturation at 100%.

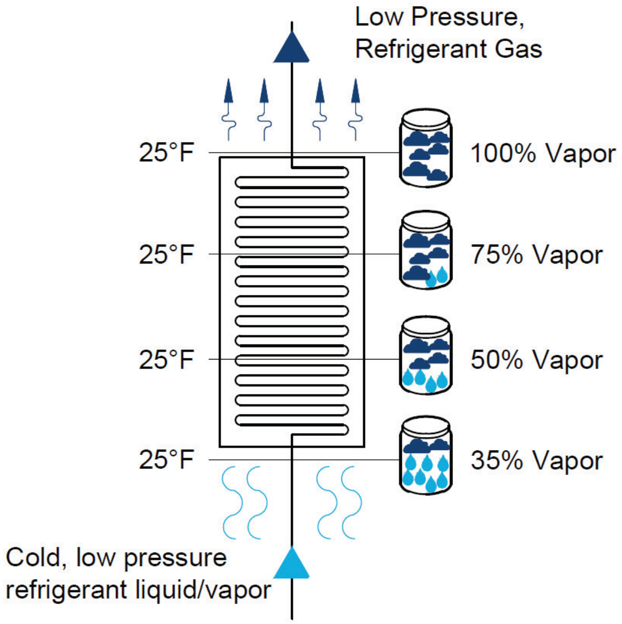


Figure 38: Ideal evaporator, zero superheat

Also introduced in the figure above is the term superheat. If additional heat were to be added to the 100% vapor refrigerant, then the heat would be used to increase the temperature and it is this increase in temperature that is called superheat.

In the figure below, an evaporator with 15 °F superheat is shown. The refrigerant reaches 100% vapor prior to leaving the evaporator. All the additional heat from this point is used to increase the temperature of the refrigerant until it reaches a temperature of 40 °F. This refrigerant has a superheat of 15 °F because the final temperature is 15 degrees passed the saturation temperature of 25 °F. It is important to note that the pressure remains constant throughout the evaporator.

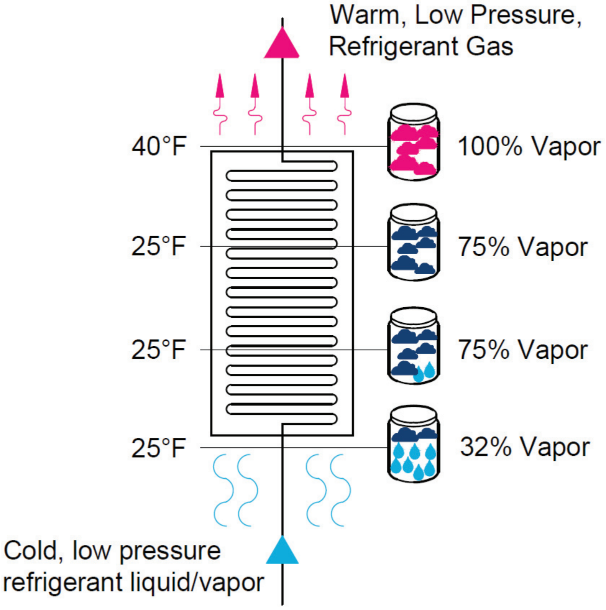

Warm, Low Pressure, Refrigerant Gas
40°F
100% Vapor
25°F
75% Vapor
25°F
75% Vapor
25°F
32% Vapor
Cold, low pressure refrigerant liquid/vapor

Figure 39: Evaporator with 15 F of superheat

On the pressure-enthalpy diagram superheat is shown as horizontal movement along the suction pressure line passed the 100% vapor curve. The figure on the following page shows the difference between 0 °F and 15 °F superheat. Point B is the 100% vapor point at a constant evaporator/suction pressure of 36.8 psia and a temperature of 25 °F. Point B' results from additional heat/enthalpy added to the refrigerant. The refrigerant moves from point B to point B', where the resulting temperature is 40 °F.

The heat gained by the evaporator is calculated by the leaving enthalpy minus the entering enthalpy, multiplied by the refrigerant mass flow rate. This is also called the *net refrigeration effect*.

$$Q_{evap}\ [Btu/hr] = (H_1 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_4 = entering\ evaporator\ enthalpy\ \left[\frac{Btu}{lb}\right]$$

$$H_1 = leaving\ evaporator\ enthalpy\ \left[\frac{Btu}{lb}\right]$$

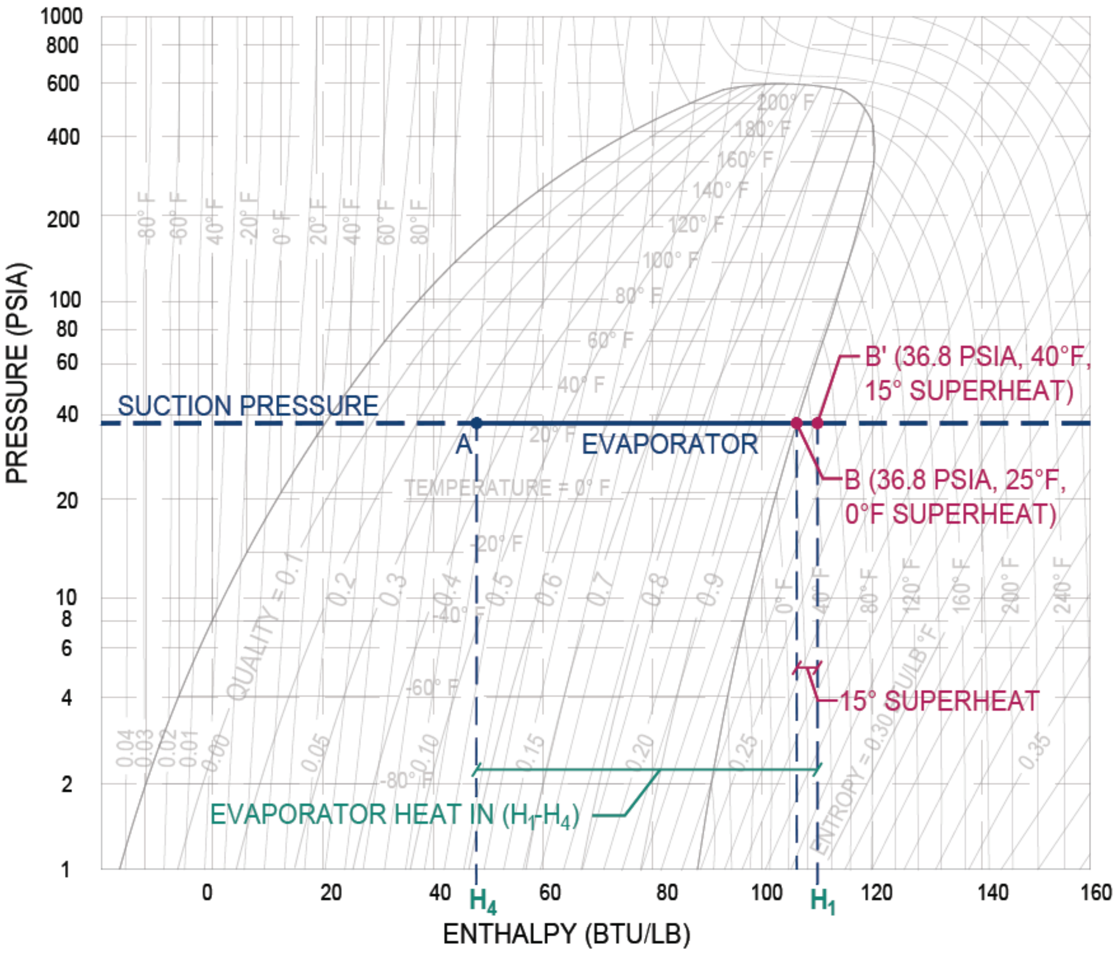


Figure 40: R134a pressure-enthalpy diagram, 0 °F vs. 15 °F superheat

8.3.2 Step 2 Compressor

The compressor is characterized by the refrigerant suction and discharge conditions. Horizontal lines are drawn across the refrigerant's pressure enthalpy diagram for the suction and discharge pressures. Then the incoming temperature of the compressor, as determined by the leaving temperature of the evaporator, is used as the starting point of the compressor, as shown by point B' in the figure below. The compressor then increases the pressure of the refrigerant up to the discharge pressure. **Ideal compression occurs at constant entropy, also known as isentropic compression.** Therefore the intersection of the constant entropy line and the discharge pressure line will identify the final condition of the refrigerant gas leaving the compressor, as shown by point C in the figure below.

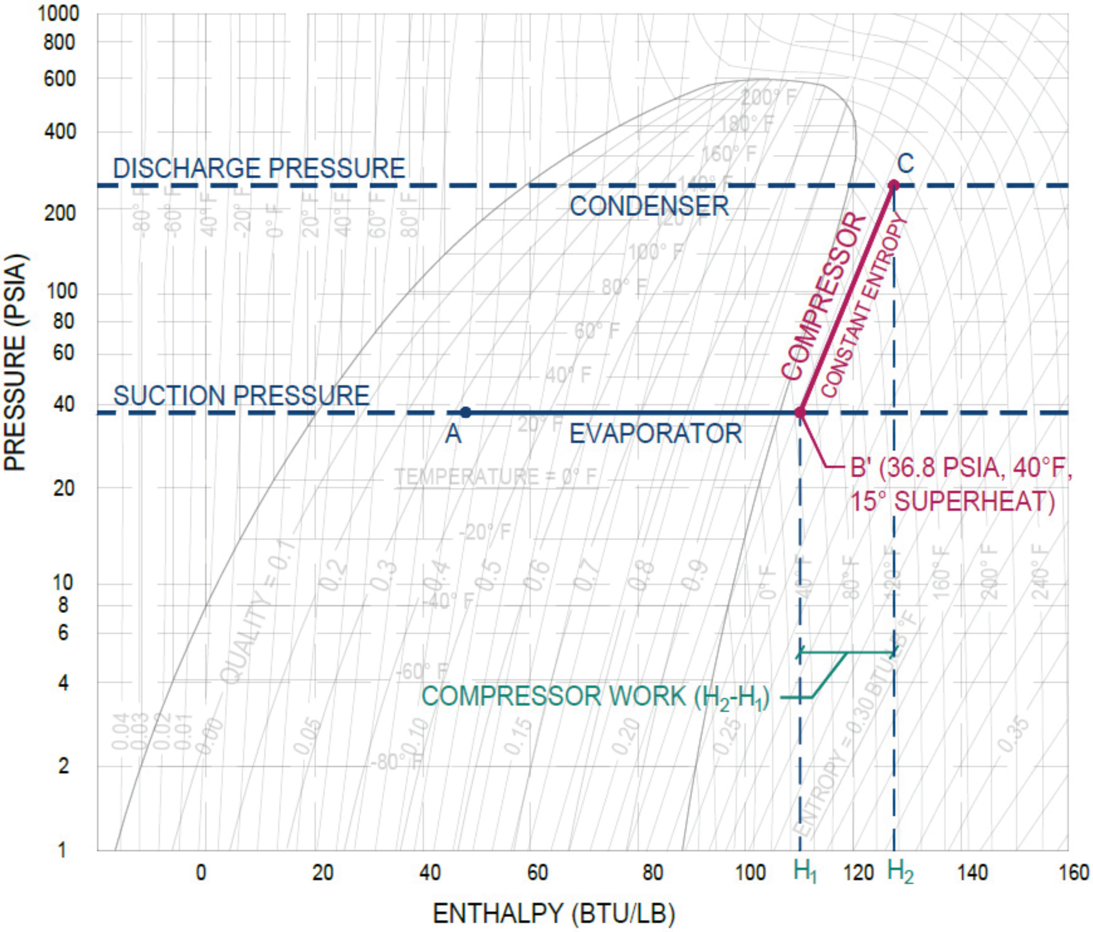


Figure 41: R134a pressure-enthalpy diagram - compressor

A common skill, that is required of a professional engineer, is to determine the work done by the compressor. This work is shown on the previous figure as the difference between the compressor leaving enthalpy (H_2) and the entering enthalpy (H_1). The equation to determine the work of the compressor is shown below. This equation multiples the refrigeration flow rate by the change in enthalpy between the discharge and suction conditions.

$$W\ [Btu/hr] = (H_2 - H_1)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_1 = entering\ compressor\ enthalpy\left[\frac{Btu}{lb}\right]$$

$$H_2 = leaving\ compressor\ enthalpy\left[\frac{Btu}{lb}\right]$$

Compressor Efficiency: An ideal compressor follows the constant entropy line, but in reality, friction losses from the compressor will produce heat. This heat is absorbed by the refrigerant, which causes the actual discharge point (C') to be slightly higher in temperature. The pressure at C' will be the same. The compressor efficiency will normally be given to you. H_1 is known from the evaporator outlet. $H_{2,actual}$ can be found by following the constant entropy line. Then, you can use the following equation to solve for the ideal enthalpy, $H_{2,ideal}$.

$$Compressor\ Efficiency\ (\%) = \frac{H_{2,ideal} - H_1}{H_{2,actual} - H_1}$$

Tip: Normally when we calculate efficiency, we intuitively divide actual over ideal, since the actual outputs we encounter are typically smaller, like in energy applications. However, by reviewing the graph, we can see that the actual enthalpy is greater, because the inefficiencies from heat is added. *So, be aware that in these calculations, compressor efficiencies are ideal divided by actual enthalpy.*

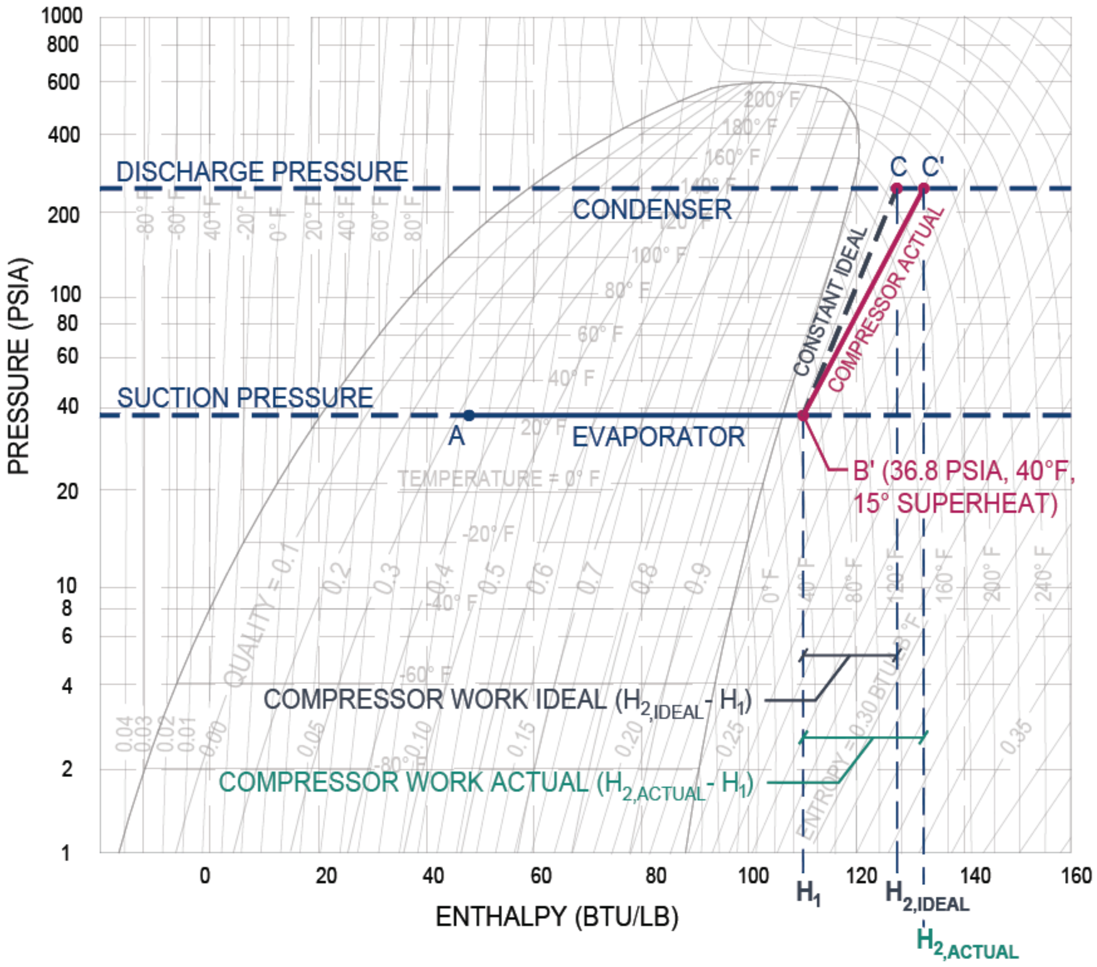


Figure 42: The actual compressor process, with inefficiencies added, is shown.

8.3.3 Step 3 Condenser

The refrigerant entering the condenser is now a hot, high pressure refrigerant gas. The condenser is shown on the pressure-enthalpy diagram as a horizontal line. This horizontal line is a line of constant pressure, corresponding to the discharge pressure of the compressor. The condenser proceeds from right to left in the following three steps:

(1) The superheated gas cools down to saturation temperature [C' 160 °F to D' 140 °F]. Cooling takes place as heat flows from the hot refrigerant gas to the condenser cooling medium.

(2) Next, the100% saturated vapor at D' is converted to 100% saturated liquid at D''. Heat is lost to the condenser cooling medium as the vapor is condensed to a liquid.

(3) Finally, the 100% saturated liquid is sub-cooled from D'' to D'''[140 °F to 115 °F]. In an ideal condenser, no sub-cooling occurs. Once the refrigerant is a fully saturated liquid, any additional heat loss results in a decrease in temperature. This cooling of the saturated liquid is referred to as sub-cooling. In this example, the refrigerant has gone through 25 °F of sub-cooling and resulted in a sub-cooled temperature of 115 °F.

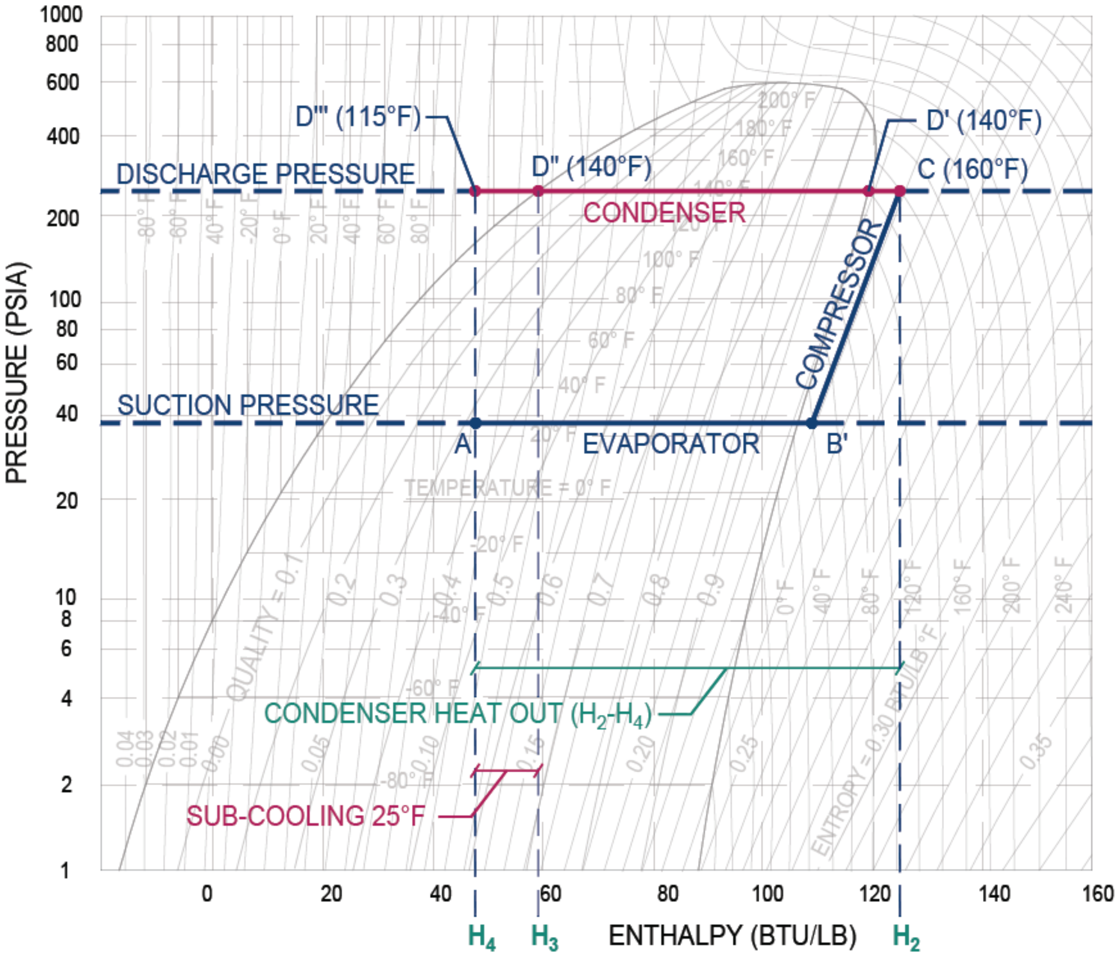


Figure 43: R134a pressure-enthalpy diagram – condenser

A common question is to determine the heat expelled by the condenser, which is shown on the previous figure as the difference between the condenser entering condition (H2) and the leaving condition (H4). The equation to determine the net condenser effect is shown below. This equation multiples the refrigeration flow rate by the change in enthalpy between the entrance and exit of the condenser.

$$Q_{cond,out}\ [Btu/hr] = (H_2 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_2 = entering\ condenser\ enthalpy\left[\frac{Btu}{lb}\right]$$

$$H_4 = leaving\ condenser\ enthalpy\left[\frac{Btu}{lb}\right]$$

8.3.4 Step 4 Expansion Device

The expansion device is the counterpart of the compressor. Similarly, the expansion device is characterized by the suction and discharge pressures. Horizontal lines are again drawn on the refrigerant's pressure-enthalpy diagram. The input condition of the expansion device is determined by the condenser output conditions.

There are two entering conditions to the expansion device shown on the following diagram. The first situation has 0 °F of sub-cooling [D’’] and the second situation has 15 °F of sub-cooling [D’’’].

The expansion device expands the high pressure refrigerant liquid adiabatically to a low pressure liquid-vapor refrigerant mixture. Adiabatic expansion indicates that there is no change in enthalpy and is characterized by a downward vertical line as shown on the below graph.

Note on the graph below as the refrigerant moves from point D to point A, the refrigerant moves from the liquid phase of the graph to the vapor-liquid mixture region. The amount of gas that is formed during this expansion is called flash gas.

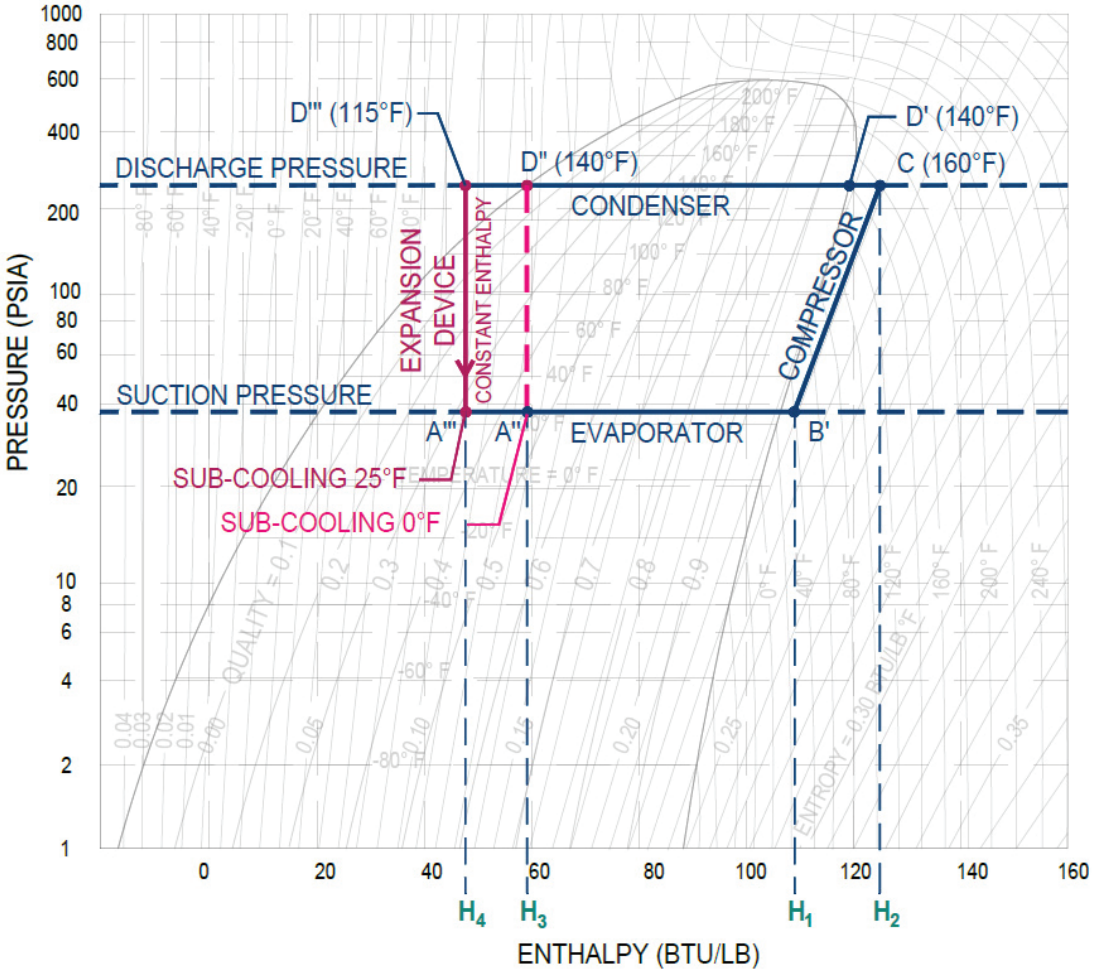


Figure 44: Pressure-enthalpy diagram – expansion device

8.3.5 Net Refrigeration/Condenser, Work and COP:

Typical questions on the FE exam involve being able to navigate the P-H diagram through a refrigeration cycle. The previous sections described each of the four steps of the refrigeration cycle in detail and this section provides an overview of the cycle. In addition, this section provides problems involving net refrigeration, net condenser effect, compressor-work and COP.

The net refrigeration effect is the amount of cooling provided by the evaporator. In order to determine the net refrigeration effect, find the incoming and leaving enthalpy conditions of the evaporator and multiply the difference by the refrigeration flow rate.

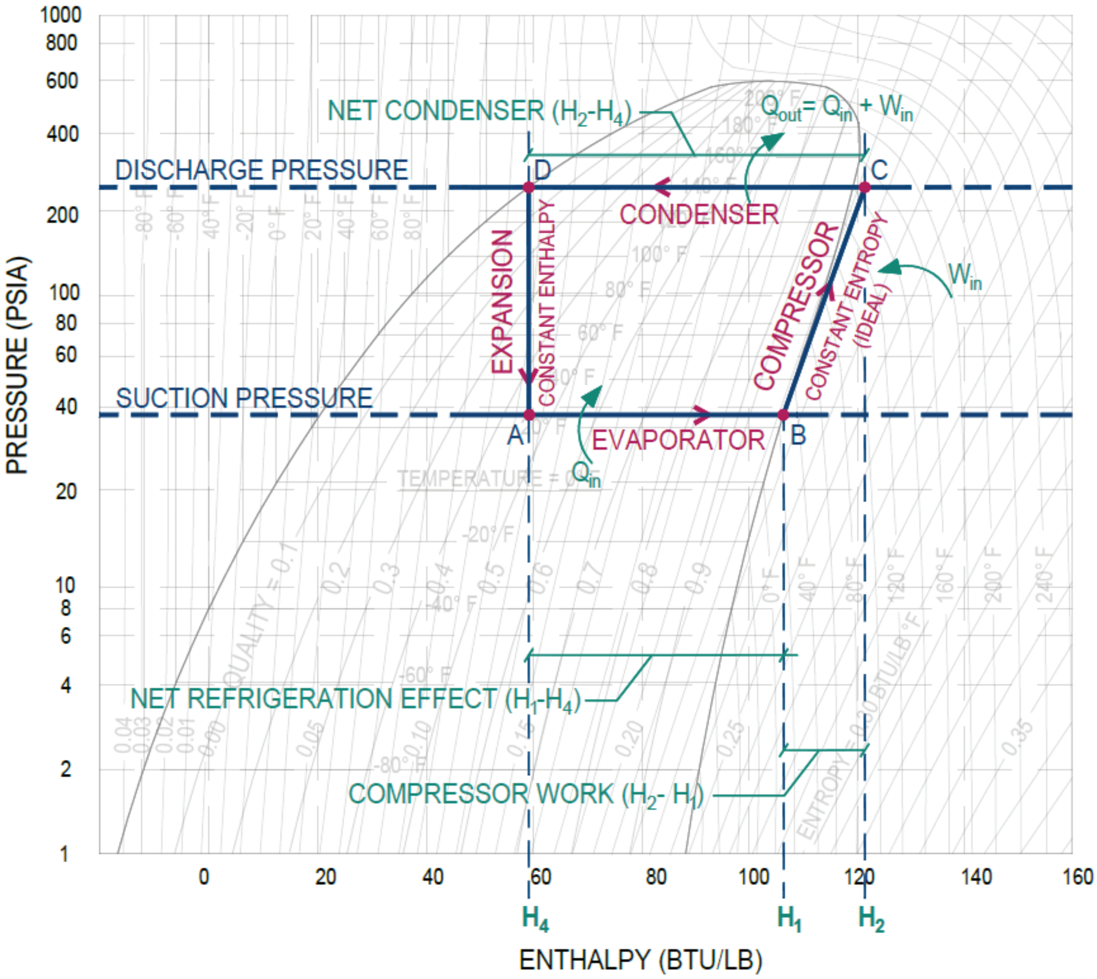

NET CONDENSER (H_2-H_4)
$Q_{out} = Q_{in} + W_{in}$
DISCHARGE PRESSURE
D
C
CONDENSER
EXPANSION
CONSTANT ENTHALPY
COMPRESSOR
CONSTANT ENTROPY (IDEAL)
W_{in}
SUCTION PRESSURE
A
B
EVAPORATOR
Q_{in}
TEMPERATURE = 0° F
QUALITY = 0.1
NET REFRIGERATION EFFECT (H_1-H_4)
COMPRESSOR WORK (H_2- H_1)
ENTROPY = 0.30 BTU/LB °F
PRESSURE (PSIA)
1000
800
600
400
200
100
80
60
40
20
10
8
6
4
2
1
0
20
40
60
80
100
120
140
160
H_4
H_1
H_2
ENTHALPY (BTU/LB)

Figure 45: Summary of refrigeration cycle; no superheat/subcooling.

The *compressor work is the amount of work provided by the compressor*. In order to determine the compressor work, find the incoming and leaving enthalpy conditions of the compressor and multiply the difference by the refrigeration flow rate.

The *net condenser effect is the amount of heat removed by the condenser*. In order to determine the net condenser effect, find the incoming and leaving enthalpy conditions of the condenser and multiply the difference by the refrigeration flow rate. When determining the leaving condition of the condenser, ensure that the appropriate amount of sub-cooling is used. Refer to the previous figure. *It is important to note that the net condenser effect is equal to sum of the net refrigeration effect and the compressor work.*

A summary of the useful equations are shown below.

Equation 1: Evaporator Net Refrigeration Effect

$$Q_{net\ refrigeration\ effect}\ [Btuh] = (H_1 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_1 = leaving\ evaporator\ enthalpy\left[\frac{Btu}{lb}\right];\ H_4 = entering\ evaporator\ enthalpy\ [\frac{Btu}{lb}]$$

Equation 2: Compressor Work

$$W_{compressor}\ [Btuh] = (H_2 - H_1)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_2 = leaving\ compressor\ enthalpy\left[\frac{Btu}{lb}\right];\ H_1 = entering\ condneser\ enthalpy\ [\frac{Btu}{lb}]$$

Equation 3: Net Condenser Effect

$$Q_{net\ condenser\ effect}\ [Btuh] = (H_2 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_2 = entering\ condenser\ enthalpy\left[\frac{Btu}{lb}\right];\ H_4 = leaving\ condneser\ enthalpy\ [\frac{Btu}{lb}]$$

Equation 4: Net Condenser Effect

$$Q_{net\ condenser\ effect}\ [Btuh] = W_{compressor}\ [Btuh] + Q_{net\ refrigeration\ effect}\ [Btuh]$$

The next term that the engineer should understand is Coefficient of Performance or COP. COP is the ratio of the amount of “Work Out” divided by the amount of “Work In”. In the refrigeration cycle, “Work Out” is equal to the net refrigeration effect. “Work In” is equal to the Compressor Work, this is the only point at which outside work is put into the system.

Equation 5: COP

$$COP = \frac{W_{out}}{W_{in}} = \frac{Q_{net\ refrigeration\ effect}\ [Btuh]}{W_{compressor}\ [Btuh]}$$

COP is a term used to describe the efficiency of a piece of equipment. Another common term is the Energy Efficiency Ratio or EER. The EER term describes the ratio of the cooling capacity in units [Btu/hr] to the input electrical power in units [Watts]. The conversion of COP to EER is shown below.

Equation 6: Relationship between COP and EER

$$COP = \frac{EER}{3.412}$$

9.0 NON-REACTING MIXTURES OF GASES

A non-reacting gas mixture is a mixture of two or more ideal gases that can be treated as a sum of the individual ideal gases. For example, the total mass of a mixture is just the sum of the gases.

$$Mass \rightarrow m_{mixture} = m_{gas,1} + m_{gas,2} + \ldots + m_{gas,n}$$

The number of particles of each type of gas will be dependent on the number of moles in the mixture. So the same equation can be re-written with the mole variable.

$$Moles \rightarrow n_{mixture} = n_{gas,1} + n_{gas,2} + \ldots + n_{gas,n}$$

These two equations can be combined with the molecular weight for each gas.

$$m_{mixture} = n_{gas,1} * MW_{gas,1} + n_{gas,2} * MW_{gas,2} + \ldots + n_{gas,n} * MW_{gas,n}$$

The second concept that you should know for the non-reacting mixtures of gases is that the ideal gas law applies to this mixture of gases. This means that the volume of the gas mixture is equal to the sum of each volume of individual gas. The same is also true for pressures.

$$Volume \rightarrow V_{mixture} = V_{gas,1} + V_{gas,2} + \ldots + V_{gas,n}$$

$$Pressure \rightarrow P_{mixture} = P_{gas,1} + P_{gas,2} + \ldots + P_{gas,n}$$

The last concept that you need to know is that the properties of the gas mixture is dependent on the individual properties of each gas and the mole fraction of each individual gas.

$$Mole\ Fraction \rightarrow X_{gas,n} = \frac{n_{gas,n}}{n_{mixture}}$$

$$Internal\ Energy \rightarrow u_{mixture} = u_{gas,1} * X_{gas,1} + u_{gas,2} * X_{gas,2} + \ldots + u_{gas,n} * X_{gas,n}$$

$$Enthalpy \rightarrow h_{mixture} = h_{gas,1} * X_{gas,1} + h_{gas,2} * X_{gas,2} + \ldots + h_{gas,n} * X_{gas,n}$$

$$Heat\ Capacity \rightarrow c_{p_{mixture}} = c_{p_{gas,1}} * X_{gas,1} + c_{p_{gas,2}} * X_{gas,2} + \ldots + c_{p_{gas,n}} * X_{gas,n}$$

$$Entropy \rightarrow s_{mixture} = s_{gas,1} * X_{gas,1} + s_{gas,2} * X_{gas,2} + \ldots + s_{gas,n} * X_{gas,n}$$

9.1 IDEAL GAS LAW

Gases are characterized as less dense, more compressible and easier to flow than liquids. The ideal gas law can be broken down into a few relationships between the properties of the ideal gas.

9.1.1 GAY-LUSSAC'S LAW

First, the ideal gases will have a direct relationship between its pressure and temperature. As the pressure increases, the temperature increases. As the temperature increases, the pressure will increase. This relationship is called, Gay-Lussac's law.

$$P \; \& \; T \rightarrow Pressure\ increases\ as\ temperature\ increases$$

The main assumption that ensures this relationship is that all other things are held the same, like the volume of the gas. As the gas heats up in an enclosed volume, the gas molecules will become more excited and will exert pressure on that fixed volume.

$$If\ V_1 = V_2, then\ \frac{P_1}{T_1} = \frac{P_2}{T_2} = Constant$$

Example: As an example, assume you have an ideal gas in a fixed volume container, at a temperature of 40 °C and a pressure of 300 kPa. If the temperature is increased to 50 °C, then what will be the new gas pressure? The main thing here is to convert your temperature to absolute temperatures.

$$\frac{300\ kPa}{(40° + 273)\ °K} = \frac{P_2}{(50° + 273)\ °K}$$

$$P_2 = 309.6\ kPa$$

9.1.2 CHARLES' LAW

The next relationship is between volume and temperature. If the pressure is held constant and the temperature increases, then the volume will also increase. This relationship is called, Charles's Law.

$$V \; \& \; T \rightarrow Volume\ increases\ as\ temperature\ increases$$

The main assumption that ensures this relationship is that all other things are held the same, like the pressure of the gas. As the gas heats up in an expandable volume, the gas molecules will become more excited and will expand its volume.

$$If\ P_1 = P_2, then\ \frac{V_1}{T_1} = \frac{V_2}{T_2} = Constant$$

9.1.3 BOYLE'S LAW

The pressure/temperature and volume/temperature relationships can be combined to the following equation. Please remember to always convert your temperatures to absolute temperatures. This relationship is called, Boyle's Law.

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

9.1.4 AVOGADRO'S LAW

Next, the relationship between volume and moles must be established. This relationship is called, Avogadro's Law. This law states that equal volumes of gas at the same temperature and pressure will have the same number of molecules, typically measured in number of moles. If the volume increases, then the number of moles will increase, based on the assumption that the pressure and temperature remain constant.

$$\frac{V_1}{n_1} = \frac{V_2}{n_2}$$

9.1.5 GAS CONSTANT

The only additional component you need for the ideal gas law is the gas constant. The gas constant is defined below and when multiplied by temperature and moles, you will get Joules or ft-lbf which are units of power.

$$SI\ Universal\ Gas\ Constant\ \rightarrow\ \bar{R} = 8.314 \frac{J}{°K - mol}\ or\ 8.314 \frac{m^3 - Pa}{°K - mol}$$

$$or\quad 8.314 \frac{m^3 - kPa}{°K - kmol}\ \ or\ 8.314 \frac{L - kPa}{°K - mol}$$

$$USCS\ Universal\ Gas\ Constant\ \rightarrow\ \bar{R} = 1{,}545 \frac{ft - lbf}{°R - lbmol}$$

9.1.6 IDEAL GAS LAW

Combining all the previous laws and the gas constant will result in the ideal gas shown below. The gas constant is multiplied by the temperature and moles, in order to get power on the right hand side of the equation. On the left hand side of the equation you have a pressure multiplied by a volume which is most commonly known as work. Work has the same units as power.

$$Ideal\ Gas\ Law \rightarrow\ PV = nRT$$

9.2 REAL (NON-IDEAL) GAS LAW

The ideal gas law makes two main assumptions that make it not suitable for real situations. These assumptions are that the attraction of the gas molecules does not affect the pressure. As the volume is decreased, the pressure will not increase linearly in response. The increase in pressure is caused by the molecules bouncing against the walls of the container, but there will be an attractive force between the molecules that will decrease this pressure. The second assumption is that the volume of individual gas molecules do not affect the volume. As the pressure increases, the volume should decrease linearly in response. But due to the size of the gas molecules, it will be more difficult to decrease the volume with this higher pressure, since the molecules will now start to hit each other. These two assumptions result in the following real gas law that modifies the pressure and volume components.

$$Ideal\ Gas\ Law \rightarrow\ PV = nRT;\ \ Real\ Gas\ Law \rightarrow \left(P + \frac{an^2}{v^2}\right)(V - nb)\ = nRT;$$

You don't need to remember the above equation, because those pressure and volume adjustments are characterized by a single factor, called compressibility. This equation is shown in the FE Reference Handbook.

$$Real\ Gas\ Law \rightarrow PV = (Z)nRT;\ Z = compressibility;$$

The compressibility characterizes the ratio of the actual molar volume of the gas to the ideal molar volume of the gas. A problem on the exam may ask you to find the compressibility factor. This is a very long process and it will be described in the following example problem.

9.2.1 COMPRESSIBILITY FACTOR

An example problem is the best way to understand the process of obtaining the compressibility factor. In this problem, you have 20 g of hydrogen in a 5 L container. The actual pressure of the gas in the container is 500 kPa. What is the actual gas temperature? What is the ideal gas temperature?

First step, you must find the following properties of hydrogen gas. The molecular mass can be found on the periodic table in the Chemistry section. The critical temperatures and pressures can be found in the Thermodynamics section of the NCEES FE Reference Handbook.

$$Molecular\ Mass = 2.016\frac{g}{mol};\ \ Critical\ Temperature\ (T_{cr}) = 33.2\ K$$

$$Critical\ Pressure\ (P_{cr}) = 1.3\ MPa = 1300\ kPa;$$

$$Moles = 20\ g * \left(\frac{mol}{2.016\ g}\right) = 9.9\ mols$$

Next, you need to calculate the reduced pressure and pseudo reduced volume. The equations are shown on the compressibility chart in the Thermodynamics section of the FE Reference Handbook. Please make sure that you use the correct units for R and make sure all the units work out.

$$Reduced\ Pressure\ (P_r) = \frac{P_{actual}}{P_{cr}} = \frac{500\ kPa}{1{,}300\ kPa} = 0.385$$

$$Psuedo\ Reduced\ Volume\ (V_r) = \frac{VP_c}{nRT_c} = \frac{(5\ L * (\frac{1\ m^3}{1000\ L})(1{,}300\ kPa)}{\left(9.9\ mols * \left(\frac{1\ kmol}{1000\ mol}\right)\right)\left(8.314\frac{kPa - m^3}{kmol - K}\right)\ (33.2\ K)}$$

$$V_r = 2.38$$

You have two values for the compressibility graph. There are three possible values that you could use to find compressibility, (1) reduced temperature, (2) reduced pressure and (3) pseudo reduced volume. You need only 2 of these 3 values, in order to find Z.

$$Compressibility\ Graph \rightarrow from\ the\ intersection\ of\ P_r\ \ and\ V_r \rightarrow Z = 0.88$$

Finally, with the Z-factor, you can find the other remaining factors from the modified ideal gas law equation.

$$Real\ Gas \rightarrow (500\ kPa)(5\ L) = (0.88)(9.9\ mols)\left(8.314\frac{kPa - L}{mol - °K}\right)T$$

$$T = 34.5\ °K;\ \ T_r = \frac{34.5}{33.2} = 1.04$$

$$Ideal\ Gas \rightarrow (500\ kPa)(5\ L) = (9.9\ mols)\left(8.314\frac{kPa - L}{mol - °K}\right)T$$

$$T = 30.4\ °K$$

10.0 PSYCHROMETRICS

The **Psychrometric Chart** describes the various properties *(Dry Bulb Temperature, Wet Bulb Temperature, Enthalpy, Humidity Ratio, Relative Humidity and Specific Volume)* of moist air. Moist air is defined as an air-water mixture. Moist air on a psychrometric chart ranges from Dry Air to Saturated Air.

Dry air is defined as having no water vapor and is located on the x-axis of the psychrometric chart. Saturated air is defined as an air-water mixture at equilibrium between the liquid and vapor phase. Saturated air is moist air in balance with its liquid and vapor phases. At saturation, air cannot hold any more moisture. It is the extreme opposite of dry air. This saturated air is defined by the exponential curve, called the saturation curve, which is clearly shown on Figure 3.

The psychrometric chart does not account for variations in pressure. Pressure is not shown on any axis, because it is constant. For the most of the FE exam and for this guide, it is assumed that the psychrometric chart is based on atmospheric pressure at sea level (14.696 psia or 1 atm or 29.921 in. Hg), unless noted otherwise. The psychrometric chart for the FE exam also only shows a range of the temperatures typically encountered.

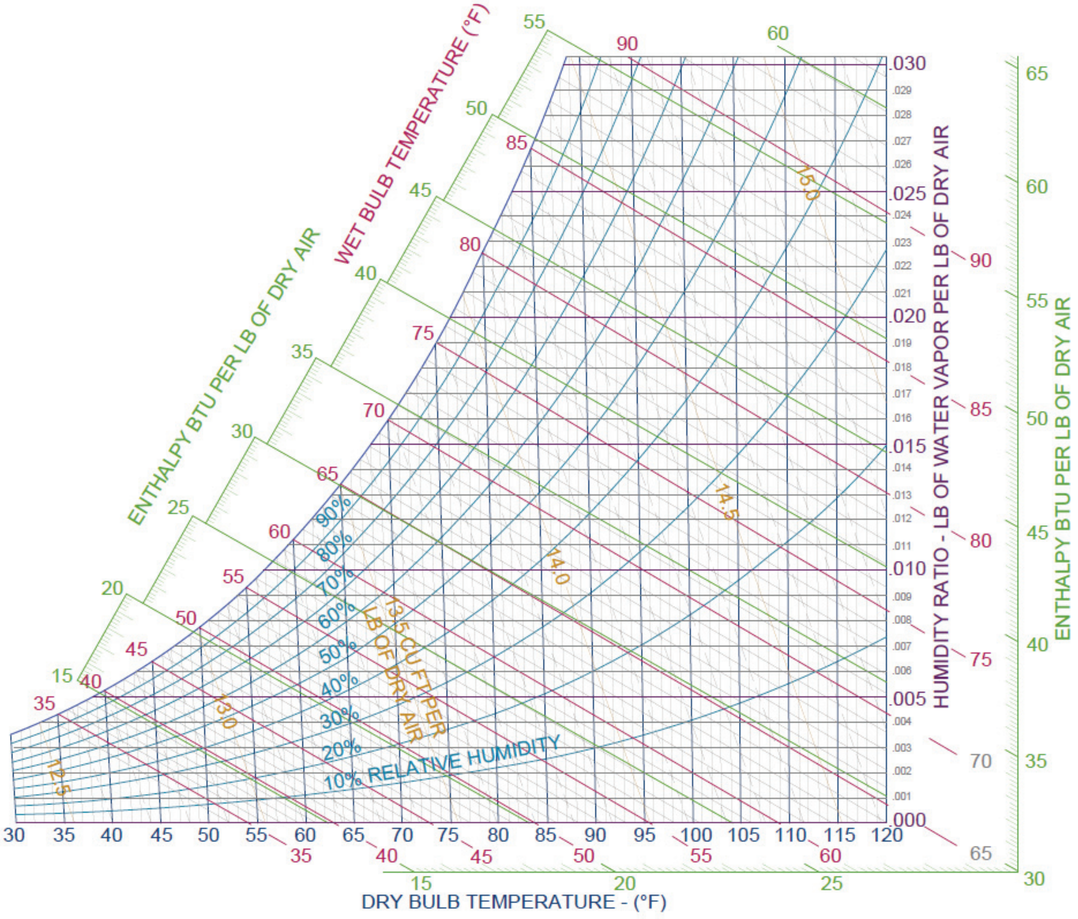


Figure 46: Psychrometric Chart at Sea Level

The following (3) Psychrometric Chart Topics will now be discussed in detail:

- Air Properties on the Psychrometric Chart: *What does each property tell of the air-water mixture? How much energy does an air mixture have? How do two different air conditions compare? Is the air hot/cold, wet/dry?*

- Movement on the Psychrometric Chart: *What causes each type of movement on the chart (Right to Left, Up and Down, Diagonal). What is sensible heating/cooling? What is humidification/dehumidification? What is the sensible heat ratio?*

- Typical Psychrometric Questions: *Mixtures of Moist Air, Humidification/Dehumidification, Heating/Cooling Questions, Condensation, etc.*

10.1 PROPERTIES OF MOIST AIR

Psychrometric Chart – Properties: The psychrometric chart as previously discussed, graphically shows the following thermodynamic properties of Moist Air, 1) Dry Bulb Temperature, 2) Wet Bulb Temperature, 3) Relative Humidity, 4) Humidity Ratio, 5) Enthalpy, 6) Specific Volume and 7) Dew Point. If any of the two above properties are known of an air mixture, then the five other corresponding properties can be found.

The following sections go into detail on each of the thermodynamic properties. It is the intent of these sections for the reader to gain an understanding of the concepts and to grasp the meaning of each property.

10.1.1 DRY BULB TEMPERATURE

This is the temperature most people are familiar with, since the Dry Bulb temperature is the temperature always shown on thermometers and thermostats. It is called dry because the temperature is not affected by the moisture in the air. It is a direct indication of the amount of sensible heat in the air at the given conditions.

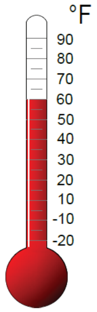


Figure 47: Dry Bulb Temperature

In practice, design dry bulb temperatures for typical offices range from 72° to 75° F. Also typical dry bulb temperatures of cool supply air from an air handling coil ranges from 52° to 55° F. For the purpose of the FE Exam, United States Customary Units or USCS are used for Psychrometrics. Please do not spend additional time on studying conversions to International System of Units (SI).

On the psychrometric chart, the dry bulb temperature is shown by vertical, parallel lines, below in blue.

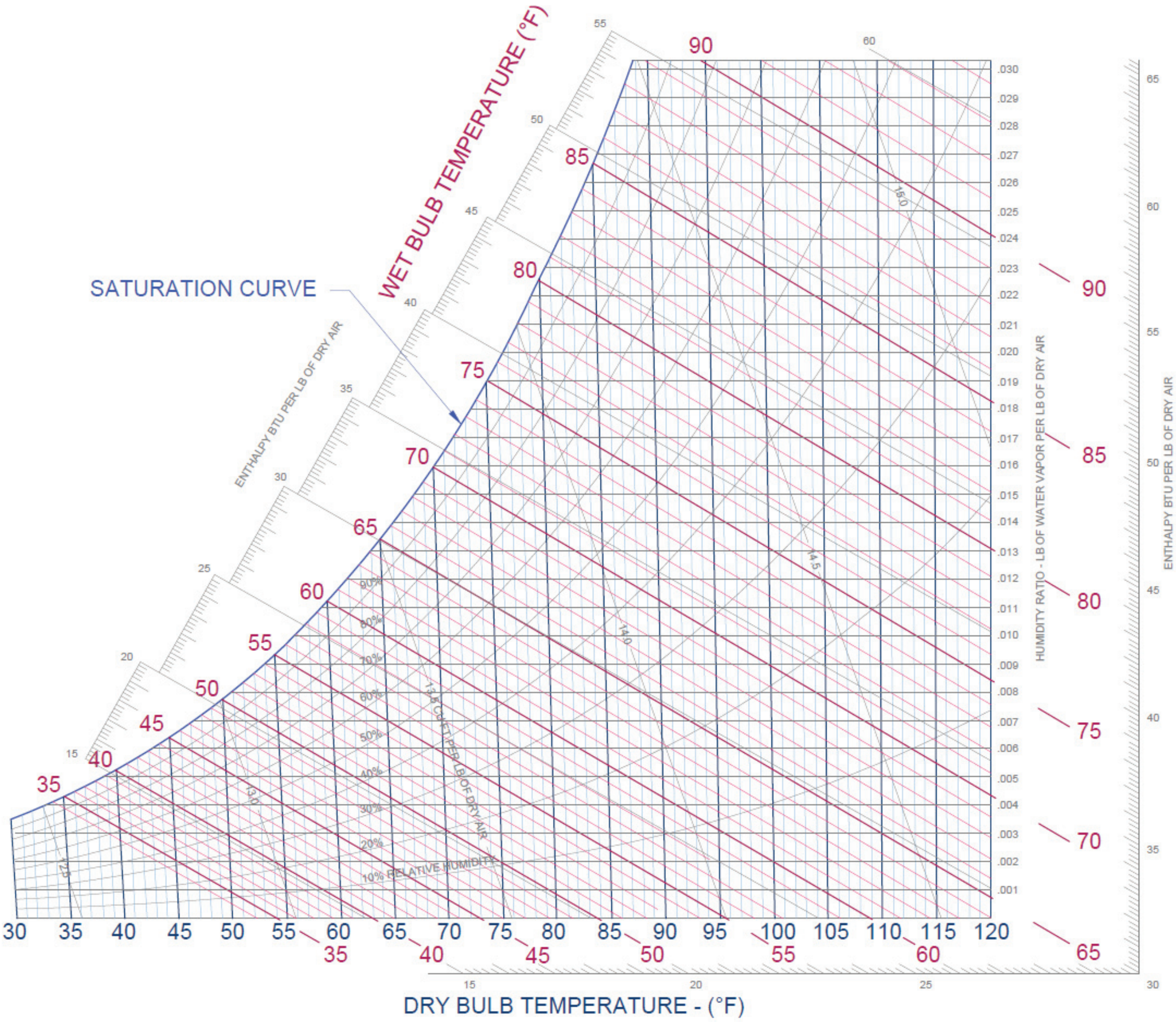


Figure 48: *Psychrometric Chart – Dry Bulb & Wet Bulb Temperature*

10.1.2 WET BULB TEMPERATURE

This property takes into account the moisture in the air. The wet bulb temperature in conjunction with the dry bulb temperature is a direct indication of the amount of moisture in the air.

The wet bulb temperature is measured through the use of a sling psychrometer, which consists of a thermometer covered by a wetted cloth wick. The psychrometer is swung and if the air is dry, then the wetted cloth wick will begin

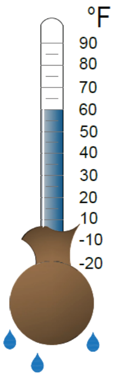


Figure 49: Wet Bulb Temperature

to evaporate. The energy lost to evaporation cools the wetted cloth wick, which will in turn decrease the temperature reading.

If the air is saturated, then the wetted cloth wick will not evaporate and the wetted cloth wick will read the same temperature as the air. The drier the air, the more evaporation will take place, which will cool the wetted cloth wick and will decrease the wet bulb temperature reading. Thus the difference between the dry bulb temperature and the wet bulb temperature describes whether or not the air is humid or dry.

For example, wet bulb temperatures significantly lower than the dry bulb temperature is an indication of dry air. A wet bulb temperature near the dry bulb temperature describes air that is nearly saturated.

On the psychrometric chart shown in the previous figure, the wet bulb temperature lines originate from the intersection of the corresponding dry bulb temperature and the saturation curve. The wet bulb lines are shown in red. If you follow the dry bulb lines shown in blue from the dry air line up to the saturation curve, you can see that the wet bulb increases until it equals the dry bulb temperature. When the wet bulb temperature equals the dry bulb temperature, then the air is saturated.

10.1.3 RELATIVE HUMIDITY

Relative humidity is the ratio of the amount of water vapor in the air mixture - compared to the maximum amount of water vapor the air can hold at that dry bulb temperature. For example, an air mixture at 100% relative humidity is shown on the saturation curve because the amount of water vapor in the air mixture is equal to the maximum amount. Dry air is shown as the x-axis, which has a relative humidity of 0%, since there is no water vapor in the air mixture.

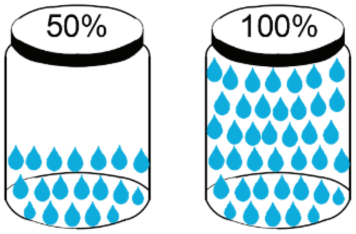


Figure 50: Relative Humidity

The figure on the right shows two equal volumes of air in a jar. One jar or volume of air contains the maximum amount of water it can hold, as illustrated as a completely filled jar of water vapor droplets. The other jar is showing only half the amount of water vapor droplets and therefore has a relative humidity of 50%.

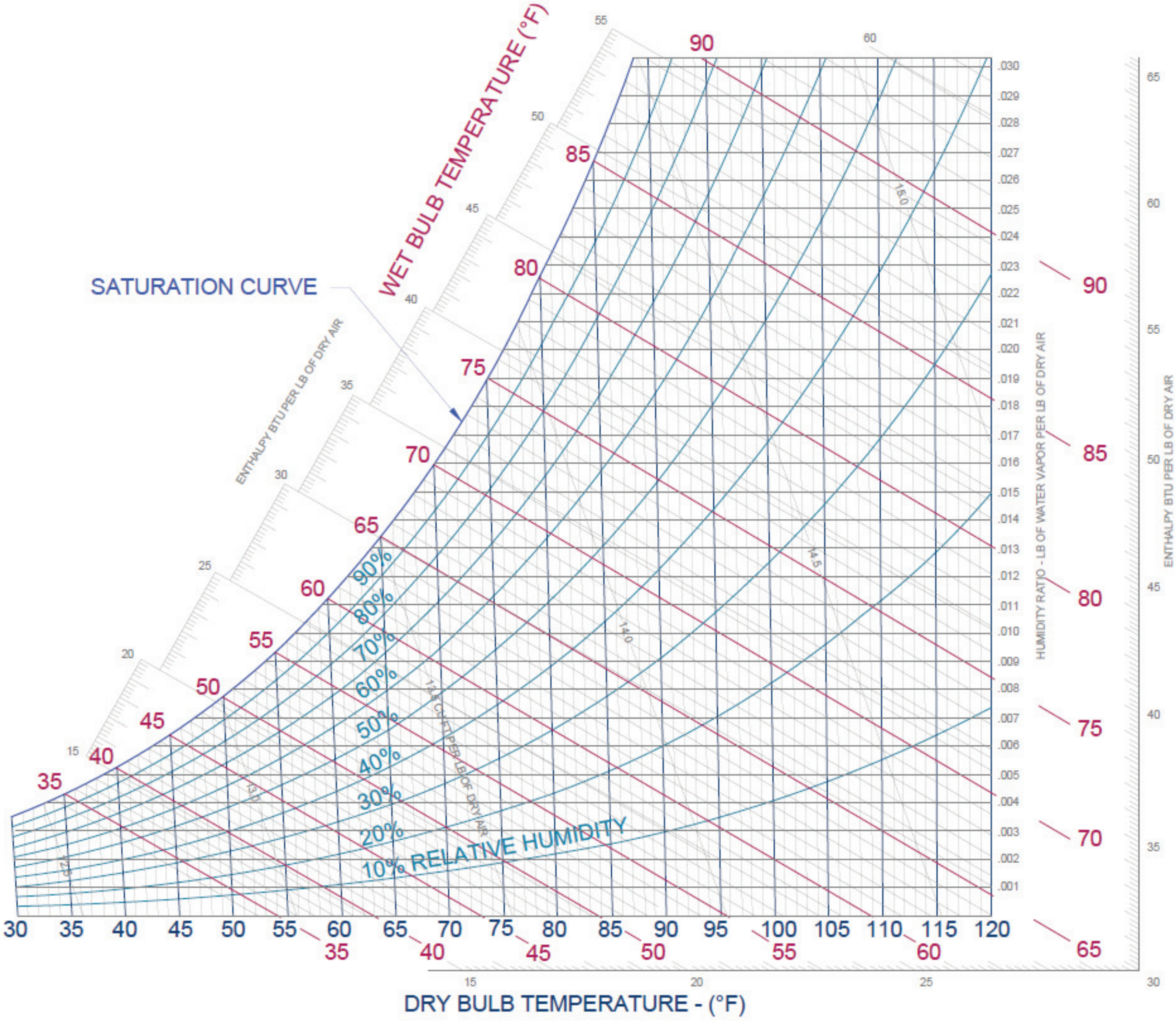


Figure 51: Psychrometric Chart – Relative Humidity, RH

Whereas the previous discussion describes relative humidity in terms of the number of water vapor particles in the air, it can also be given in terms of pressure. Relative humidity can be calculated with the following equation. Partial pressure is described in the next paragraph. Saturated water pressure can be found using the steam tables.

$$Relative\ Humidity\ (\%) = \frac{P_{water\ vapor}}{P_{saturated\ water\ @T_{DB}}} * 100\%$$

$$where\ P_{water\ vapor} = partial\ pressure\ of\ water\ vapor\ (psia)\ in\ the\ air;$$

$$P_{saturated\ water@T_{DB}} = saturated\ pressure\ of\ water\ at\ the\ dry\ bulb\ temp.of\ the\ air\ (psia)$$

Partial Pressure: To better understand the above equation, we must define partial pressures. Partial pressure is the pressure contributed by each gas type in a gas mixture. Thus, the total pressure of the mixture is equal to the sum of the partial pressures: $P_{total} = P_{gas,A} + P_{gas,B} + P_{gas,C}$, where $P_{gas,i}$ is the partial pressure of gas *i*. For air, the mixture contains dry air and water vapor particles. Since air is maintained at atmospheric pressure, the total pressure is equal to the pressure of the atmosphere, which can vary by elevation. At sea level, the total pressure is 14.7 psia. The partial pressure of water vapor is the pressure contributed by the water vapor particles. The total pressure of air is calculated as follows.

$$P_{total\ air} = P_{atmosphere} = P_{water\ vapor} + P_{dry\ air}$$

One way to visualize this is that the atmosphere exerts a pressure on the air particles. Therefore, the pressure of the dry air and water vapor particles in the air will resist the atmospheric pressure, as shown below.

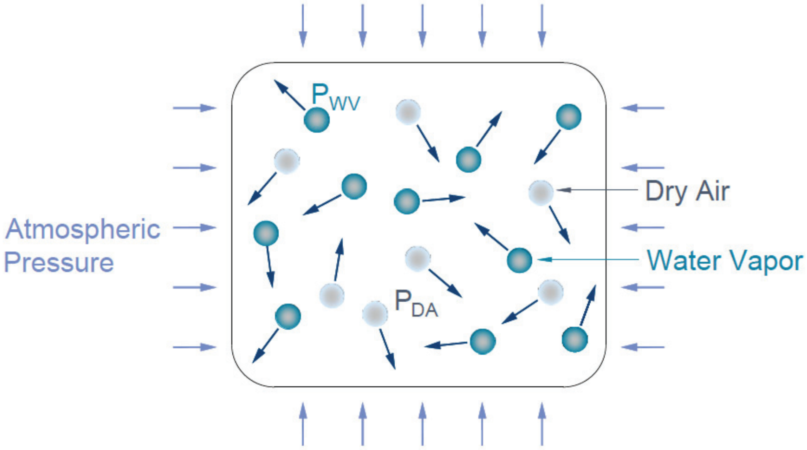


Figure 52: Partial pressures of water vapor and dry air.

10.1.4 HUMIDITY RATIO

This property describes the amount of water content in the air, expressed as the water vapor per unit mass of dry air $\left[\frac{lbm\ of\ water\ vapor}{lbm\ of\ dry\ air}\right]$ or $\left[\frac{kg\ of\ water\ vapor}{kg\ of\ dry\ air}\right]$. Sometimes the humidity ratio is expressed as $\left[\frac{grains\ of\ water\ vapor}{1\ lb\ of\ dry\ air}\right]$, where the symbol of these units is [W].

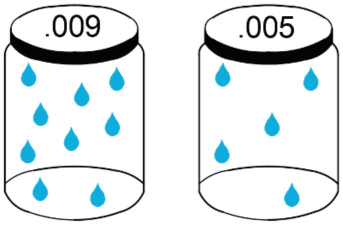


Figure 53: Humidity ratio or specific humidity

Figure 7 illustrates air with a humidity ratio of .009 $\left[\frac{lbm\ of\ water\ vapor}{lbm\ of\ dry\ air}\right]$, as shown graphically by (9) water vapor droplets. The jar on the right, holding a volume of air, has a humidity ratio of .005 $\left[\frac{lbm\ of\ water\ vapor}{lbm\ of\ dry\ air}\right]$, as illustrated by (5) water vapor droplets in the same volume of air. Mass can be interchanged with volume, if it is assumed that the density is constant. Thus, the jar with the larger number of droplets for the same volume of air has a higher humidity ratio. Humidity ratio is shown in the psychrometric chart as horizontal lines.

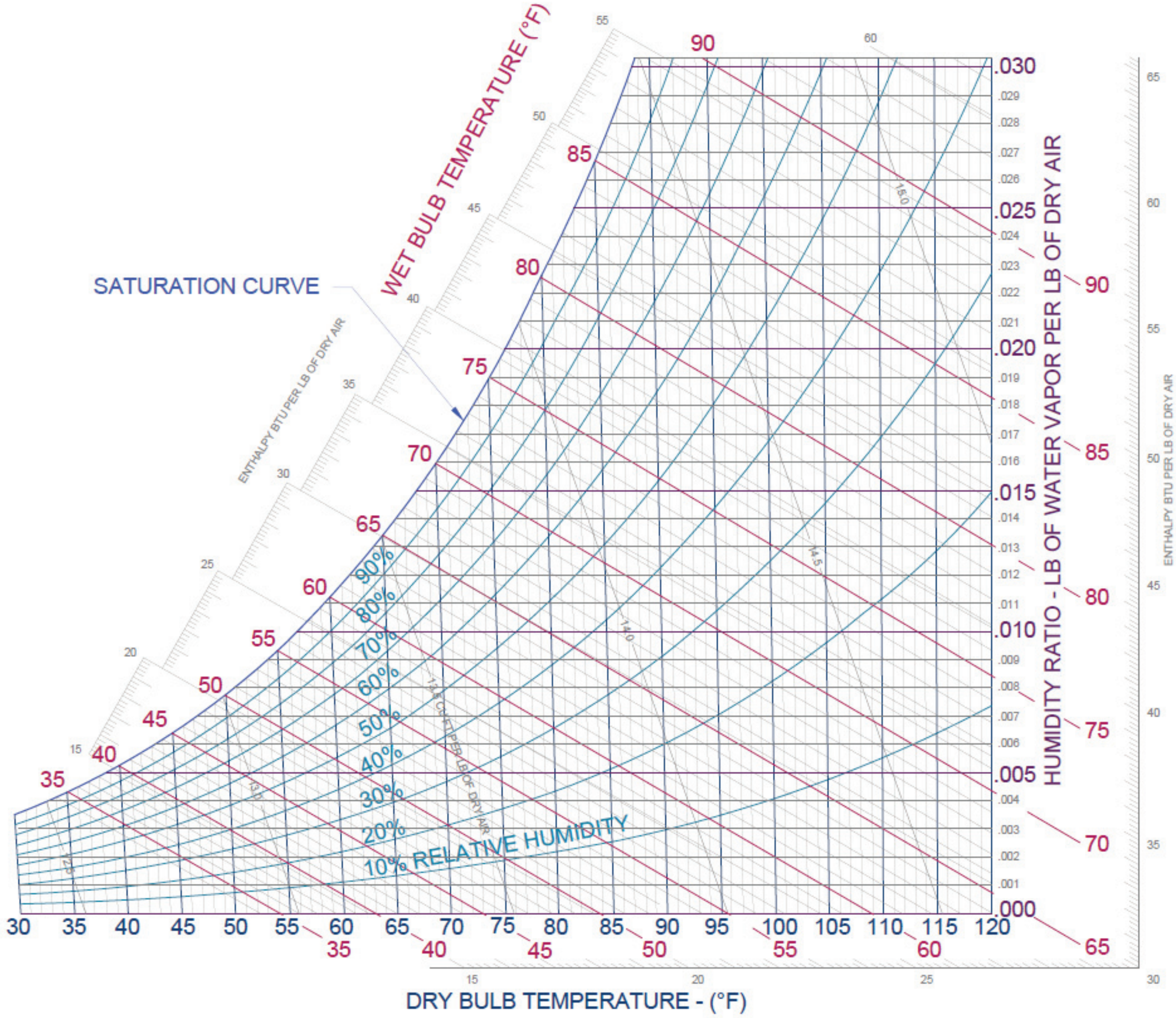


Figure 54: Psychrometric Chart – Humidity Ratio, W

Two less commonly used ratios of humidity are specific humidity and absolute humidity.

Specific humidity is similar to humidity ratio, except that the denominator is equal to the total mass of the air, which includes dry air plus water vapor. Specific humidity is calculated as the mass of the water vapor divided by the total mass of the air and is given in units of $\left[\frac{lbm\ of\ water\ vapor}{lbm\ of\ total\ (dry+water\ vapor)\ \ air}\right]$ or $\left[\frac{kg\ of\ water\ vapor}{kg\ of\ total\ (dry+water\ vapor)\ air}\right]$. Specific humidity is represented by the symbol, γ.

Absolute humidity is the mass of the water vapor divided by the total volume of air. This is also known as the *water vapor density*. Absolute humidity is given in units of $\left[\frac{lbm\ of\ water\ vapor}{ft^3\ of\ air}\right]$ or $\left[\frac{grams\ of\ water\ vapor}{m^3\ of\ air}\right]$ and represented by the symbol d_v.

10.1.5 ENTHALPY

Enthalpy is the summation of the amount of sensible and latent energy in the air. [Btu/lb of dry air]. Enthalpy is represented by the symbol = $[h]$. In HVAC, enthalpy is used to determine the amount of energy that is in moist air. It is important to recognize that on the psychrometric chart, enthalpy is shown as downward sloping lines in green. These enthalpy lines are closely aligned (but not completely) to the downward sloping wet bulb lines. Enthalpy shall be covered more closely in the following sections. This section is meant to introduce the term and its role on the psychrometric chart to the reader.

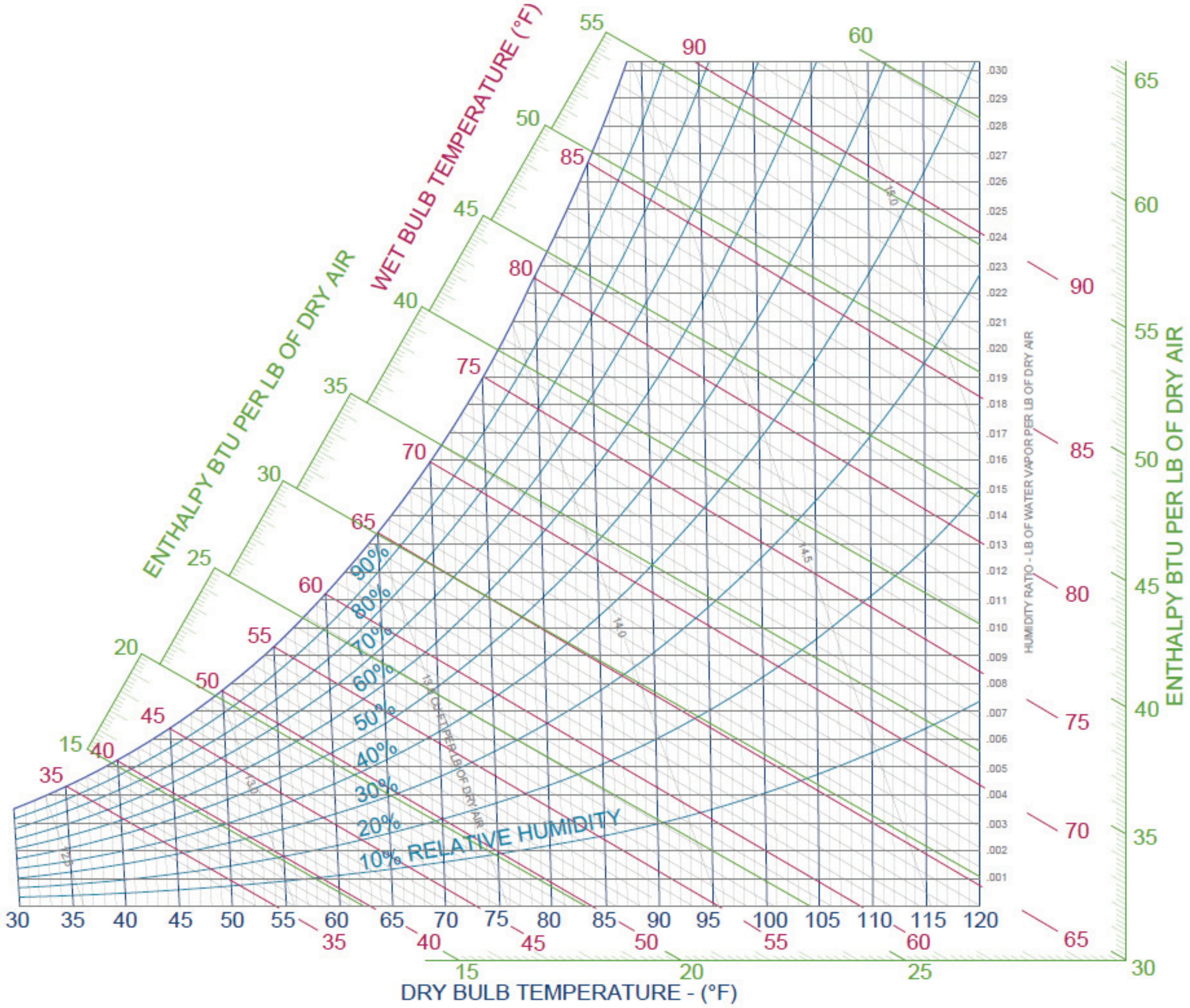


Figure 55: Psychrometric Chart - Enthalpy

10.1.6 SPECIFIC VOLUME

Specific volume is used to define the volume taken up by air at specific conditions. The units are defined as [ft^3/lb of dry air] and the symbol for specific volume is [v]. On the psychrometric chart, specific volume is shown as downward sloping lines, shown in orange on the next figure.

For example, air at 1 atmosphere, 68° F DB and 60° F WB would have a specific volume of 13.5 ft^3/lb and air at 1 atmosphere, 88° F DB and 66° F WB would have a specific volume of 14.0 ft^3/lb.

Density is simply the inverse of specific volume. At standard conditions, the density of air is 1/13.33 [ft^3/lb] or 0.075 [lb/ft^3]. Standard pressure is taken at sea level, 14.7 psia, but standard temperature can vary based on the moisture content or the reference source. Standard temperatures of dry air at the standard density are 69°F, 0% RH or 60°F, 100% RH for moist air. For fan testing, the Air Movement and Control Association International, Inc. (AMCA) sets the standard temperature at 68°F, 50% RH. All conditions have a density that rounds to 0.075 lb/ft^3. For exam purposes, air at standard conditions based on the *NCEES FE Reference Handbook* is 69°F, 14.7 psia, 0.075 lb/ft^3 for dry air.

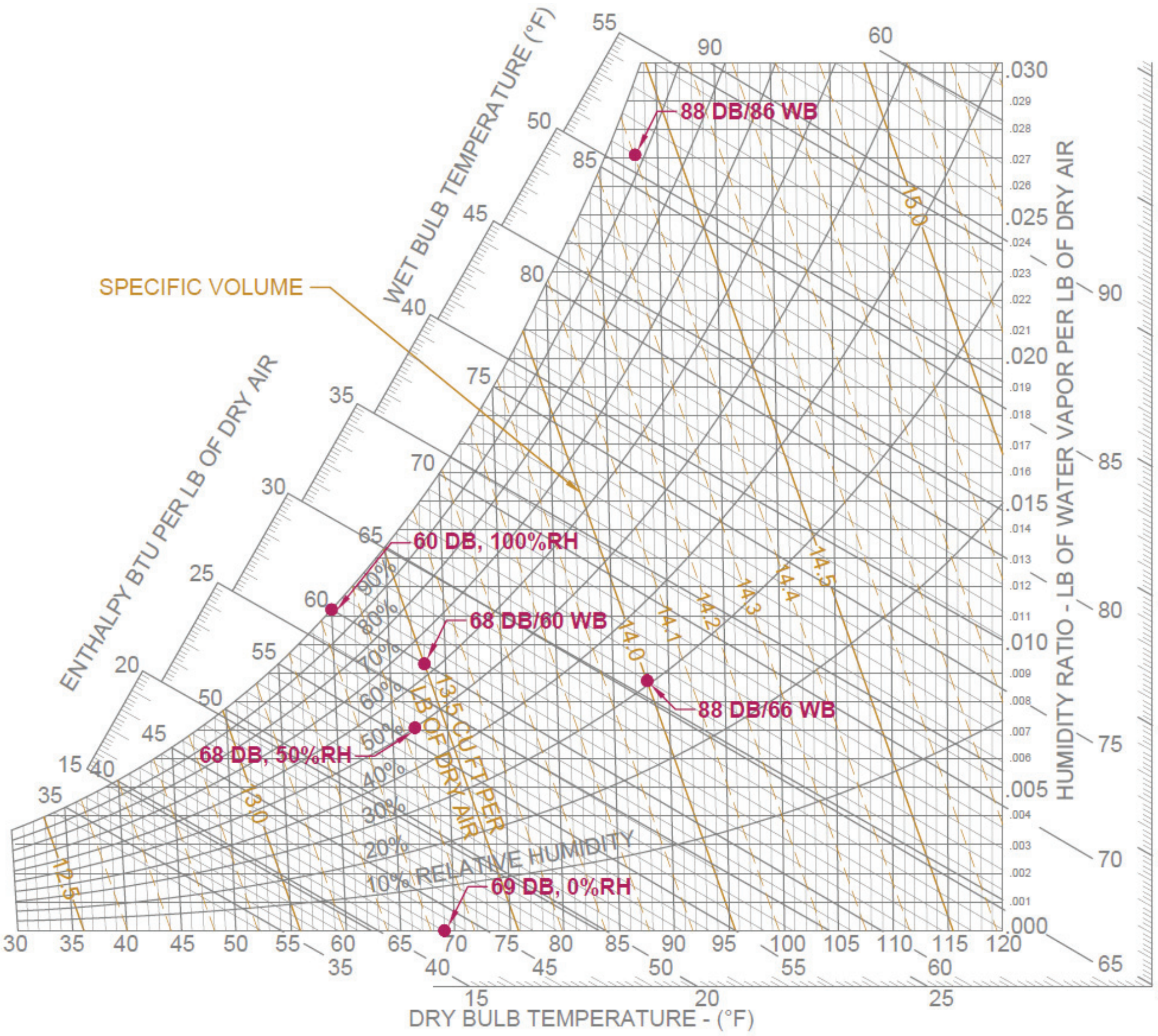


Figure 56: Psychrometric chart - Specific Volume. Standard conditions at 13.3 ft^3/lbm

10.1.7 DEW POINT

The dew point can be found by first finding a certain condition on the graph and moving horizontally left to the saturation curve. This is the temperature at which the air mixture must be cooled in order for condensation to first occur. In HVAC, this skill is often used to determine under what conditions condensation will occur on a window, pipe, duct or equipment surface. One example is the fairly new HVAC system called Chilled Beams. Chilled beams is the system of cooling spaces through the use of pipes containing cool water suspended below a ceiling, the pipes are often situated into a passive (no fan) or active (fan) heat exchanger. As warm air passes over the chilled beam, it cools and falls to the floor, thereby cooling the space below. Condensation is a concern because if the chilled beam is at a temperature below the dew point of the air, then as the air hits the coil, water may condense on the chilled beam and drip into the space.

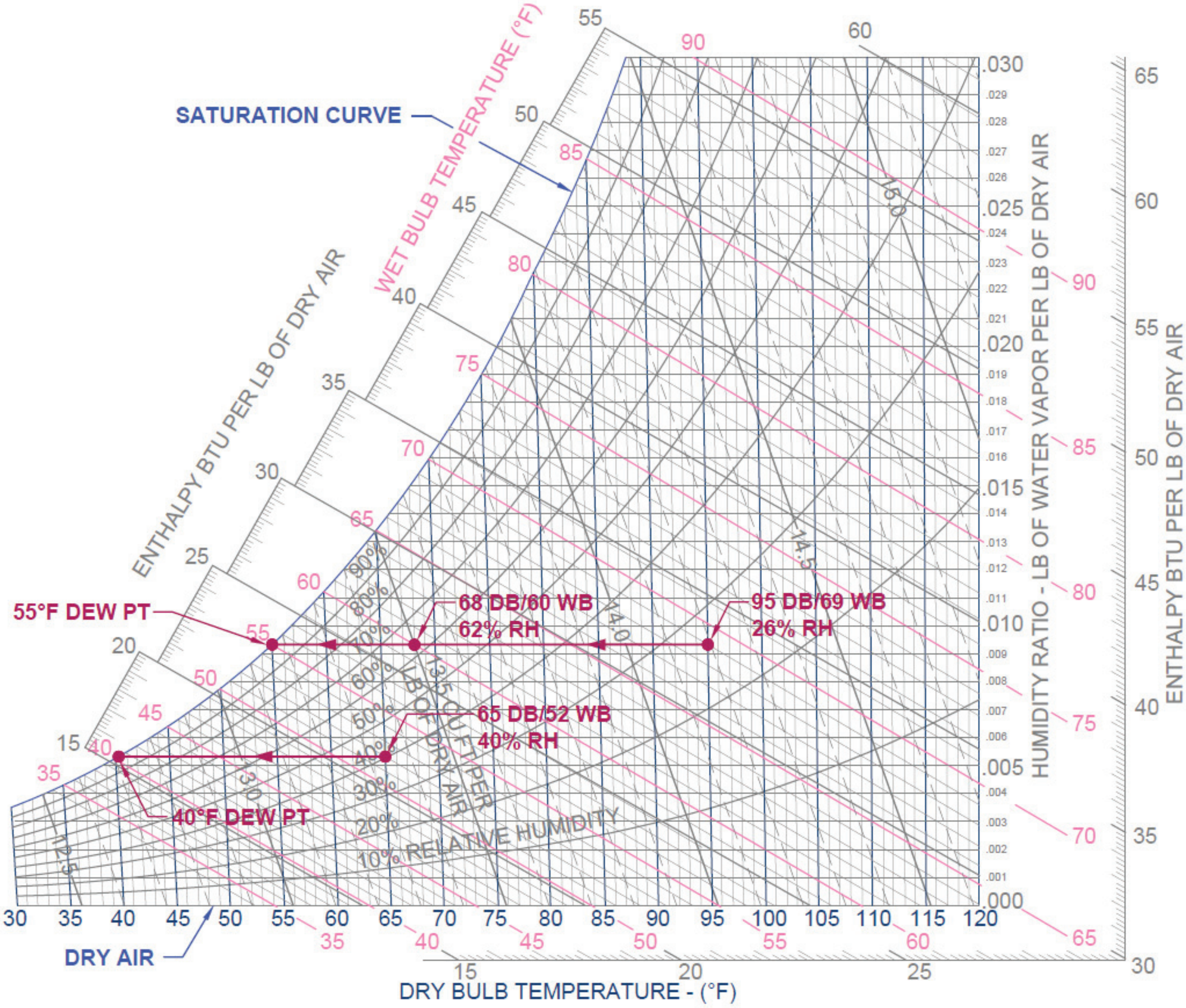


Figure 57: Psychrometric Chart - Dew Point

11.0 HEATING, VENTILATING AND AIR-CONDITIONING (HVAC) PROCESSES

11.1 SENSIBLE HEATING/COOLING

Sensible heating and cooling is defined as the removal or addition of heat to an air mixture, with no effect on the moisture content. Its sole effect is on the increase or decrease of the dry bulb temperature.

$$Q_{sensible}\left[\frac{Btu}{h}\right] = \dot{m}c_p\Delta T$$

$$where\ c_p = 0.24\ \frac{Btu}{lbm_{dry}\cdot ℉},\ specific\ heat\ of\ air^{*}$$

$$and\ \dot{m}\ \left(\frac{lbm}{hr}\right) = \dot{V}\left(\frac{ft^3}{min.}\right) * \frac{60\ minutes}{1\ hour} * \frac{0.075\ lbm}{ft^3}$$

$$Q_{sensible}\left[\frac{Btu}{h}\right] = \frac{60min}{hr} * 0.075\frac{lbm}{ft^3} * 0.24\frac{Btu}{lbm\cdot ℉} * CFM * \Delta T$$

$$\boldsymbol{Q_{sensible}}\left[\frac{Btu}{h}\right] = \ 1.08 * CFM * \Delta T \approx \mathbf{1.1} * \boldsymbol{CFM} * \Delta\boldsymbol{T}$$

**Note:* The full equation includes the specific heat of vapor, which is added to the specific heat of dry air as shown in the calculation below. The contribution from the specific heat of vapor is very small and excluded in the previous analysis for simplicity. You should not waste your time including this in your calculations, unless you have extreme temperature and humidity conditions, which is not very likely. As an example of how small the changes are, 90F, 65% RH, where W = 0.020 lb_w/lb_d, which is higher than average, the heat equation factor is 1.12. For winter conditions when W = 0.002 lb_w/lb_d and $c_{p,wet}$ = 0.45 Btu/lbm-F, the factor is 1.08.

$$c_p = c_{p,dry\ air} + c_{p,vapor} = 0.24 + c_{p,vapor}$$

$$c_{p,vapor}\ =\ c_{p,wet}\left(\frac{Btu}{lbm_{wet}\cdot ℉}\right) * W\left(\frac{lb_{wet}}{lb_{dry}}\right)$$

$$c_{p,vapor@75F,50\%}\ =\ 0.46\ \frac{Btu}{lbm_{wet}\cdot ℉} * 0.0093\left(\frac{lb_{wet}}{lb_{dry}}\right) = .0043\frac{Btu}{lbm_{dry}\cdot ℉}$$

Substituting this into the heat equation results in a 1.10 factor instead of 1.08 from the previous derivation.

$$\boldsymbol{Q_{sensible}}\left[\frac{Btu}{h}\right] = \mathbf{1.10} * \boldsymbol{CFM} * \Delta\boldsymbol{T}$$

On a Psychrometric Chart, sensible heating and cooling is shown as a horizontal line. It is horizontal because the amount of water vapor in the air is not changed, thus the Humidity Ratio remains the same. A horizontal movement increases or decreases the dry bulb temperature. As the dry bulb temperature increases with sensible heating, the air's capacity to hold water also increases. The opposite is true with sensible cooling. In addition, sensible heating is shown on the graph below to decrease relative humidity, while sensible cooling increases relative humidity.

$$Relative\ Humidity = \frac{Amount\ of\ Water\ Vapor\ in\ Air}{Maximum\ amount\ of\ water\ vapor\ in\ Air}\ \frac{(no\ change)}{(increases)}$$

$$Sensible\ Heating\ causes\ a\ decrease\ in\ relative\ humidity.$$

$$Relative\ Humidity = \frac{Amount\ of\ Water\ Vapor\ in\ Air}{Maximum\ amount\ of\ water\ vapor\ in\ Air}\ \frac{(no\ change)}{(decreases)}$$

$$Sensible\ Cooling\ causes\ an\ increase\ in\ relative\ humidity.$$

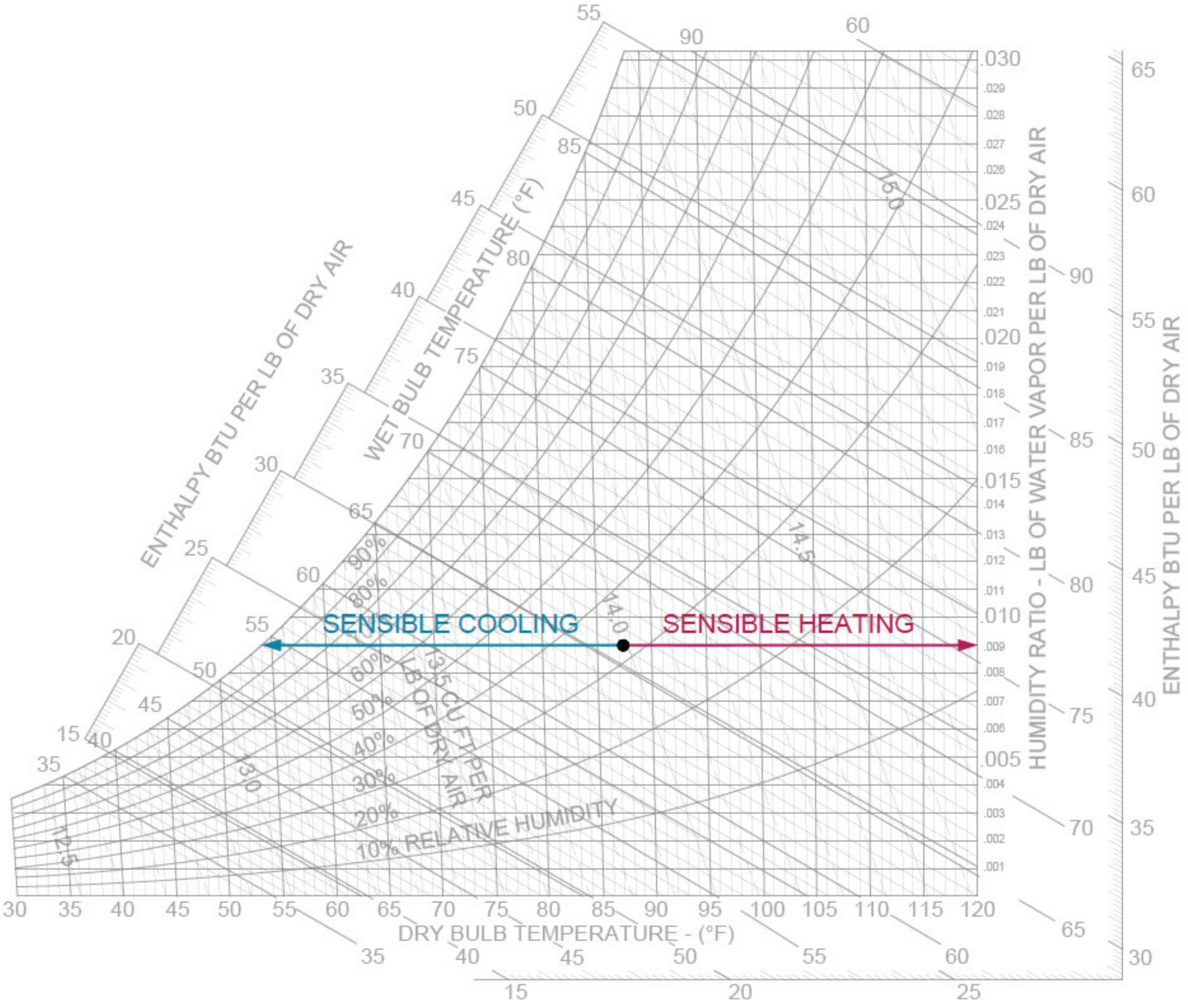


Figure 58: Sensible heating/cooling - horizontal movement

Sensible Heating is characterized by the following: (1) Increase in dry bulb temperature, (2) Decrease in relative humidity, (3) No change in humidity ratio, (4) Increase in enthalpy, (5) Increase in specific volume [hotter air takes up more volume], (6) No change in dew point, (7) Increase in wet bulb.

Sensible Cooling is characterized by the following: (1) Decrease in dry bulb temperature, (2) Increase in relative humidity, (3) No change in humidity ratio, (4) Decrease in enthalpy, (5) Decrease in specific volume, (6) No change in dew point, (7) Decrease in wet bulb.

Quantifying sensible movement on the psychrometric chart is best described through the use of an example problem.

Example Problem: Sensible Heating

Background: 1,000 CFM of conditioned air at **55° F DB/53° F WB**, passes through an electric resistance heater that provides 12,000 Btu/h of heat to the air, assume that 100% of the air is in contact with the heater (bypass factor = 0). Assume a constant density of 0.075 lb/ft^3.

Question: What are the final conditions of the air? Dry bulb, wet bulb, relative humidity, humidity ratio?

Solution: The electric heater will only provide sensible heating of the air, therefore the following equation can be used:

$$Q_{sensible}\left[\frac{Btu}{h}\right] = 1.1 * \Delta T * CFM$$

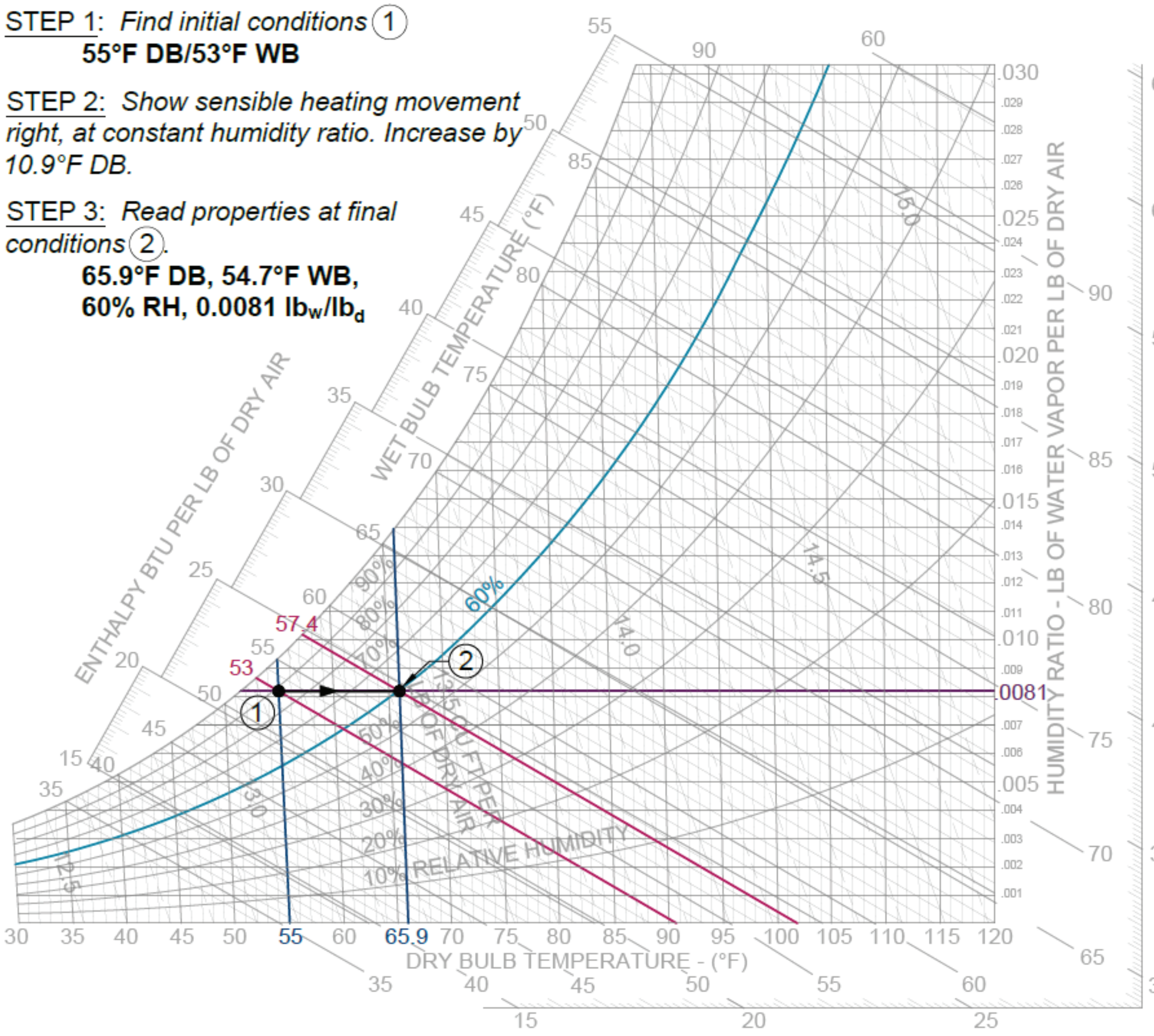
$$12{,}000\left[\frac{Btu}{h}\right] = 1.1 * \Delta T * 1{,}000\ CFM$$

$$\Delta T = 10.9°\text{F} = T_{DB,final} - T_{DB,initial}$$

$$10.9°\text{F} = T_{final} - 55°\text{F}$$

$$T_{final} = 65.9°\ \text{F}$$

We now know the final dry bulb temperature and since only sensible heating occurred, there is no change in water content, thus humidity ratios are equal $W_{final} = W_{initial}$. In order to get the final conditions of the air, the psychrometric chart must be used. Knowing two values allows the other 5 properties to be determined, see the following page.



STEP 1: *Find initial conditions* ①

55°F DB/53°F WB

STEP 2: *Show sensible heating movement right, at constant humidity ratio. Increase by 10.9°F DB.*

STEP 3: *Read properties at final conditions* ②.

65.9°F DB, 54.7°F WB, 60% RH, 0.0081 lb_w/lb_d

Figure 59: Sensible heating example

11.2 LATENT HEATING/COOLING

Latent heat energy is the amount of energy required to produce a phase change, water (liquid) to water (vapor).

Latent heating and cooling is defined as the removal or addition of moisture (water vapor) to an air mixture. Latent heating is more commonly known as humidification and latent cooling is known as dehumidification. In HVAC, common latent heat sources include people, equipment and outdoor air.

Latent heating and cooling is calculated with the latent heat of vaporization, or the energy required to change phases from liquid to vapor. It does not account for actual temperature changes to the air. Since typical HVAC cooling occurs from 75°F, 50% RH to around 50°F, convention is to use a latent heat of vaporization of 1076 Btu/lb.

$$H_v = h_{g\,@75F} - h_{f\,@50F} = 1094\,Btu/lb - 18\,Btu/lb = \mathbf{1076\,Btu/lb}$$

You will notice other sources use the heat of vaporization at 60°F, $h_{fg,60F}$ = 1060 Btu/lb. This is just a different reference point. For the sake of the exam, you should use the heat of vaporization given in the *NCEES FE Reference Handbook,* unless your temperature conditions vary widely. The amount of latent heating/cooling is determined through the following two ways [constants shown for standard conditions].

(1) ΔW_{LB}: Change in pounds of water vapor per pound of dry air.

Step 1: $\boldsymbol{Q_{latent} = \dot{m} * H_v}$

Latent Energy is equal to change in mass flow rate
of water multiplied by the heat of vaporization.

$$Q_{latent} = latent\ heat\left[\frac{Btu}{Hr}\right]$$

$$\dot{m} = mass\ flow\ rate\ of\ water\left[\frac{lb\ of\ H20}{Hr}\right]$$

$$H_v = heat\ of\ vaporization,\ 1076\left[\frac{Btu}{lb\ of\ H20}\right]$$

Step 2: $\dot{\boldsymbol{m}} = CFM\left[\frac{\cancel{ft^3}}{\cancel{min}}\right] * .075\left[\frac{lbs\ of\ dry\ air}{\cancel{ft^3}}\right] * 60\left[\frac{\cancel{min}}{hr}\right] * \Delta\boldsymbol{W_{LB}}\left[\frac{lb\ of\ H20}{lb\ of\ dry\ air}\right]$

Find the mass flow rate of water in air, using the specific volume of air and the specific humidity.

$$.075\left[\frac{lbs\ of\ dry\ air}{ft^3}\right] = Density\ of\ air\ at\ 60°F/58°F\ at\ 1\ atm$$

$$60\left[\frac{min}{hr}\right] = conversion\ minutes\ to\ hour$$

$$\Delta W_{LB}\left[\frac{lb\ of\ H20}{lb\ of\ dry\ air}\right] = change\ in\ specific\ humidity$$

Step 3: $\boldsymbol{Q_{latent} = 4,840 * \Delta W_{LB} * CFM}$

Consolidated constants.

(2) ΔW_{GR}: Change in grains of water vapor per pound of dry air.

Step 1: $\boldsymbol{Q_{latent} = \dot{m} * H_v}$

Latent Energy is equal to change in mass flow rate of water multiplied by the heat of vaporization.

$$Q_{latent} = latent\ heat\left[\frac{Btu}{Hr}\right]$$

$$\dot{m} = mass\ flow\ rate\ of\ water\left[\frac{lb\ of\ H20}{Hr}\right]$$

$$H_v = heat\ of\ vaporization,\ 1076\left[\frac{Btu}{lb\ of\ H20}\right]$$

Step 2: $\boldsymbol{\dot{m}} = CFM\left[\frac{\cancel{ft^3}}{\cancel{min}}\right] * .075\left[\frac{lbs\ of\ dry\ air}{\cancel{ft^3}}\right] * 60\left[\frac{\cancel{min}}{hr}\right] * \Delta \boldsymbol{W_{GR}}\left[\frac{grain\ of\ H20}{lb\ of\ dry\ air}\right] * \left[\frac{1\ lb\ of\ H20}{7000\ grains\ of\ H20}\right]$

Find the mass flow rate of water in air, using the specific volume of air and the specific humidity.

$$.075\left[\frac{lbs\ of\ dry\ air}{ft^3}\right] = Density\ of\ air\ at\ 60°F\ DB/58°F\ at\ 1\ atm$$

$$60\left[\frac{min}{hr}\right] = conversion\ minutes\ to\ hour$$

$$\Delta W_{GR}\left[\frac{grain\ of\ H20}{lb\ of\ dry\ air}\right] = \ change\ in\ specific\ humidity$$

$$*\left[\frac{1\ lb\ of\ H20}{7000\ grains\ of\ H20}\right] = \ conversion\ lb\ to\ grains$$

Step 3: $\boldsymbol{Q_{latent} = \ 0.69 * \Delta W_{GR} * CFM}$

Consolidated constants.

Exam Tip: Ensure that these quick equations can be used, know the signs.

In most cases, the test taker will be able to use the quick equations shown as part of the exam guide, but it is still important to understand where these quick equations come from. In the event that the exam provides a non-standard condition, the test taker should be able to rapidly move from the quick equation back to the original roots [Step 1]. Certain signs include, (1) The exam states the density of the air in question, (2) the exam gives hint to extreme temperatures outside of the norm, or (3) the exam indicates a non-standard pressure. For the purposes of this exam, it is assumed that standard conditions are used. With your understanding of the derivation of the quick conditions and standard conditions, feel confident that you will be able to solve similar type problems at differing conditions.

On a Psychrometric Chart, latent heating and cooling is shown as a vertical line. Latent heating or humidification is shown as an upward movement and latent cooling or dehumidification is shown as a downward movement.

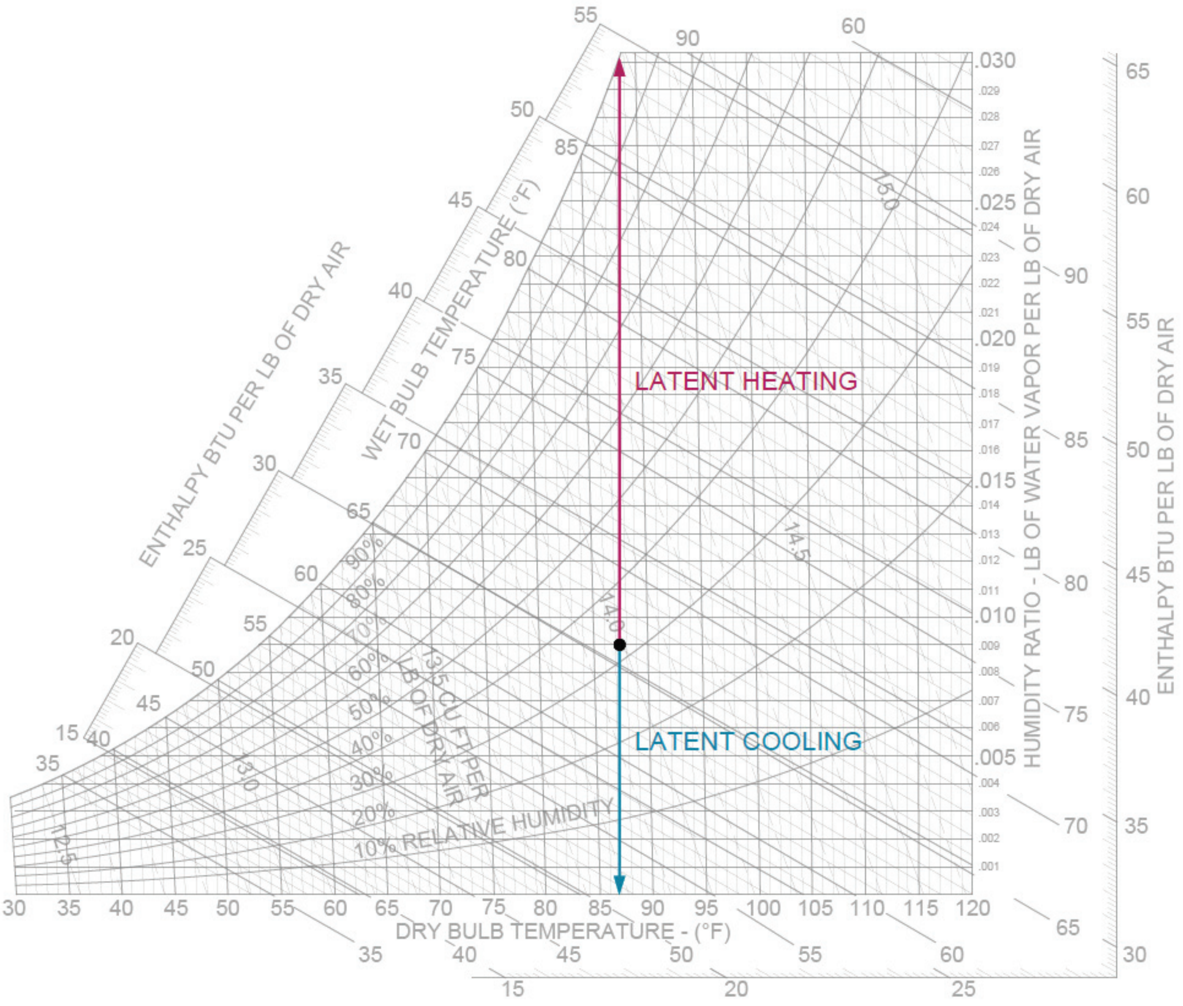


Figure 60: Latent heating and cooling/ humidification and dehumidification

Quantifying latent movement on the psychrometric chart is best described through the use of an example problem.

Example Problem - Latent Heating, Humidification

Background: 1,000 CFM of dry air at 66°F DB/30% Relative Humidity, passes through a humidifier that provides 12,000 Btu/h of latent heat to the air, assume that 100% of the air is in contact with the humidifier (bypass factor = 0). Assume a constant density of 0.075 lb/ft^3.

Question: What are the final conditions of the air? Dry bulb, wet bulb, relative humidity, humidity ratio?

Solution: The humidifier will only provide latent heating of the air; therefore the following equation can be used:

$$\boldsymbol{Q_{latent} = 4,840 * \Delta W_{LB} * CFM}$$

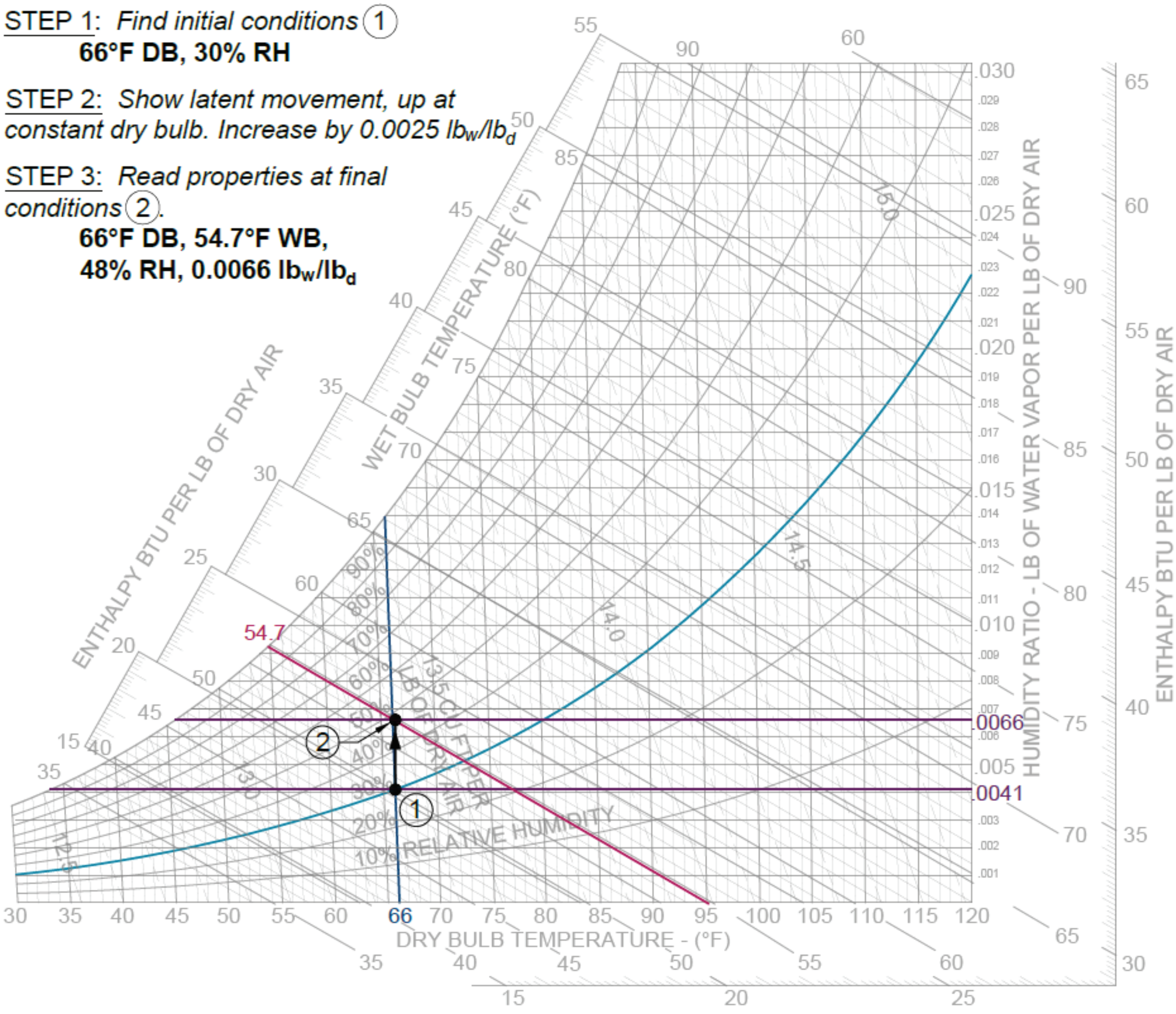
$$12,000\left[\frac{Btu}{h}\right] = \boldsymbol{4,840 * \Delta W_{LB} * 1,000}$$

$$\boldsymbol{\Delta W_{LB}}\left[\frac{lb\ of\ H20}{lb\ of\ dry\ air}\right] = .0025 = W_{final} - W_{initial}$$

At 66°F, 30% RH, the humidity ratio is 0.0041 lb_w/lb_d. Solve for the final humidity ratio.

$$.0025\left[\frac{lb\ of\ H20}{lb\ of\ dry\ air}\right] = W_{final} - .0041 \Rightarrow W_{final}\left[\frac{lb\ of\ H20}{lb\ of\ dry\ air}\right] = .0066$$

We now know the final humidity ratio and since the air has only undergone latent heating, there is no change in dry bulb temperature, dry bulbs are equal $T_{DB,final} = T_{DB,initial}$. In order to get the final conditions of the air, the psychrometric chart must be used. Knowing two values allows the other 5 properties to be determined. See the figure below.



STEP 1: *Find initial conditions* ①
66°F DB, 30% RH

STEP 2: *Show latent movement, up at constant dry bulb. Increase by 0.0025 lb_w/lb_d*

STEP 3: *Read properties at final conditions* ②.
66°F DB, 54.7°F WB,
48% RH, 0.0066 lb_w/lb_d

Figure 61: Latent heating, humidification example

11.3 TOTAL HEATING/COOLING

When both latent and sensible heating/cooling occurs, the total load is the sum of the latent and cooling loads.

$$Total\ heat\ \left(\frac{Btu}{h}\right) = Sensible\ heat\ \left(\frac{Btu}{h}\right) + Latent\ heat\ \left(\frac{Btu}{h}\right)$$

The total heat equation can be calculated using enthalpy with the following general equation.

$$\boldsymbol{Q_{total}\left[\frac{Btu}{h}\right] = \dot{m} * \Delta h}$$

$$\Delta h = difference\ between\ entering\ and\ leaving\ conditions$$

For standard temperature and pressure conditions, the equation simplifies to the following.

$$\dot{m}\left[\frac{lb}{h}\right] = CFM\left[\frac{\cancel{ft^3}}{\cancel{min}}\right] * 0.075\left[\frac{lb}{\cancel{ft^3}}\right] * 60\left[\frac{\cancel{min}}{hr}\right]$$

$$\boldsymbol{Q_{total}\left[\frac{Btu}{h}\right] = 4.5 * CFM * \Delta h}$$

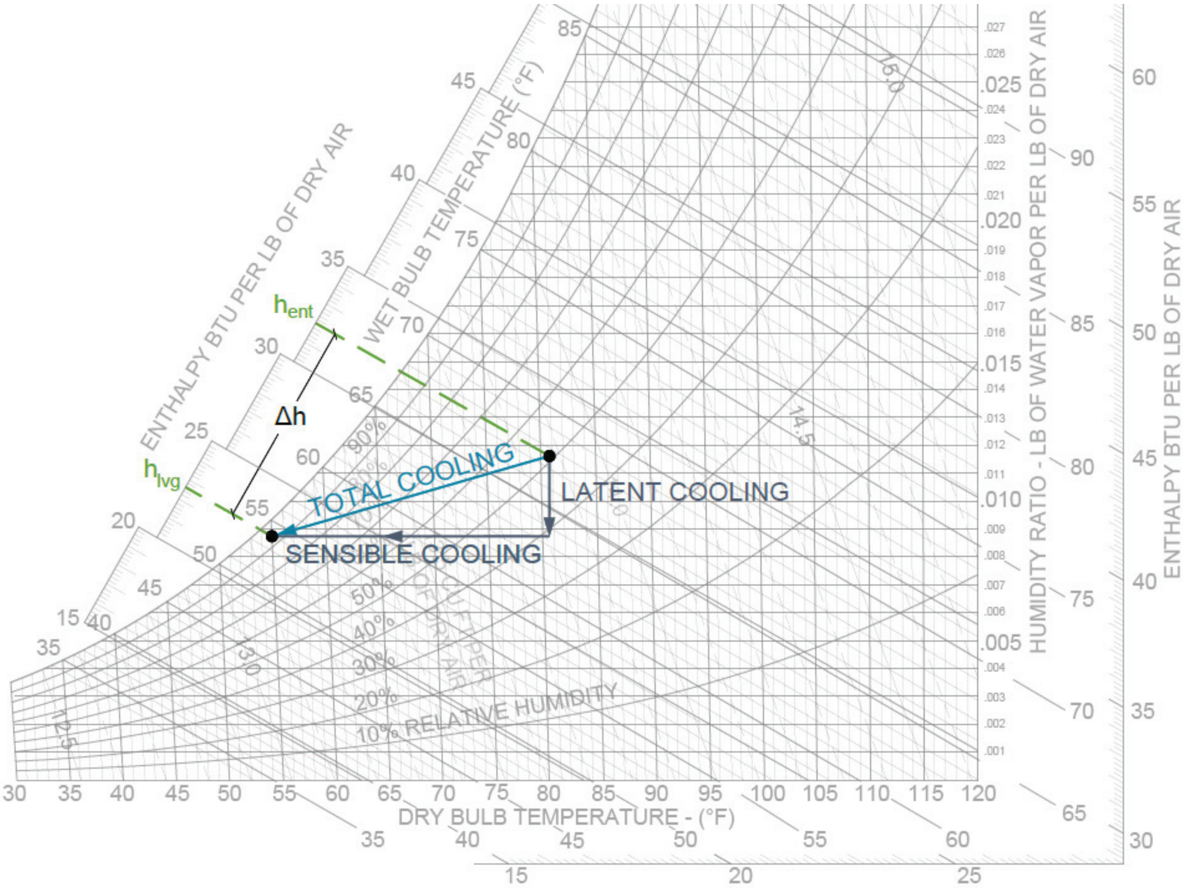


Figure 62: Total cooling/heating is calculated as the sum of latent and sensible cooling/heating or the difference in enthalpy times the mass flow rate.

11.4 SENSIBLE HEAT RATIO

The sensible heat ratio (SHR) is used when there is diagonal movement on the psychrometric chart. Diagonal movement would indicate both sensible heating/cooling and latent heating/cooling. SHR is defined as the ratio of sensible heat to total heat.

$$SHR = \frac{Sensible\ heat\ \left(\frac{Btu}{h}\right)}{Total\ heat\ \left(\frac{Btu}{h}\right)} = \frac{Sensible\ heat\ \left(\frac{Btu}{h}\right)}{Sensible{+}Latent\ heat\ \left(\frac{Btu}{h}\right)}$$

The sensible heat ratio (SHR) is used in HVAC to describe the ratio of the sensible space cooling to total cooling required for a space. Typically, there will be more sensible cooling required than latent cooling. On the psychrometric chart it describes the slope of the line as shown in the figure below. If the ratio is known, this slope can also be plotted using the compass provided with the psychrometric chart. This slope is a tool to finding the final air conditions due to the cooling or heating loads.

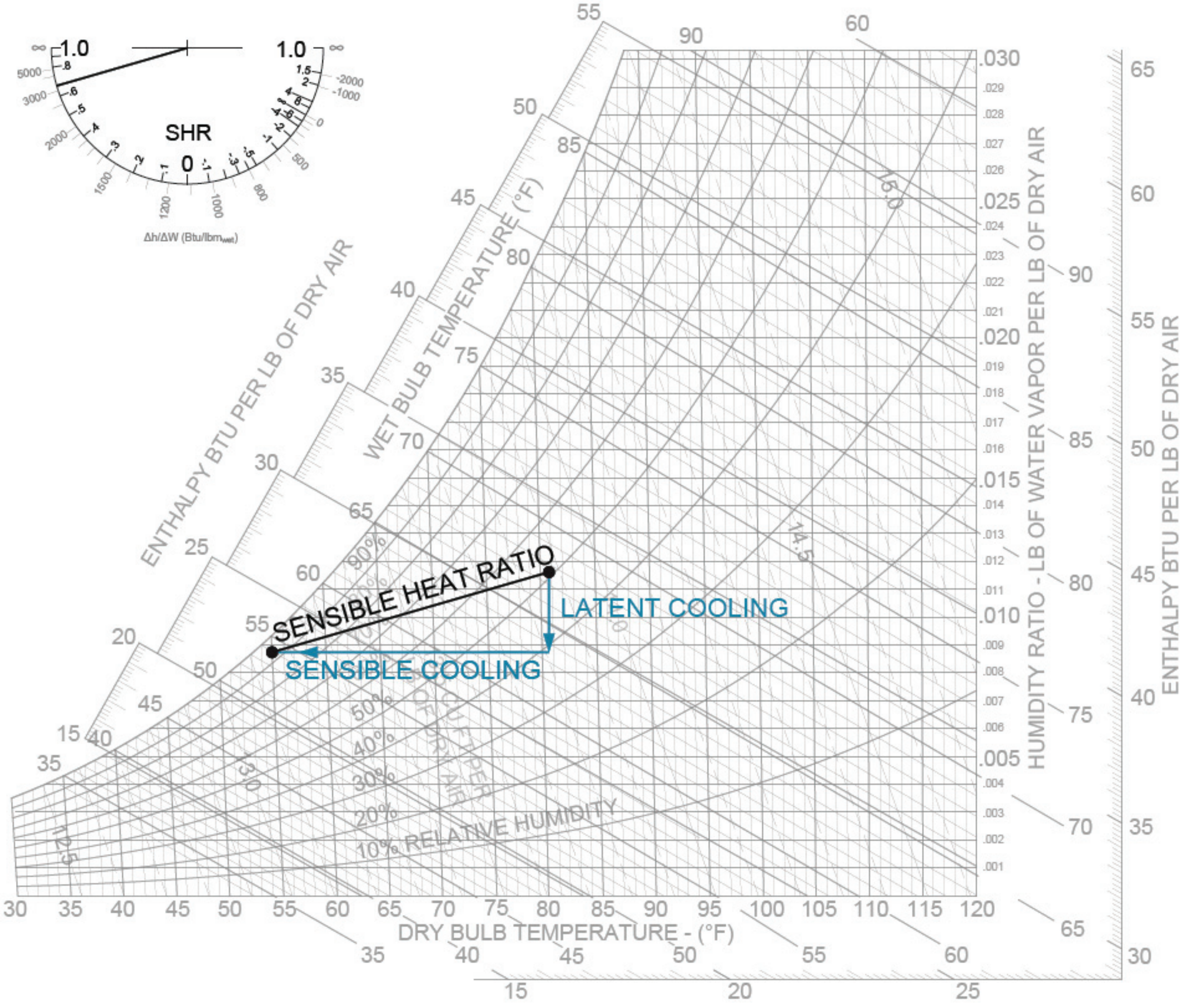


Figure 63: Sensible Heat Ratio (SHR)

In order to determine the sensible heat ratio, you must be able to determine the total amount of heat and the sensible heat through the use of a psychrometric chart. As a reminder, the sensible, latent, and total heats can be found with the following simplified equations for standard temperature and pressure conditions.

$$Q_{sensible}\left[\frac{Btu}{h}\right] = 1.08 * \Delta T * CFM$$

$$Q_{latent}\left[\frac{Btu}{h}\right] = 4{,}840 * \Delta W_{LB} * CFM \quad \text{- or -} \quad Q_{latent}\left[\frac{Btu}{h}\right] = 0.69 * \Delta W_{GR} * CFM$$

$$Q_{total}\left[\frac{Btu}{h}\right] = 4.5 * \Delta \mathrm{h} * \mathrm{CFM}$$

In air conditioning, sensible heat ratios range from 0.60 to 0.99, where 0.60 type buildings include spaces with high humid outside air loads and 0.99 type buildings include computer rooms/server rooms with high sensible loads and limited outside air and other latent loads.

Example: Calculate the sensible heat ratio for the following problem:

Problem: Discharge air at 55°F DB/53°F WB from a coil enters a space with both sensible heating and latent heating. The following space conditions of the air are 75°F DB/50% Relative Humidity.

Solution: In this problem, the starting condition is the supply air at 55°F DB/53°F WB. Then, sensible and latent heat is picked up from the load in the space, which raises the temperature to its final condition: 75°F DB/50% RH. These conditions are plotted on the psychrometric chart.

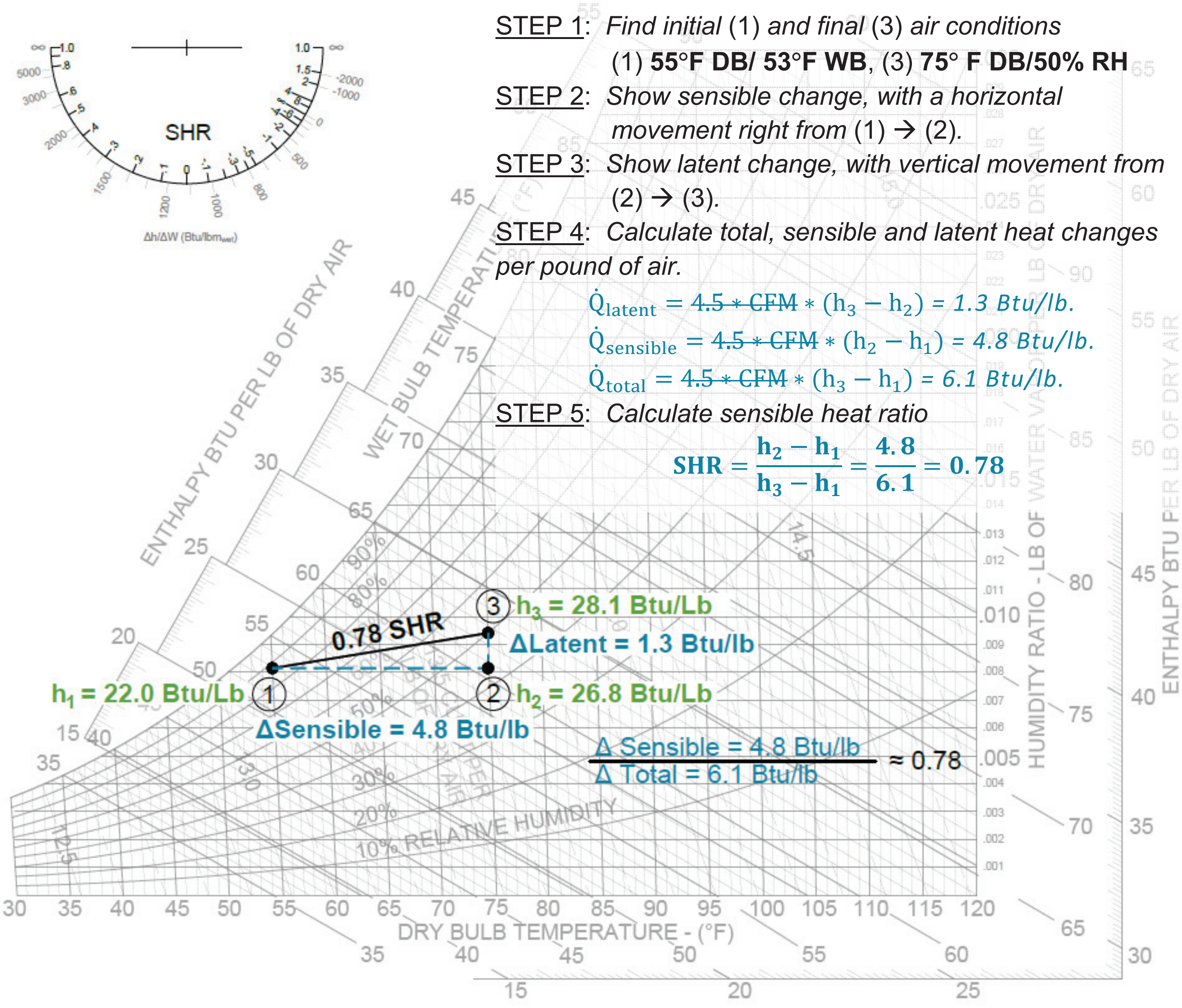
The sensible heat ratio is calculated as the sensible heat divided by the total heat. You can find the sensible and total heats individually with the following equation:

$$\mathrm{SHR} = \frac{Q_{sensible}}{\mathrm{Q_{total}}} = \frac{1.08 * \Delta T * CFM}{4.5 * \Delta \mathrm{h} * \mathrm{CFM}}$$

Or, you can realize that $Q = \dot{m} * \Delta h$ applies for all heat equations, whether sensible, latent, or total. The mass flow rate does not change, so simplify the equations using only enthalpies.

$$\mathrm{SHR} = \frac{Q_{sensible}}{\mathrm{Q_{total}}} = \frac{\cancel{\dot{m}} * \Delta h_{sensible}}{\cancel{\dot{m}} * \Delta h_{total}} = \frac{\Delta h_{sensible}}{\Delta h_{total}}$$

During the exam, you should use whichever approach is fastest.



STEP 1: *Find initial* (1) *and final* (3) *air conditions*

(1) **55°F DB/ 53°F WB**, (3) **75° F DB/50% RH**

STEP 2: *Show sensible change, with a horizontal movement right from* (1) → (2).

STEP 3: *Show latent change, with vertical movement from* (2) → (3).

STEP 4: *Calculate total, sensible and latent heat changes per pound of air.*

$$\dot{Q}_{latent} = \cancel{4.5 * CFM} * (h_3 - h_2) = 1.3\ Btu/lb.$$

$$\dot{Q}_{sensible} = \cancel{4.5 * CFM} * (h_2 - h_1) = 4.8\ Btu/lb.$$

$$\dot{Q}_{total} = \cancel{4.5 * CFM} * (h_3 - h_1) = 6.1\ Btu/lb.$$

STEP 5: *Calculate sensible heat ratio*

$$\mathbf{SHR = \frac{h_2 - h_1}{h_3 - h_1} = \frac{4.8}{6.1} = 0.78}$$

SHR
Δh/ΔW (Btu/lbm_w)
h_1 = 22.0 Btu/Lb
h_2 = 26.8 Btu/Lb
h_3 = 28.1 Btu/Lb
0.78 SHR
ΔLatent = 1.3 Btu/lb
ΔSensible = 4.8 Btu/lb
Δ Sensible = 4.8 Btu/lb / Δ Total = 6.1 Btu/lb ≈ 0.78
ENTHALPY BTU PER LB OF DRY AIR
WET BULB TEMPERATURE (°F)
HUMIDITY RATIO - LB OF WATER VAPOR PER LB OF DRY AIR
10% RELATIVE HUMIDITY
DRY BULB TEMPERATURE - (°F)
30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110 115 120

Figure 64: Sensible Heat Ratio Example

11.5 MIXING OF TWO AIR STREAMS

A common skill is determining the output conditions of the mixture of two air streams. The important concept to first understand is that the output conditions of the mixed airstream will be most similar to the air stream that has the most volumetric flow rate.

For example, if 2,000 CFM of 80°F DB/65°F WB is mixed with 100 CFM of 50°F DB/45°F WB, then the resulting mixed air conditions will be most similar to the 80°F DB/65°F WB air. This seems obvious but often times in test situations, the test taker may forget the obvious and simply rely on the calculations, which can be mistakenly entered into a calculator. This realization, will give the engineer an insight into the most probable answer of the four possible choices.

The second concept is that the dry bulb and humidity ratios change linearly. For example, if 2,000 CFM of 80°F DB is mixed with 2,000 CFM of 50°F DB, then the resulting temperature will be located equally in between 80°F DB and 50°F DB. The resulting temperature will be 65°F DB.

For example, if 3,000 CFM of 80°F DB is mixed with 1,000 CFM of 50°F DB, then the resulting temperature will be 72.5°F DB, corresponding to the location on the graph indicated by 75% of the total mixture is from 80°F DB air.

Refer to the following graph for more information.

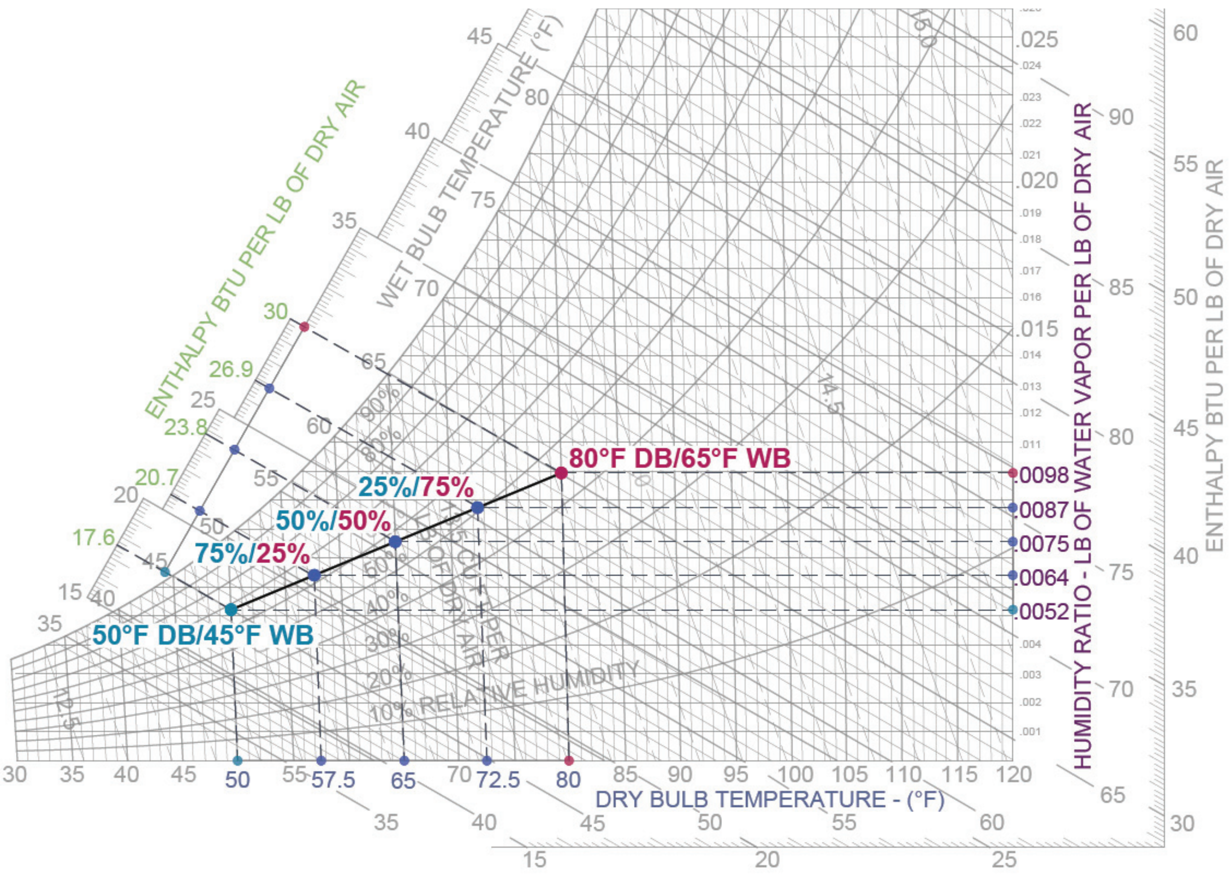


Figure 65: Mixing of two airstreams

Once an understanding of the graph is shown, then the following equations can be used.

$$T_{mix,DB} = T_{1,DB} * \%_1 + T_{2,DB} * \%_2$$

$$T_{mix,DB} = \frac{T_{1,DB} * CFM_1 + T_{2,DB} * CFM_2}{CFM_1 + CFM_2}$$

This linear relationship for dry bulb temperatures is also true for humidity ratios and enthalpies, as shown in the following equations.

Humidity Ratio Mixing Equation

$$\mathrm{W_{mix}} = \mathrm{W_1} * \%_1 + \mathrm{W_2} * \%_2$$

$$\mathrm{W_{mix}} = \frac{\mathrm{W_1} * \mathrm{CFM_1} + \mathrm{W_2} * \mathrm{CFM_2}}{\mathrm{CFM_1} + \mathrm{CFM_2}}$$

Enthalpy Mixing Equation

$$\mathrm{h_{mix}} = \mathrm{h_{1,DB}} * \%_1 + \mathrm{h_2} * \%_2$$

$$\mathrm{h_{mix}} = \frac{\mathrm{h_1} * \mathrm{CFM_1} + \mathrm{h_2} * \mathrm{CFM_2}}{\mathrm{CFM_1} + \mathrm{CFM_2}}$$

This is also known as the lever rule. Realize that this rule does not apply to wet bulb and relative humidity properties.

12.0 COMBUSTION AND COMBUSTION PRODUCTS

Most power plants are driven by a chemical reaction called combustion, which usually involves fuel sources that are compounds of hydrogen and carbon and an oxidizer (air). Combustion is the process by which the fuel is oxidized and as a result releases heat. The process also releases water and carbon dioxide.

The equation for combustion is as follows:

$$Fuel + Air \rightarrow Heat + CO_2 + H_2O$$

When you substitute the basic formula for a hydrocarbon fuel and air, then the resulting equation is as follows.

$$A * C_xH_y + B * (O_2 + 3.76N_2) \rightarrow Heat + C * CO_2 + D * H_2O$$

$$where\ A, B, C\ \&\ D\ are\ numerical\ values\ that$$

$$describe\ the\ ratio\ of\ each\ compound\ in\ the\ equation$$

12.1 FUEL

Fuel is simply any material that can be burned to release heat. But the most common fuels in the Thermal & Fluids application are the hydrocarbon fuels. This category includes fuels like coal, gasoline and natural gas. These fuels power the power plants and cars in the United States of America. The fuels are a combination of carbon and hydrogen.

$$Hydrocarbon\ fuels = C_xH_y$$

$$where\ x\ and\ y = a\ quantiy\ of\ each\ element$$

For example, these are the formulas for gasoline, diesel and methane.

$$Gasoline = C_8H_8\ ;\ \ Diesel = C_{12}H_{26}\ ; Methane = C_1H_4$$

12.2 AIR

Air is about 21% oxygen (by volume) and 78% nitrogen and about 1% other gases. The FE exam may ask you to calculate the amount of air used in a combustion process and air should take into account both oxygen and nitrogen.

$$Air = \ 21\%O_2 + 79\%N_2$$

$$Air = (O_2 + 3.76N_2)$$

For example, when air is used in the propane combustion equation, the following equation is used.

12.3 STOICHIOMETRY

Stoichiometry of chemical reactions means that species react in exact proportions. The proportions that react are governed by the specific fuel's combustion equation. When completing these types of problems, remember that the proportions that are in the equations are based on moles and not weight. *If a problem gives you weights, you need to first convert the weight to moles before you can use the stoichiometry equations.*

In order to convert weight to moles you need the molecular weight, so make sure you remember the periodic table in the *NCEES FE Reference Handbook*. Here are some of the common molecular weights used in the combustion equations.

Element	Molecular Weight (g/mol)
Carbon [C]	12
Nitrogen [N]	14
Oxygen [O]	16
Sulfur [S]	16
Hydrogen [H]	1

There are two basic principles to understand with the stoichiometric questions you encountered on the exam, first there is a conservation of elements. The quantity of elements on the left hand side must equal the quantity of elements on the right side for each type of element.

Propane:

$$C_3H_8 + 5(O_2 + 3.76N_2) \rightarrow 3CO_2 + 4H_2O + 18.8N_2$$

Element	Left Side	Right Side	Equal?
C	3	3	Yes
H	8	4*2	Yes
O	5*2	6+4	Yes
N	5*3.76	18.8	Yes

The second principle is that mass must be conserved. The amount of fuel and air entering the combustion chamber must equal the mass of the products released from the combustion chamber. This will be discussed in the air to fuel ratio section.

12.4 AIR TO FUEL RATIO

Mass will always be conserved in a combustion reaction. For example, if there is 1 lb of fuel and 20 lbs of air added to a combustion chamber, then the total weight of the products will be 21 lbs.

The air to fuel ratio is important because it describes the minimum amount of air which will allow the complete combustion of fuel. There are two important ratios used in the exam, the theoretical air to fuel ratio and the actual air to fuel ratio.

The theoretical air to fuel ratio is solely dependent on the stoichiometry. The actual air to fuel ratio is what is done in practice.

For example, in the propane equation the theoretical air to fuel ratio will be as follows.

$$Balanced\ propane\ combustion\ equation \rightarrow$$

$$C_3H_8 + 5(O_2 + 3.76N_2) \rightarrow 3CO_2 + 4H_2O + 18.8N_2$$

$$Fuel = C_3H_8 \rightarrow Mass = 3 * \left(12\frac{g}{mol}\right) + 8 * \left(1\frac{g}{mol}\right) = 44\frac{g}{mol}$$

$$Air = 5(O_2 + 3.76N_2) \rightarrow Mass = 5 * 2 * \left(16\frac{g}{mol}\right) + 5 * 3.76 * 2 * \left(14\frac{g}{mol}\right) = 686.4\frac{g}{mol}$$

$$Theoretical\ air\ to\ fuel\ ratio = \frac{686.4}{44} = 15.6\ to\ 1$$

The actual air to fuel ratio is something that is determined in practice. An exam question may ask you to derive the actual air to fuel ratio based on the mass of the products or the exam may give you air to fuel ratio and ask you to derive the products.

12.5 EXCESS AIR

There is a minimum amount of air that supplies sufficient O2 for the complete combustion of the fuel, which is called the theoretical air. However, theoretical air is difficult to achieve in practice and as such excess air is always supplied to the combustion process.

Excess air is equal

$$\%\, Excess\ Air = \frac{Actual\ Air}{Theoretical\ Air} - 100\%$$

13.0 PRACTICE PROBLEMS

13.1 PROBLEM 1 – EVAPORATOR

Background: A chiller uses R-134a refrigerant, with a flow rate of 50 lb/min (0.378 kg/s) and has a suction pressure of 40 PSIA (275.79 kPa) and a discharge pressure of 200 PSIA (1378.95 kPa), with no sub-cooling and no super heating.

Problem: What is the net refrigeration effect of the evaporator?

(a) 10 Tons

(b) 11.5 Tons

(c) 13.5 Tons

(d) 16 Tons

13.2 PROBLEM 2 – EVAPORATOR

Background: A chiller uses R-134a refrigerant, with a flow rate of 50 lb/min (0.378 kg/s). The chiller is used to provide chilled water leaving the evaporator at a temperature of 45 °F (7.22 °C).

Problem: Which suction pressure should the evaporator operate at to provide a 10 °F differential between the chilled water and refrigeration liquid temperature? Refrigeration temperature shall be 35 °F (1.67 °C)

(a) 40 psia (275.79 kPa)

(b) 45 psia (310.264 kPa)

(c) 50 psia (344.738 kPa)

(d) 55 psia (379.212 kPa)

13.3 PROBLEM 3 – EVAPORATOR

Background: A chiller with a refrigerant flow rate of 50 lb/min (0.378 kg/s) of R-134a has a suction pressure of 40 PSIA and a discharge pressure of 200 PSIA. The evaporator provides 20 °F of super-heating and the condenser provides no sub-cooling.

Problem: What is the net refrigeration effect?

(a) 11 Tons

(b) 12.25 Tons

(c) 13 Tons

(d) 14.25 Tons

13.4 PROBLEM 4 – CONDENSER

Background: A chiller operates with a suction pressure of 35 PSIA and a discharge pressure of 225 PSIA. The refrigerant flow rate is 100 lb/min of R-134a. The compressor provides 200,000 Btu/h of work. The COP of the chiller is 4.0. What is the net condenser effect?

Problem: What is the net condenser effect?

(a) 83.3 Tons

(b) 93.6 Tons

(c) 120.3 Tons

(d) 139.3 Tons

13.5 PROBLEM 5 – EXPANSION DEVICE

Background: A temperature expansion valve, senses that the temperature at the end of the evaporator has increased from 10 °F super heat [normal operation] to 15 °F super heat [current].

Problem: What best describes the situation and the temperature expansion valve's next action?

(a) Cooling load is decreasing, TXV should decrease refrigerant flow to decrease super heat.

(b) Cooling load is decreasing, TXV should increase refrigerant flow to decrease super heat.

(c) Cooling load is increasing, TXV should decrease refrigerant flow to decrease super heat.

(d) Cooling load is increasing, TXV should increase refrigerant flow to decrease super heat.

13.6 PROBLEM 6 – EXPANSION DEVICE

Problem: Which of the following is NOT a primary function of a thermal expansion valve in a refrigeration cycle?

(a) Control the amount of refrigerant in the refrigeration cycle.

(b) Ensure the correct amount of sub-cooling is provided by the condenser.

(c) Maintain the correct level of superheat from the evaporator.

(d) Ensure no liquid is passed from the evaporator to the compressor.

13.7 PROBLEM 7 – COP

Background: A chiller with an R-134a. The chiller operates with a suction pressure of 40 PSIA and a discharge pressure of 200 PSIA. The chiller is rated at 100 tons of useful cooling. The compressor also provides 84 KW of useful work to the system.

Problem: What is the COP?

(a) 3.5

(b) 4

(c) 4.2

(d) 4.6

13.8 PROBLEM 8 – COP

Background: A chiller with an R-134a refrigerant flow rate of 75 lb/min. The chiller operates with a suction pressure of 40 PSIA and a discharge pressure of 200 PSIA. The evaporator produces 15 °F of superheat and the condenser produces 10 °F of sub-cooling.

Problem: What is the COP?

(a) 4

(b) 4.5

(c) 5

(d) 5.5

13.9 Problem 9 - Steam Production

Water (50 °C, 22 kg/hr) is converted to saturated steam (no super-heat) at a pressure of 101.35 kPa. What is the total production of the boiler (kW)?

(a) 10.3 KW

(b) 13.2 KW

(c) 15.1 KW

(d) 22.0 KW

13.10 Problem 10 - Steam Air Coils

0.00623 kg/s of steam (saturated vapor) enters a heating coil at a pressure of 101.325 KPa and leaves as a saturated liquid. 700 CFM of air enters the coil at 60 °F. Assume 100% efficient heat transfer. What is the resulting exiting temperature of the air?

(a) 105 °F

(b) 110 °F

(c) 115 °F

(d) 125 °F

13.11 PROBLEM 11 - STEAM - HOT WATER COILS

A hot water coil has an incoming water temperature of 70 °F and an outgoing temperature of 140 °F. Hot water is flowing through the coil at a rate of 40 GPM. What steam flow rate is required to properly heat the water [kg/s]? Assume saturated steam at a pressure of 101.325 KPa, with no super heat and sub-cooling and 100% effective heat exchange.

(a) 0.111 kg/s

(b) 0.149 kg/s

(c) 0.174 kg/s

(d) 0.182 kg/s

13.12 PROBLEM 12 – BRAYTON CYCLE

Background: Air is compressed at a gas turbine from 14.7 psi and 77 °F to 73.5 psi. The resulting temperature is most nearly? Assume isentropic compression.

(a) 390 °F

(b) 420 °F

(c) 500 °F

(d) 560 °F

13.13 PROBLEM 13 – BRAYTON CYCLE

Background: Air is compressed at a gas turbine from 14.7 psi and 77 °F to 73.5 psi. If the resulting temperature is actually 440 °F, then what is the efficiency of the compressor?

(a) 79%

(b) 83%

(c) 87%

(d) 89%

13.14 PROBLEM 14 – COMBUSTION

Given the combustion equation below.

$$C_4H_{10} + 6.5(O_2 + 3.76N_2) \rightarrow 4CO_2 + 5H_2O + 24.44N_2$$

What is the air to fuel ratio? Assume the equation is balanced.

(a) 10.1

(b) 12.9

(c) 14.8

(d) 15.4

13.15 PROBLEM 15 – RANKINE CYCLE

Background: A steam power plant boiler operates at a pressure of 1 MPa. Steam exits the turbine at 14.7 psi (101.4 kPa). If the boiler heats the steam to 200 °C, then what is the enthalpy of the steam leaving the turbine? Assume an isentropic turbine.

The answer is most nearly,

(a) 417 KJ/kg

(b) 1975 KJ/kg

(c) 2427 KJ/kg

(d) 2791 KJ/kg

13.16 PROBLEM 16 – PSYCHROMETRIC CHART

Background: The return air temperature to an air conditioning coil is 80°F DB and 65% Relative Humidity. Assume pressure = 1 atm.

Problem: What is the Enthalpy [$\frac{Btu/h}{lbm\ of\ dry\ air}$]?

(a) 45

(b) 40

(c) 35

(d) 30

Problem: What is the Humidity Ratio [$\frac{lbm\ of\ water\ vapor}{lbm\ of\ dry\ air}$]?

(a) .014

(b) .012

(c) .010

(d) .008

Problem: What is the Wet Bulb (F°)?

(a) 80° F

(b) 75° F

(c) 71° F

(d) 65° F

Problem: What is the Specific Volume [$\frac{ft^3}{lbm\ of\ dry\ air}$]?

(a) 11

(b) 12

(c) 13

(d) 14

13.17 Problem 17 – Air Mixtures

Background: Return air is often mixed with outside air prior to entering an air handler, where it is then cooled/heated to the appropriate air delivery temperature.

Problem: Determine the mixed air condition of the following.

1,000 CFM return air at 78°F DB/54% Relative Humidity is mixed with 1,000 CFM outside air at 88°F DB/70% Relative Humidity, what is the resulting state of air?

(a) 80°F DB/54% Relative Humidity

(b) 83°F DB/70% Relative Humidity

(c) 83°F DB/64% Relative Humidity

(d) 86°F DB/54% Relative Humidity

14.0 SOLUTIONS

14.1 SOLUTION 1 - EVAPORATOR

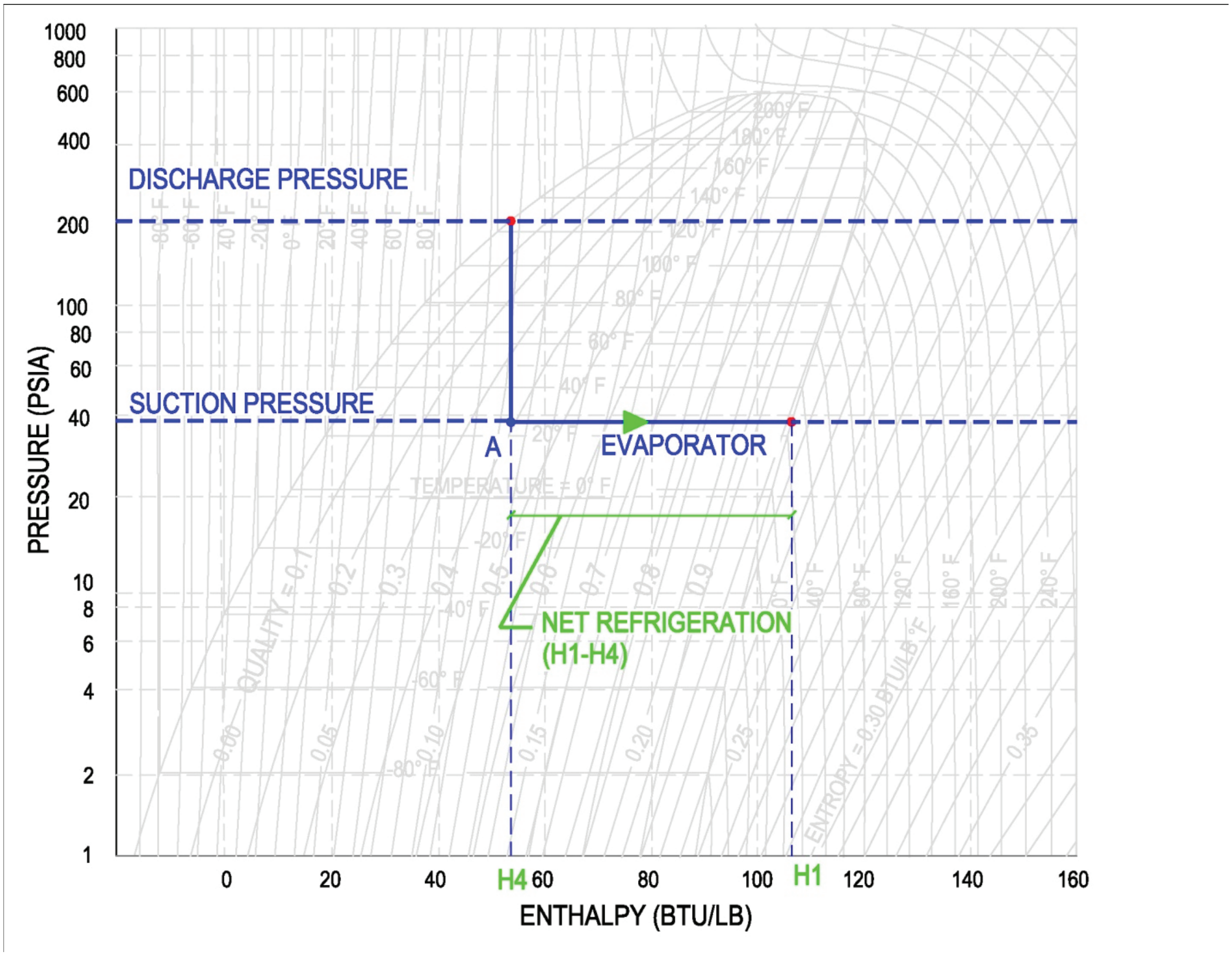
$$Q_{net\ refrigeration\ effect}\ [Btuh] = (H_1 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_1 = leaving\ evaporator\ enthalpy\left[\frac{Btu}{lb}\right] = 107\ Btu/lb$$

$$H_4 = entering\ evaporator\ enthalpy\left[\frac{Btu}{lb}\right] = 54\ Btu/lb$$

$$Q_{net\ refrigeration\ effect}\ [Btu] = (107 - 54)\left[\frac{Btu}{lb}\right] * 50\left[\frac{lb}{min}\right] * 60\left[\frac{min}{hr}\right]$$

$$Q_{net\ refrigeration\ effect} = 159{,}000\ Btuh * \frac{1\ ton}{12{,}000\ Btuh} = 13.25\ Tons$$



14.2 SOLUTION 2 – EVAPORATOR

In order to provide a 10 degree differential between the chilled water temperature at 45 °F, the refrigerant liquid in the evaporator must be 35 F. According to the R-134A P-H diagram, 35 °F corresponds to a pressure of ~45 PSIA.

(a) 40 PSIA - corresponding evaporator temperature = 29 °F.

(b) 45 PSIA - corresponding evaporator temperature = 34.9 °F.

(c) 50 PSIA - corresponding evaporator temperature = 40.3 °F.

(d) 55 PSIA - corresponding evaporator temperature = 45.2 °F.

14.3 SOLUTION 3 – EVAPORATOR

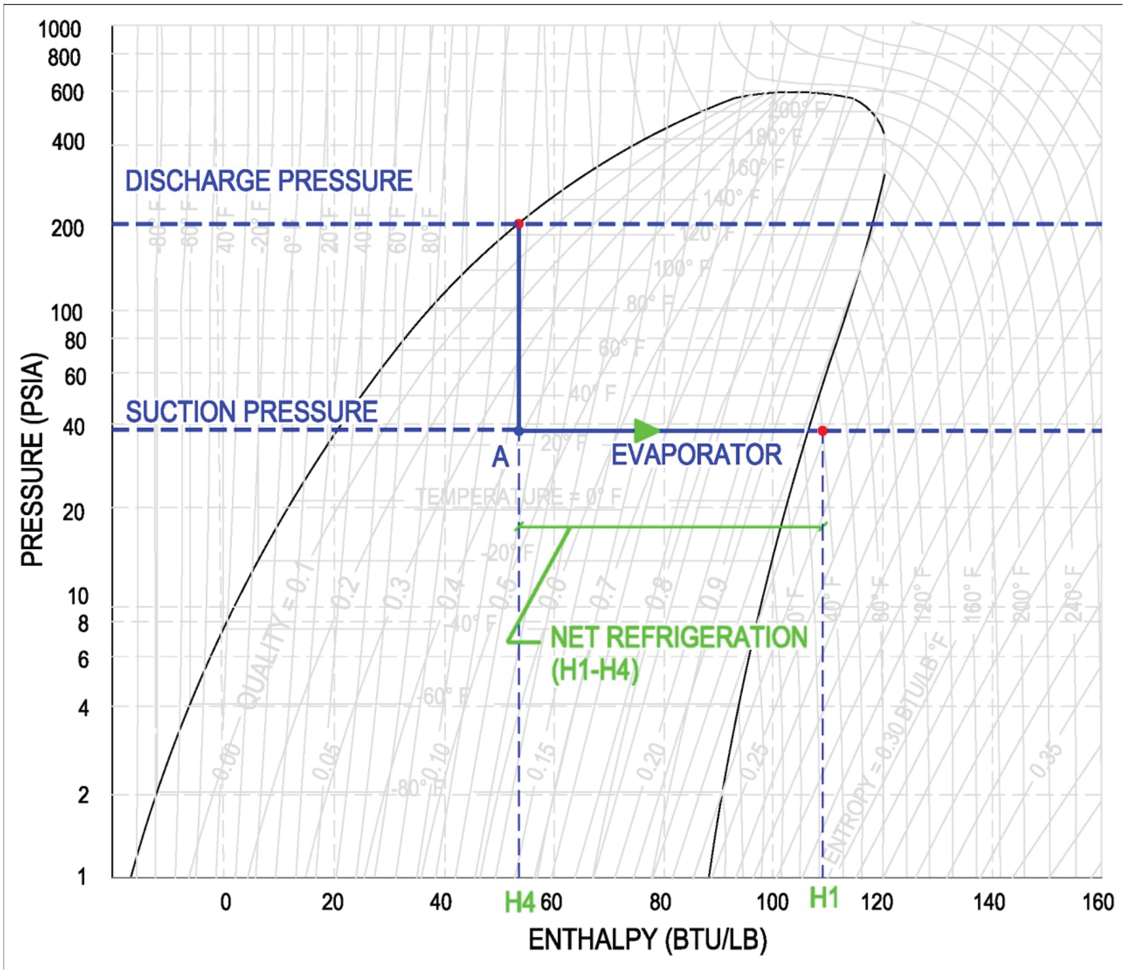
$$Q_{net\ refrigeration\ effect}\ [Btuh] = (H_1 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$H_1 = leaving\ evaporator\ enthalpy\left[\frac{Btu}{lb}\right] = 111\ Btu/lb$$

$$H_4 = entering\ evaporator\ enthalpy\left[\frac{Btu}{lb}\right] = 54\ Btu/lb$$

$$Q_{net\ refrigeration\ effect}\ [Btu] = (111 - 54)\left[\frac{Btu}{lb}\right] * 50\left[\frac{lb}{min}\right] * 60\left[\frac{min}{hr}\right]$$

$$Q_{net\ refrigeration\ effect} = 171{,}000\ Btuh * \frac{1\ ton}{12{,}000\ Btuh} = 14.25\ Tons$$



14.4 Solution 4 – Condenser

Step 1: Determine the net refrigeration effect.

$$COP \;=\; 4.0 = \frac{Q_{net\ refrigeration\ effect}\ [Btuh]}{200{,}000\ [Btuh]}$$

$$Q_{net\ refrigeration\ effect} \;=\; 800{,}000\ Btuh$$

Step 2: Determine the net condenser effect.

$$Q_{net\ condenser\ effect} \;=\; Q_{net\ refrigeration\ effect} + W_{compressor}$$

$$Q_{net\ condenser\ effect} = 800{,}000 + 200{,}000 = 1{,}000{,}000\ Btu/h\ or\ 83.3\ Tons$$

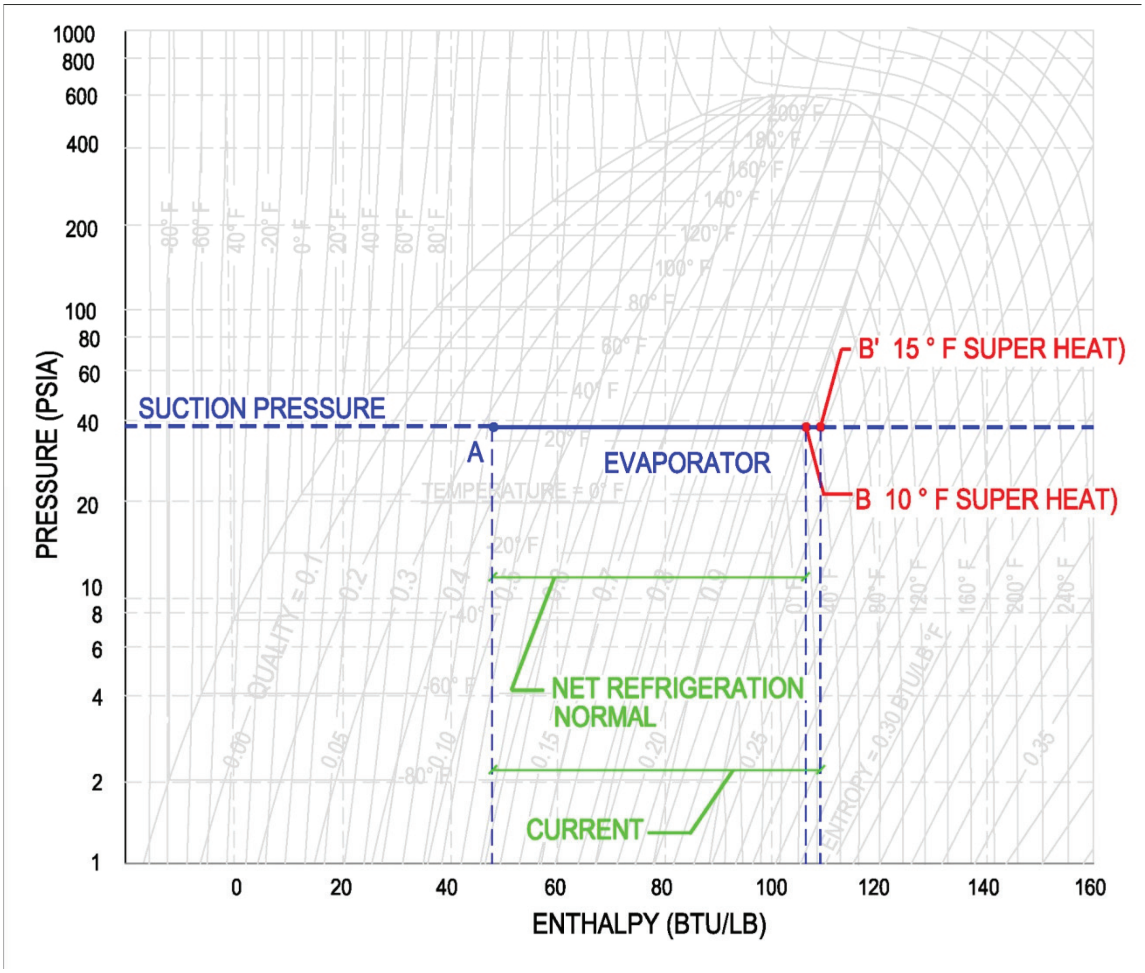
14.5 SOLUTION 5 – EXPANSION DEVICE

If the amount of super heat is increasing from normal conditions, then the net refrigeration effect is increasing, as point B is moving further to the right. Cooling load is increasing.

If the cooling load is increasing, then the TXV should provide more refrigerant to increase net refrigeration effect.

The correct answer is (D).

$$\overset{\uparrow}{Q}_{net\ refrigeration\ effect}\ [Btuh] = (H_1 - \overset{\downarrow}{H_4})\left[\frac{Btu}{lb}\right] * (Refrig\ \overset{\uparrow}{Flow}\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$



14.6 SOLUTION 6 – EXPANSION DEVICE

The TXV is a metering device that regulates the amount of refrigerant to the evaporator

The amount of sub-cooling provided by the condenser is determined by the condensers

The TXV has a sensor that monitors the superheat and meters the refrigerant to maintain a specific level of superheat thus ensuring a superheated vapor, with no liquid.

The correct answer is most nearly, **(b) Ensure the correct amount of sub-cooling is provided by the condenser.**

14.7 SOLUTION 7 – COP

$$COP = \frac{Q_{net\ refrigeration\ effect}}{W_{compressor}}$$

This question involves converting the useful cooling of the chiller and the work done by the compressor to the same units.

$$W_{compressor} = 84\ kw * \frac{1\ Ton}{3.517\ kW} = 23.9\ tons$$

$$Q_{net\ refrigeration\ effect} = 100\ tons$$

$$COP = \frac{100\ tons}{23.9\ tons} = 4.18$$

Correct Answer: C

14.8 SOLUTION 8 – COP

First find point H1, which is the enthalpy entering the compressor, by following the suction pressure line to a superheat temperature of 15 F.

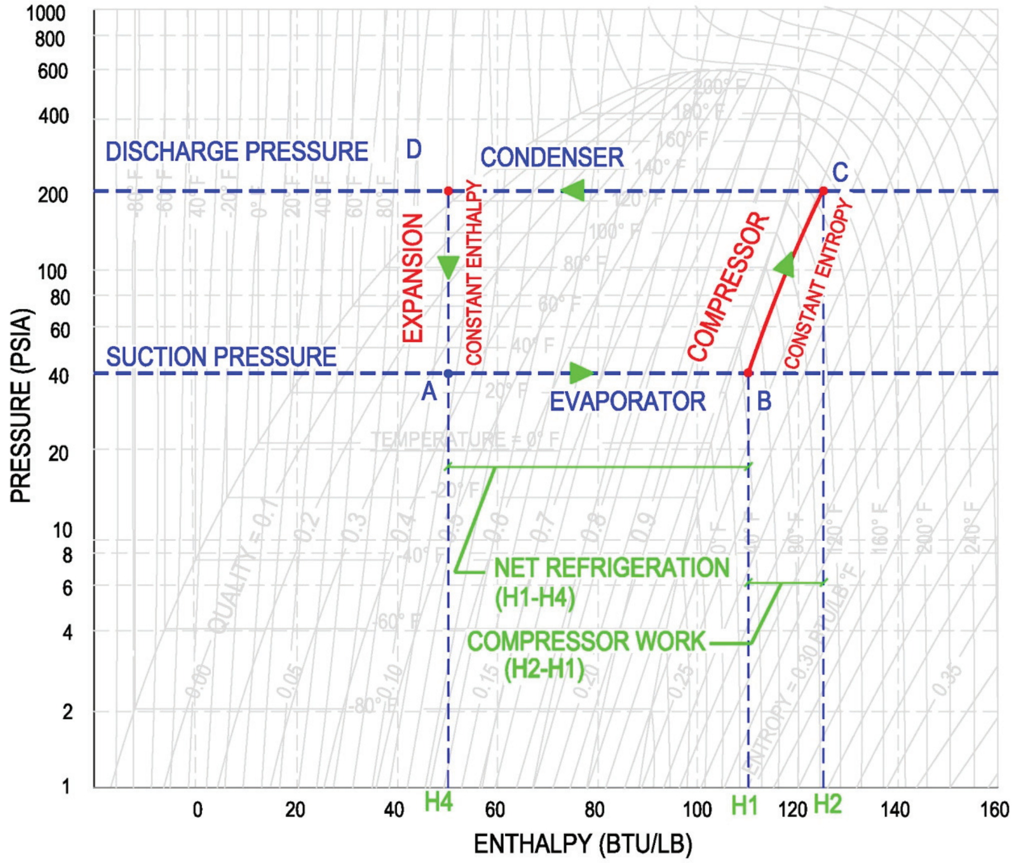
$$H_1 = 110 \frac{Btu}{lb}$$

Second find point H2, which is the enthalpy leaving the compressor, by following the constant entropy line from point entering the compressor to the intersection of the discharge pressure.

$$H_2 = 125 \frac{Btu}{lb}$$

Third find point H4, which is the enthalpy entering the evaporator, by following the constant enthalpy line by following the discharge pressure line to a sub-cooled temperature of 10 F.

$$H_4 = 50.5 \frac{Btu}{lb}$$



$$W_{compressor}\ [Btuh] = (H_2 - H_1)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$W_{compressor}\ [Btuh] = (125 - 110) = 15$$

$$Q_{net\ refrigeration\ effect}\ [Btuh] = (H_1 - H_4)\left[\frac{Btu}{lb}\right] * (Refrig\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right]$$

$$Q_{net\ refrigeration\ effect}\ [Btuh] = (110 - 50.5) = 59.5$$

$$COP = \frac{59.5}{15} = 3.97$$

The correct answer is most nearly, **(a) 4.**

14.9 SOLUTION 9 - STEAM PRODUCTION

Water (50 °C, 22 kg/hr) is converted to saturated steam (no super-heat) at a pressure of 101.35 kPa. What is the total production of the boiler (kW)?

The enthalpy of the incoming feed-water at 50 °C and a pressure of 101.35 kPa is a sub-cooled liquid. Use your temperature to find the enthalpy.

$$h_{feedwater} = 209.33 \frac{kJ}{kg}$$

The enthalpy of the outgoing steam at 101.35 kPa, saturated is shown below. Use the saturated vapor value as a function of pressure.

$$h_{steam} = 2{,}676.1 \frac{kJ}{kg}$$

The total production of the boiler (KW) is found by multiplying the difference between the incoming and outgoing boiler enthalpies by the mass flow rate of the steam.

$$Q = \dot{m} * (h_{steam} - h_{feedwater})$$

$$Q = 22 \frac{kg}{hr} * (2{,}676.1 - 209.3)$$

$$Q = 54{,}270 \frac{kJ}{hr}$$

$$Q = 54{,}270 \frac{kJ}{hr} * \frac{1hr}{3600s} = 15.08\, KW$$

The correct answer is most nearly, **(c) 15.1 KW.**

14.10 SOLUTION 10 - STEAM AIR COILS

0.00623 kg/s of steam (saturated vapor) enters a heating coil at a pressure of 101 KPa and leaves as a saturated liquid. 700 cubic feet per minute of air enters the coil at 60 F. Assume 100% efficient heat transfer. What is the resulting exiting temperature of the air?

Steam at 101 KPa has an enthalpy of vaporization of 2,257 kJ/kg.

$$Q = \dot{m}_{steam} * h_{fg} = 0.075\frac{lbm}{ft^3} * 700\frac{ft^3}{min} * 60\frac{min}{hr} * 0.24\frac{Btu}{lbm - F} * \Delta T$$

$$0.00623\frac{kg}{s} * 2{,}257\frac{kJ}{kg} = 756\frac{Btu}{lbm - F} * \Delta T$$

$$14.06\frac{kJ}{s} * \frac{Btu}{1.055\ kJ} * \frac{3600\ s}{hr} = 756 * \Delta T$$

$$\Delta T = 63.5\ °\mathrm{F}$$

The final temperature of the air is 124 °F.

$$60°\mathrm{F} + 64\ °\mathrm{F} = 124\ °\mathrm{F}$$

14.11 SOLUTION 11 - STEAM HOT WATER COILS

A hot water coil has an incoming water temperature of 70 F and an outgoing temperature of 140 F. Hot water is flowing through the coil at a rate of 40 GPM. What steam flow rate is required to properly heat the water [lb/hr]? Assume saturated steam at a pressure of 101 KPa, with no super heat and sub-cooling and 100% effective heat exchange.

Steam at 101 KPa has an enthalpy of evaporation of 2,257 kJ/kg.

An energy balance is conducted on the heat loss through condensing the steam and the heat gained by the water.

$$Heat\ loss\ through\ steam\ condensation = Heat\ gained\ by\ water$$

$$\dot{m}_{steam} * h_{fg} = \dot{m} * c_p * \Delta T$$

$$\dot{m}_{steam} * 2{,}257\frac{\mathrm{kJ}}{\mathrm{kg}} = \frac{8.34\ lbm}{gal} * 40\frac{gal}{min} * 60\frac{min}{hr} * \frac{1.0\ Btu}{lbm - F} * 70\ F$$

$$\dot{m}_{steam} * 2{,}257\frac{\mathrm{kJ}}{\mathrm{kg}} = 1{,}401{,}120\frac{Btu}{hr} * \left(\frac{1\ hr}{3600\ s}\right) * \left(\frac{1.055\ kJ}{Btu}\right)$$

$$\dot{m}_{steam} = 0.182\frac{kg}{s}$$

14.12 SOLUTION 12 – BRAYTON CYCLE

Background: Air is compressed at a gas turbine from 14.7 psi and 77 degrees F to 73.5 psi. The resulting temperature is most nearly? Assume isentropic compression.

The compression process is isentropic, meaning that there is no change in entropy.

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}$$

$$T_1 = 77\ F + 460 = 537\ °R$$

$$P_1 = 14.7\ psi; P_2 = 73.5\ psi; k = 1.4\ for\ air$$

$$\frac{T_2}{537} = \left(\frac{73.5}{14.7}\right)^{\frac{1.4-1}{1.4}}$$

$$T_2 = 850\ °R$$

$$T_2 = 850\ °R - 460 = 390\ °\mathrm{F}$$

The correct answer is most nearly, **(a) 390 °F.**

14.13 SOLUTION 13 – BRAYTON CYCLE

Background: Air is compressed at a gas turbine from 14.7 psi and 77 °F to 73.5 psi. If the resulting temperature is actually 440 °F, then what is the efficiency of the compressor?

From the previous question, the ideal resulting temperature is 390 F.

$$n_c = \frac{h_1 - h_{2,isentropic}}{h_1 - h_{2,actual}} = \frac{ideal}{actual} = \frac{\text{no compressor heat gain}}{\text{with compressor heat gain}}$$

$$\text{where } n_c = \text{adiabatic efficiency of the compressor}$$

$$\text{but we don't have } h\text{, so we can use a different efficiency equation}$$

$$n_c = \frac{ideal\ Q}{actual\ Q} = \frac{\cancel{\dot{m}c_p}(T_{2,ideal\ exit} - T_{1,ideal,enter})}{\cancel{\dot{m}c_p}(T_{2,actual\ exit} - T_{1,actual,enter})} =$$

$$n_c = \frac{\cancel{\dot{m}c_p}(850\ °R - 537\ °R)}{\cancel{\dot{m}c_p}(900\ °R - 537\ °R)} = 86.2\%$$

The correct answer is most nearly, **(c) 87%.**

14.14 SOLUTION 14 – COMBUSTION

$$C_4H_{10} + 6.5(O_2 + 3.76N_2) \rightarrow 4CO_2 + 5H_2O + 24.44N_2$$

What is the air to fuel ratio? Assume the equation is balanced.

$$First\ find\ the\ mass\ of\ air, 6.5(O_2 + 3.76N_2)$$

$$6.5 * (16 * 2 + 3.76 * 14 * 2) = 892$$

$$Second\ find\ the\ mass\ of\ fuel, C_4H_{10}$$

$$4 * 12 + 10 * 1 = 58$$

$$Air\ to\ fuel\ ratio = \frac{m_{air}}{m_{fuel}} = \frac{892}{58} = 15.4$$

The correct answer is most nearly, **(d) 15.4.**

14.15 SOLUTION 15 – RANKINE CYCLE

Background: A steam power plant boiler operates at a pressure of 1 MPa. Steam exits the turbine at 14.7 psi (101.4 kPa). If the boiler heats the steam to 200 °C, then what is the enthalpy of the steam leaving the turbine? Assume an isentropic turbine.

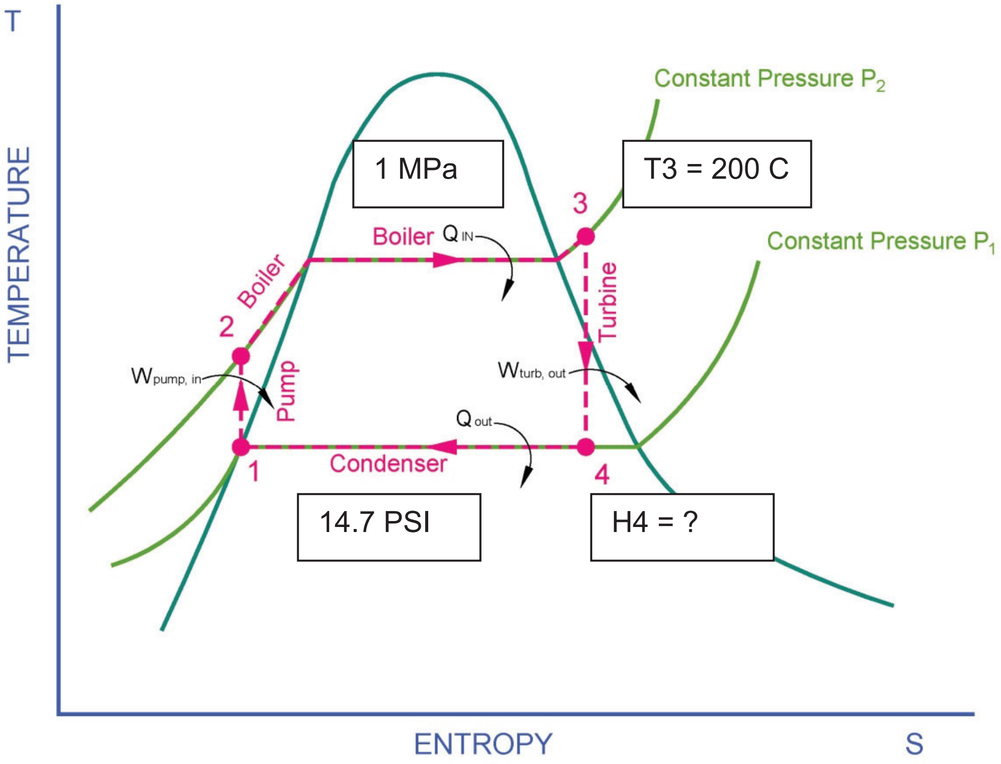
The answer is most nearly,

(a) 417 KJ/kg

(b) 1975 KJ/kg

(c) 2427 KJ/kg

(d) 2791 KJ/kg


T
TEMPERATURE
Constant Pressure P_2
1 MPa
T3 = 200 C
3
Boiler Q_{IN}
Constant Pressure P_1
Boiler
2
Turbine
$W_{pump, in}$
Pump
$W_{turb, out}$
Q_{out}
1
Condenser
4
14.7 PSI
H4 = ?
ENTROPY
S

The first step is to find the entropy of h_3 with the steam tables. If we look at the standard steam tables

In your NCEES FE Reference Handbook, we find that at a pressure of 1 MPa, saturated steam is at a temperature of 180 °C.

Therefore we must use the superheated steam tables, we find that

$$h_3 = 2827.9\ kj/kg; s_3 = 6.6940$$

$$Since\ the\ turbine\ process\ is\ isentropic, s_4 = s_3 = 6.6940$$

$$We\ also\ know\ that\ the\ pressure\ at\ step\ 4\ is\ 101\ kPa\ and\ according\ to\ the\ steam\ tables$$

$$s_f = 1.3069\ and\ s_g = 7.3549, but\ s_4 = 6.6940; so\ s_4 is\ governed\ by\ the\ following\ equation$$

$$s_4 = s_f + x * s_{fg}; where\ s_{fg} = entropy\ of\ evaporation\ at\ 6.0480\ amd\ x =\ steam\ quality$$

$$s_4 = 6.6940 = 1.3069 + x * 6.0480$$

$$Solving\ for\ x\ ;\ x = 0.89$$

$$now\ that\ we\ have\ x, we\ can\ use\ a\ similar\ equation\ to\ find\ h_4$$

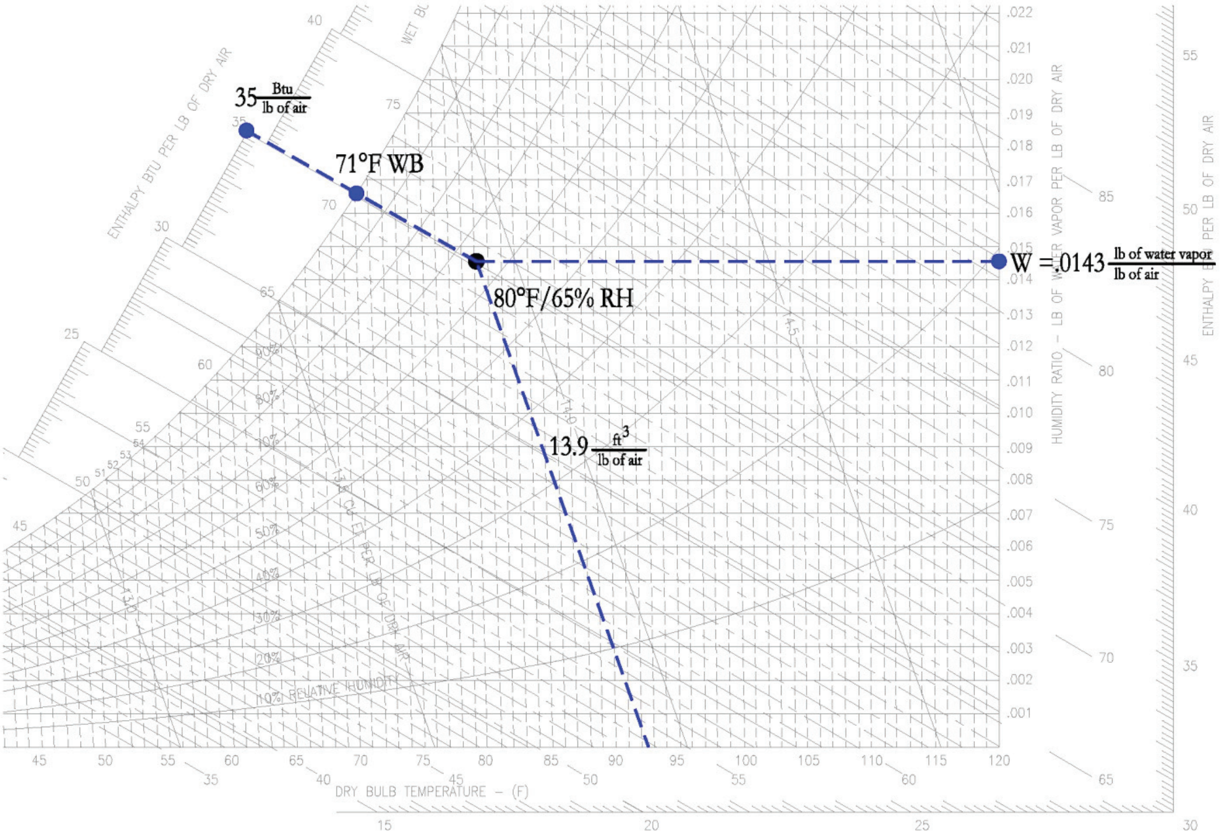
$$h_4 = h_f + x * h_{fg};$$

$$h_4 = 419 + .89 * 2257$$

$$h_4 = 2{,}427\ kJ/kg$$

The correct answer is most nearly, **(c) 2,382 kJ/kg.**

14.16 SOLUTION 16 – PSYCHROMETRIC CHART



Problem: What is the Enthalpy $[\frac{Btuh}{lbm\ of\ dry\ air}]$?

(c) 35

Problem: What is the Humidity Ratio $[\frac{lbm\ of\ water\ vapor}{lbm\ of\ dry\ air}]$?

(a) .014

Problem: What is the Wet Bulb (°F)?

(c) 71 °F

Problem: What is the Specific Volume $[\frac{ft^3}{lbm\ of\ dry\ air}]$?

(d) 14

14.17 SOLUTION 17 – AIR MIXTURES

Background: Return air is often mixed with outside air prior to entering an air handler, where it is then cooled/heated to the appropriate air delivery temperature.

1,000 CFM of 88°F DB/70% RH; 1,000 CFM of 78°F DB/54% RH

$$\mathrm{T_{mix,DB}} = \frac{88_{1,DB} * 1{,}000_1 + 78_{2,DB} * 1{,}000_2}{1{,}000_1 + 1{,}000_2}$$

$$\mathrm{T_{mix,DB}} = \frac{166{,}000}{2{,}000} = 83\ ^\circ\mathrm{F\ DB}$$

Next, plot the conditions on the psychrometric chart in order to determine the relative humidity of the mixed condition. Plot the two entering conditions and connect the two points with a line. The mixed condition is found at the intersection of the mixed dry bulb temperature and the line connecting the two entering conditions.

The correct answer is most nearly, **(c) 83°F DB/64% Relative Humidity**

12 – Heat Transfer

MECHANICAL FE TEXTBOOK

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Section 12.0 – Heat Transfer

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1.0 INTRODUCTION

Heat Transfer accounts for approximately 7 to 11 questions on the Mechanical FE exam. The heat transfer principles tested on the FE Mechanical exam are shown in the outline below. There are three main areas of heat transfer: conduction, convection and radiation. Conduction is the transfer of heat through contact. In this type of heat transfer, common skills needed include finding overall heat transfer coefficients, finding insulation values and temperature transitions through materials. Convection is the transfer of heat through a moving fluid. This is most commonly seen in heat exchangers as moving hot fluids transfer heat to cool fluids. The main skill needed in this area include finding the convective heat transfer coefficient through the Nusselt number, Reynolds number and/or Rayleigh number. The final type is radiation, which will require finding the radiative heat transfer coefficient and finding the radiative heat transfer between an object and its surroundings.

The last two topics are heat exchangers and transient processes. For heat exchangers, you must be able to calculate the log mean temperature difference for various types of heat exchangers and the heat transfer between the hot and cold fluid.

Section 12.0 Heat Transfer (7 to 11 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Topic 1.0	Introduction
12A	Topic 2.0	Conduction
12B	Topic 3.0	Convection
12C	Topic 4.0	Radiation
12D	Topic 5.0	Transient Processes
12E	Topic 6.0	Heat Exchangers
	Topic 7.0	Boiling
	Topic 8.0	Condensation
	Topic 9.0	Practice Exam Problems
	Topic 10.0	Practice Exam Solutions

2.0 CONDUCTION

Conduction is the method of heat transfer through material(s) in physical contact. The driving force in conduction is a temperature difference on either side of the material(s). For example, if the end of a metal rod is placed in a fire, heat will be conducted through the metal rod to the other end. Heat transfer due to conduction is most commonly calculated for wall and roof heat loads. The outside of a wall or roof is heated by the outdoor conditions. Then the heat is conducted from the outside of the wall through the wall material and to the inside of the wall, where the heat is transferred to the space. The formula for calculating heat transfer due to conduction through a material is as follows:

$$Q = \frac{k * A * (T_{hot} - T_{cold})}{t}$$

$$where\ Q = quantity\ of\ heat\ transferred\ \left[W\ or\ \frac{Btu}{hr}\right]$$

$$k = thermal\ conductivity\ of\ material\ \left[\frac{W}{m * °K}\ or\ \frac{Btu}{hr * ft * °\mathrm{F}}\right]$$

$$T_{hot} - T_{cold} = temperature\ difference\ [°K\ or\ °\mathrm{F}]$$

$$t = thickness\ of\ material\ [m\ or\ ft]$$

$$A = area\ of\ heat\ transferl\ [m^2\ or\ ft^2]$$

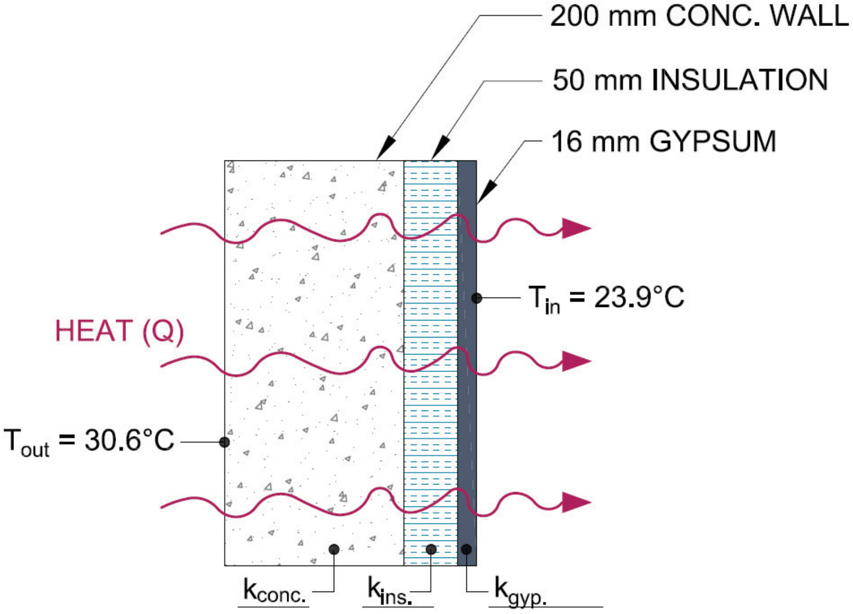


Figure 1: Conduction through a building wall

The amount of heat transferred is linearly dependent on the difference in temperature between the inside and outside surfaces of the wall. The conduction equation shows that as the temperature difference increases, the heat load also increases. The same is also true for the area available for heat transfer and the thermal conductivity. On the other hand, the amount of heat transferred is inversely related to the thickness of the wall or roof material.

2.1 THERMAL CONDUCTIVITY

Thermal conductivity is a measure of how well a material conducts and promotes heat transfer. Metals are excellent conductors and thus have a high conductivity. For example, aluminum has a thermal conductivity of $128\,\frac{Btu}{hr*ft*°\mathrm{F}}$ and iron has a conductivity of $\sim 30\,\frac{Btu}{hr*ft*°\mathrm{F}}$. Poor conductors include materials like wood (Douglas fir $0.0833\,\frac{Btu}{hr*ft*°\mathrm{F}}$) and insulation materials (Cellular Glass $0.0275\,\frac{Btu}{hr*ft*°\mathrm{F}}$; Glass Fiber $0.0221\frac{Btu}{hr*ft*°\mathrm{F}}$).

It is important to note that often times, thermal conductivity is given in units of $\frac{Btu*in}{hr*ft^2*°\mathrm{F}}$ or $\frac{W*mm}{m^2*°C}$. This value is the thermal conductivity per inch thickness of material. Insulation, masonry, plastering and wood materials are often presented with thermal conductivity per inch of materials. As an example, cellular glass has a unit thermal conductivity of 0.33$\frac{Btu*in}{hr*ft^2*°\mathrm{F}}$), which means that for an inch in thickness of cellular glass material, the thermal conductivity is 0.33. For 2" of thickness the thermal conductivity is halved to 0.165$\frac{Btu*in}{hr*ft^2*°\mathrm{F}}$.

$$Thermal\ Conductivity\ Units:\ \left[\frac{W-mm}{m^2*°K}\right] or\ [\frac{Btu-in}{hr*ft^2*°\mathrm{F}}]$$

Besides thermal conductivity, materials can also be classified by their R-Value or their U-Factors as shown below.

2.2 U-FACTOR

U-Factor stands for the overall heat transfer coefficient and it is representative of a material's ability to conduct heat. Similar to thermal conductance, a higher U-factor value has a higher ability to conduct and transfer heat. U-factor is related to thermal conductance by the following formula.

$$U=\frac{k}{t}\left[\frac{W}{m^2*°K}\right] or\ [\frac{Btu}{hr*ft^2*°\mathrm{F}}]$$

$$Q=U*A*(T_{hot}-T_{cold})$$

The units of thickness (t) may be in inches, millimeters, feet or meters. You should pay attention to the units, in order to ensure you use the correct values.

This equation assumes that U does not vary based on temperature. For purposes of the exam, this is a safe assumption.

2.3 R-VALUE

R-Value stands for thermal resistance and it is representative of a material's ability to resist heat. The R-Value is the inverse of the U-Factor and thermal conductance, which are measures of a materials ability to conduct heat. The relationship between the R-Value, U-Factor and thermal conductance is shown in the following formula.

$$R = \frac{1}{U} = \frac{t}{k} \left(\frac{m^2 * °K}{W} \; or \; \frac{hr * ft^2 * °F}{Btu}\right)$$

$$Q = \frac{1}{R} * A * (T_{hot} - T_{cold})$$

This equation assumes that R does not vary based on temperature. For purposes of the exam, this is a safe assumption.

R-values are typically used in the Thermal & Fluids field to describe building insulation materials. Often times, R-values are shown as a function of thickness, similar to the table below.

Thickness	R-Value
1” (25 mm)	5
1.5” (38 mm)	7.5
2” (50 mm)	10
2.5” (64 mm)	12.5

Notice that the unit R-Value is 5 for 1” (25 mm) of insulation. The corresponding R-values for various inches of thicknesses are found by simply multiplying the thickness in inches by the unit R-value, refer to the below equation.

$$t * R_{1"} = R_{t"}$$

$$t = thickness\ of\ insulation\ in\ inches\ or\ milimeters$$

$$R_{1"} = unit\ R - value$$

$$R_{t"} = R - value\ of\ insulation\ with\ a\ thickness\ of\ t"$$

2.4 THERMAL CONDUCTION THROUGH PIPES

When calculating the conductive heat transfer through pipes, the following equation should be used.

$$Q = \frac{2k\pi L * (T_{inner} - T_{outer})}{\ln(\frac{r_{outer}}{r_{inner}})}$$

$$where\ Q = quantity\ of\ heat\ transferred\ \left[W\ or\ \frac{Btu}{hr}\right]$$
$$k = thermal\ conductivity\ of\ material\ \left[\frac{W}{m * °C} or \frac{Btu}{hr * ft * °F}\right]$$
$$T_{inner} = Temperature\ at\ the\ inner\ pipe\ wall\ [°C\ or\ °F]$$
$$T_{outer} = Temperature\ at\ the\ pipe\ exterior\ [°C\ or\ °F]$$
$$r_{inner} = inner\ radius\ of\ pipe\ [m\ or\ ft]$$

$$r_{outer} = outer\ radius\ of\ pipe\ [m\ or\ ft]$$
$$L = pipe\ length\ [m\ or\ ft]$$

For a pipe with multiple layers of different materials, the equation becomes:

$$Q = \frac{2k\pi L * (T_{inner} - T_{outer})}{\frac{\ln\left(\frac{r_2}{r_{inner}}\right)}{k_i} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{k_{ii}} + \cdots + \frac{\ln\left(\frac{r_{outer}}{r_n}\right)}{k_m}}$$

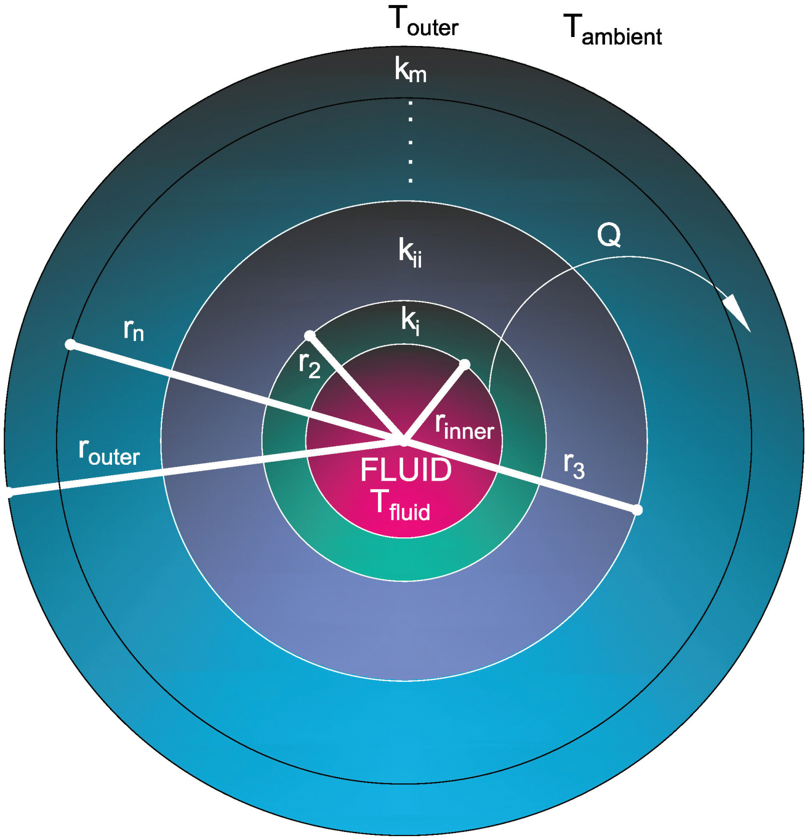


Figure 2: Heat transfer takes place from the fluid to the exterior through the various layers around the pipe.

2.5 THERMAL RESISTANCE

A must have skill is to be able to calculate the overall heat transfer coefficient (U-factor) for a wall, roof, duct or pipe. This skill will be described and explained through the following example.

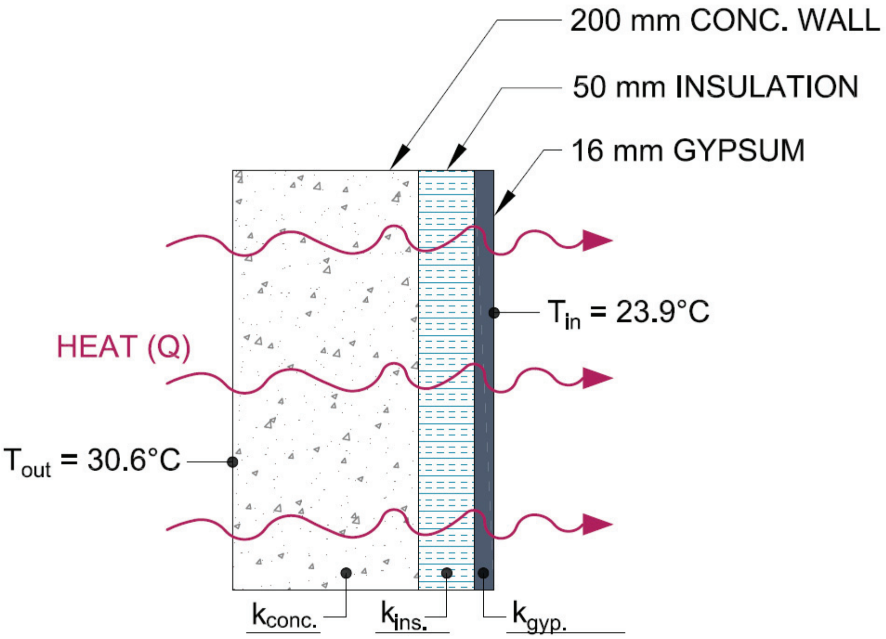


Figure 3: Overall Heat transfer Coefficient

It is important to be able to follow the flow of heat from the beginning to the end of this diagram [from left to right]. The diagram shows how the temperature starts from a high temperature of 30.6 °C down to 23.9 °C.

(1) The first method of heat transfer is due to convection. Warm outdoor air moves across the outer surface of the concrete wall causing the outer surface of the wall to heat up. In reality, there would also be radiation loads acting upon the surface of the wall, but for simplicity it is assumed that there are no radiation loads.

(2) Next the heat travels from the outer surface of the concrete wall to the inside surface.

(3) The heat then moves from the outer surface of the insulation and through the insulation.

(4) Next, the heat moves from the outer surface of the gypsum board and through the board.

(5) Finally the outer surface of the gypsum board transmits heat via convection and radiation to the indoor air.

In order to find the overall heat transfer coefficient, all of the resistances must be summed. It is recommended that each method of heat transfer should be converted to its equivalent R-Value in order to simplify the calculation, because R-Values in series are simply added together.

$$R_{series} = R_1 + R_2 \ldots + R_n$$

R-Values in parallel follow a different equation which is highlighted below.

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_n}$$

First, convert all resistances to R-values.

$$R_{conv,outdoor} = \frac{1}{h_{conv,outdoor}}$$

$$R_{conc}, R_{ins} \text{ and } R_{gyp} \text{ do not need to be converted.}$$

$$R_{conv,indoor} = \frac{1}{h_{conv,indoor}}$$

$$R_{rad} = \frac{1}{h_{rad}}$$

Next, notice that the radiation and convection heat transfer modes are arranged in parallel. Convert these two items to a single term.

$$\frac{1}{R_{combined,indoor}} = \frac{1}{R_{conv,indoor}} + \frac{1}{R_{rad}}$$

$$R_{combined,indoor} = \frac{R_{conv,indoor} * R_{rad}}{R_{conv,indoor} + R_{rad}}$$

Now that all terms are in series, the terms can be summed together.

$$R_{total} = R_{conv,outdoor} + R_{conc} + R_{ins} + R_{gyp} + \frac{R_{conv,indoor} * R_{rad}}{R_{conv,indoor} + R_{rad}}$$

In order to find the overall heat transfer coefficient (U-factor), simply take the inverse of the total R-value.

$$U_{overall} = \frac{1}{R_{total}}$$

3.0 CONVECTION

Convection is the second mode of heat transfer and is defined as the transfer of heat through the movement of fluids. Convective heat transfer can be found whenever a moving fluid passes over a surface at a different temperature.

Convective heat transfer has a similar equation to conductive heat transfer, except the U-Factor or R-Value is replaced with the convective heat transfer coefficient. This convective heat transfer coefficient characterizes the moving fluid by taking into account its viscosity, thermal conductance, temperature, velocity and it also characterizes the surface that the fluid is moving upon.

$$Q_{conv} = h_{conv} * A_{surface} * (T_{hot} - T_{cold})$$

Most exam questions will focus around finding the heat transfer coefficient. As an engineer you should be able to find this heat transfer coefficient for common situations. The following paragraphs will go through some of the common situations, but you should be able to use the convective heat transfer coefficients for the other situation presented in the *NCEES FE Reference Handbook*. Some of the other situations not covered in this book are natural convection over a vertical flat plate, film condensation of pure vapors on a vertical surface and parallel flow over a flat plate. But if you can understand the general process below, then you can solve any of the convective heat transfer problems on the exam.

1. Determine geometric situation (parallel flat plate, outside horizontal tube, inside circular tube, etc.)
2. Determine if the flow is turbulent (Reynolds Number (Re) greater than a value and/or Prandtl (Pr) number greater than a value.)
3. Find Nusselt number (Nu)
4. Find hydraulic diameter (D_h)
5. Determine convective heat transfer coefficient (h_{conv})

3.1 DETERMINE GEOMETRIC SITUATION

On the exam, the problems should indicate your geometric situation and based on this information you must find the appropriate Nusselt number equation and other restrictions placed on the equation. These restrictions will most likely have to do with the Reynolds Number, Prandtl Number and Rayleigh Number.

3.2 REYNOLDS, PRANDTL & RAYLEIGH NUMBERS

The second step in finding the convective heat transfer coefficient is to find the appropriate numbers that are required for the geometric situation. First, the Reynolds number is the ratio of the inertial forces to the viscous forces. The Reynolds number describes whether or not a fluid is turbulent or laminar. A higher number implies a turbulent flow, typically above 2,300.

$$Re = \frac{D * v}{\nu}$$

$$D = diameter\ of\ pipe\ (m\ or\ ft);\ v = velocity\ \left(\frac{m}{s}\ or\ \frac{ft}{s}\right);$$

$$\nu = kinematic\ viscosity\ (\frac{m^2}{s} or \frac{ft^2}{s})$$

Secondly, the Prandtl number describes the fluid and its ability to store heat and transfer heat through conduction.

$$Pr = \frac{\mu * c_p}{k}$$

$$c_p = specific\ heat\ \left(\frac{J}{kg * K}\ or\ \frac{Btu}{lbm * °F}\right); k = thermal\ conductivity\ \left(\frac{W}{m * K}\ or\ \frac{Btu}{h * ft * °F}\right);$$

$$\mu = dynamic\ viscosity\ (Pa - S\ or \frac{kg}{m - s}\ or\ \frac{lbm}{ft - hr})$$

$$where\ Pa = \frac{kg}{m * s^2}\ in\ SI\ derived\ units$$

The Prandtl numbers for typical fluids are as follows:

Fluid	Pr
Air	0.7
Water	7
Seawater (32 °F)	13.4
Seawater (68 °F)	7.2

Third, the Rayleigh number is only used for natural convection problems. Natural convection is the heat transfer due to naturally moving air over an object. Air naturally moves upward when heated or more generally when a fluid moves upward due to naturally occurring density differences within a fluid due to temperature differences. The Rayleigh number is composed of two numbers, the Grashof number which characterizes the fluid's buoyancy and viscosity and the Prandtl number which was discussed previously. However, the *NCEES FE Reference Handbook* uses a different equation that is a little less intuitive.

This paragraph will try to give you more insight into the equation shown by breaking it down into its parts. The basic Rayleigh number is shown below and as you can see, the driving force is the difference in density caused by the hot and cold fluid and this density is multiplied by gravity to get the driving force. The length term is the characteristic length which is the length of the flat plate that is hot, which is causing the difference in temperatures which is causing the difference in densities. Diffusivity is the rate at which the fluid can move and the dynamic viscosity is the fluid's resistance to that movement.

$$Ra = \frac{\Delta p * g * L^3}{Diffusivity * \mu}$$

Through some manipulation the difference in density can be converted to a difference in temperature and the diffusivity and dynamic viscosity can be converted to the Prandtl Number and the kinematic viscosity of the fluid to get the equation shown below.

$$Ra = \frac{\beta * g * (T_{surface} - T_\infty) * L^3}{v^2} * Pr$$

On the exam, the question must give you all of the fluid properties, unless the fluid is water vapor, because those values are included in the *NCEES FE Reference Handbook.*

$$\beta = coefficient\ of\ thermal\ expansion\ \left(\frac{1}{°K}\ or\ \frac{1}{°R}\right) T_{\infty} = surroundings\ temperature\ (°K\ or\ °R)$$

$$T_{surface} = surface\ temperature\ of\ object\ (°K\ or\ °R); g = 9.81\frac{m}{s^2};$$

$$v = kinematic\ viscosity\ (\frac{m^2}{s}\ or\ \frac{ft^2}{s})$$

3.3 DETERMINE NUSSELT NUMBER

Next, you need to use the fluid properties and the calculated Reynolds, Rayleigh and/or Prandtl numbers to find the Nusselt number. The equation for the Nusselt number will vary based on the geometric condition and the number values.

3.4 DETERMINE HYDRAULIC DIAMETER OR CHARACTERISTIC LENGTH

The next step is to find the hydraulic diameter for the geometric condition. For circular tubes, pipes or cylinders, the hydraulic diameter is simply the inner diameter or outer diameter. If the geometric condition is a plate, then it is simply the length of the plate. The difficulty is if the geometric condition is a non-circular duct that is transferring convective heat similar to a pipe, cylinder or tube. This non-circular duct must be converted to an equivalent hydraulic diameter through the below equation.

$$D_{hydraulic}(ft\ or\ m) = \frac{4 * Cross\ Sectional\ Area\ (ft^2\ or\ m^2)}{Perimeter\ (ft\ or\ m)}$$

3.5 DETERMINE CONVECTIVE HEAT TRANSFER COEFFICIENT

The last step is to solve for the convective heat transfer coefficient for the specific geometry situation. The following equation is most common for convective heat transfer from pipes and cylinders. In this equation you take the Nusselt number, hydraulic diameter and the thermal conductivity of the fluid and solve for the convective heat transfer coefficient.

$$Nu = \frac{D * h_{conv}}{k}$$

$$where\ D = diameter\ of\ pipe\ (ft), k = thermal\ conductivity\ of\ the\ fluid,$$

The following are more examples that combine 3.3 Determine Nusselt number and 3.5 Determine Convective Heat Transfer Coefficient into a single step.

3.5.1 Turbulent flow inside circular pipe (heating)

$$Nu = \frac{D * h_{conv}}{k} = .023 * Re^{.8} * Pr^{.4}$$

This equation works when the temperature of the wall of the pipe is greater than the temperature of the fluid. The Reynolds number must also be greater than 2300 to qualify as turbulent flow.

3.5.2 Turbulent flow inside circular pipe (cooling)

$$Nu = \frac{D * h_{conv}}{k} = .023 * Re^{.8} * Pr^{.3}$$

This equation works when the temperature of the wall of the pipe is less than the temperature of the fluid. The Reynolds number must also be greater than 2300 to qualify as turbulent flow.

3.5.3 Laminar flow inside circular pipe (heating)

$$Nu = \frac{D * h_{conv}}{k} = 4.36$$

This equation works when it is assumed that there is uniform heat transfer through the pipe and laminar flow. The Reynolds number must be less than 2300 to qualify as laminar flow.

3.5.4 Laminar flow inside circular pipe (cooling)

$$Nu = \frac{D * h_{conv}}{k} = 3.66$$

This equation works when it is assumed that the surface temperature of the pipe is constant and there is laminar flow. The Reynolds number must be less than 2300 to qualify as laminar flow.

3.5.5 Other situations

There are many other types of situations, each with their own Nusselt equation. You do not need to memorize these specific situations but you should be aware of these equations in your *NCEES FE Reference Handbook*.

3.6 DETERMINE CONVECTIVE HEAT TRANSFER

Once you have the convective heat transfer coefficient then you can use the below equation to calculate the convective heat transfer.

$$Q = h_{conv} * A * \Delta T$$

$$h_{conv} = convective\ heat\ transfer\ coefficient[\frac{Btu}{hr * ft^2 * °F}]$$

$$A = area\ of\ heat\ transfer\ [ft^2]$$

$$\Delta T = temperature\ difference\ between\ hot\ and\ cold\ areas\ of\ heat\ transfer\ [°F]$$

4.0 RADIATION

Radiation is the mode of heat transfer that requires no substance to transmit heat. All objects above absolute zero radiate or project heat from its surface. The amount of heat that is transferred is governed by the surface temperature of the object. The term used to describe radiation is emissive power.

$$Q = \sigma A T^4$$

$$Q = radiation\ heat\ (Btuh\ or\ W); A = area\ (ft^2\ or\ m^2);$$

$$\sigma = 0.1713\ x\ 10^{-8}\left(\frac{Btu}{h * ft^2 * R^4}\right)\ or\ \ 5.67\ x\ 10^{-8}\left(\frac{W}{m^2 * K^4}\right);$$

$$T =\ surface\ temperature\ (°R\ or\ °K\ )$$

4.1 EMISSIVITY

The above equation assumes an ideal, 100% radiative body. However, in practice this does not occur. Not all of the energy from the hot object will be radiated to the atmosphere. Some will be kept in the body. Emissivity is a factor that describes how much of the radiative energy will be emitted into the atmosphere.

$$\varepsilon = emissivity$$

For example, a solar hot water heating system has absorbing material because of its low emissivity. The absorbing material does not release heat through radiation. The material keeps the heat internally. Black asphalt has a high emissivity. It greatly radiates heat to the atmosphere.

The radiation equation is revised to incorporate emissivity.

$$Q = \varepsilon \sigma A T^4$$

4.2 TEMPERATURE DIFFERENCE BETWEEN TWO OBJECTS

The below equation is used to calculate the heat transfer between two objects with varying temperatures due to radiation.

$$Q = \varepsilon * \sigma * A * (T_{obj1}^4 - T_{obj2}^4)$$

This equation is typically used to calculate the radiation from a hot object to its surroundings. An example could be the radiation heat from a motor operating in a room at a certain temperature.

$$Q = \varepsilon_{motor} * \sigma * A_{surface\ area\ of\ motor} * (T_{motor}^4 - T_{room\ temp.}^4)$$

5.0 TRANSIENT PROCESSES

Most questions on heat transfer will assume a steady state, but if you encounter a question that has a time component, then you need to use the transient equations. In the *NCEES FE Reference Handbook*, the only equations on transient heat transfer processes are for conduction. Thus, this section will only focus on those equations, skills and concepts revolving around those calculations and not radiation. The first concept that you need to know is the Biot Number. The Biot Number allows you to simplify the calculation so that the process can be completed without a computer program and an iterative process. .

5.1 BIOT NUMBER

The Biot number is used as a representation of the heat transfer resistances inside of a substance as compared to the surface of the substance. It tells you whether or not you can use the transient conduction lumped capacitance equations shown in the *NCEES FE Reference Handbook.* A very small Biot number around 0.1 (<1) indicates that the thermal resistance between the surface of the substance and the environment is greater than the thermal resistance inside of the substance.

$$Bi = \frac{hV}{kA}$$

$$h = convective\ heat\ transfer\ coefficient \left(\frac{W}{m^2 * K}\right);\ V = volume\ (m^3);$$

$$k = thermal\ conductivity\ of\ body\ (\frac{W}{m * K});\ A = surface\ area\ of\ body\ (m^2);$$

This number is important because at low Biot numbers, you can use the following equations to calculate the convective heat transfer from a body's surface to an environment and you can also calculate the change in temperature of a body over time.

5.2 CONVECTIVE HEAT TRANSFER BETWEEN SURFACE AND ENVIRONMENT

The equations below assume that the Biot number is really low, which means that you can treat the object that is undergoing heat transfer as a single lumped model. This is as opposed to having to worry about the heat transfer through the individual layers of the object. For example, a very thick steak cooking over an open flame would have a high convective heat transfer coefficient and a large volume as compared to its surface area. This means that the limiting factor in heat transfer is the conduction through the layers of the steak and thus you cannot use the below equations.

The following equation basically says that the heat transfer on the surface is the limiting factor and that this convective heat transfer is equal to the heat gained by the object (mass).

$$Q = hA * (T - T_{\infty}) = -pV\left(c_p\right) * \frac{\Delta T}{\Delta t}$$

$$h = convective\ heat\ transfer\ coefficient\ \left(\frac{W}{m^2 * {}^{\circ}K}\right); T_{\infty} = environment\ temperature\ ({}^{\circ}K)$$

$$A = surface\ area\ of\ body\ (m^2); T = surface\ temperature\ ({}^{\circ}K)$$

$$p = density\ of\ body\ \left(\frac{kg}{m^3}\right); V = volume\ of\ body\ (m^3); c_p = heat\ capacity\ \left(\frac{J}{kg - s}\right)$$

$$\Delta T = change\ in\ temperature\ of\ body\ ({}^{\circ}K);\ \Delta t = change\ in\ time\ (s)$$

Both sides of the equation must equal and it should be noted that the units of watts (W) is equal to joule per second (J/s). So both sides of the equations result in watts.

This equation uses the properties of the body and the convective heat transfer of the fluid around the body, to calculate the change in temperature of the body as a function of time.

$$(T - T_{\infty}) = (T_i - T_{\infty})e^{-h*A*t/p*V*c_p}$$

$$T_i = initial\ body\ temperature\ ({}^{\circ}K); t = time\ duration\ (s)$$

6.0 HEAT EXCHANGERS

Heat exchangers are mechanical devices designed to exchange or transfer heat from a hot fluid to a cold fluid. Heat exchangers are used heavily throughout the Thermal & Fluids field. For example, a condenser and boiler both are heat exchangers. A cooling or heating coil is a heat exchanger that transfers heat from one fluid to another fluid.

6.1 TYPES OF HEAT EXCHANGERS

There are many different types of heat exchangers, for example there are heat exchanger types that are based on their construction. These types include, shell and tube, plate frame and micro-channel. But for exam purposes it is more important to understand the two flow classifications of heat exchangers, (1) parallel flow and (2) counter-flow heat exchangers. These two classifications describe the relationship between the direction of flow of the cold and hot fluids.

(1) Parallel flow heat exchanger: This heat exchanger has both the cold and hot fluids entering at the same end of the heat exchanger. At the beginning of the heat exchanger there is a large difference

between the cold and hot fluids and at the end of the heat exchange the difference between cold and hot is reduced, refer to the figures below.

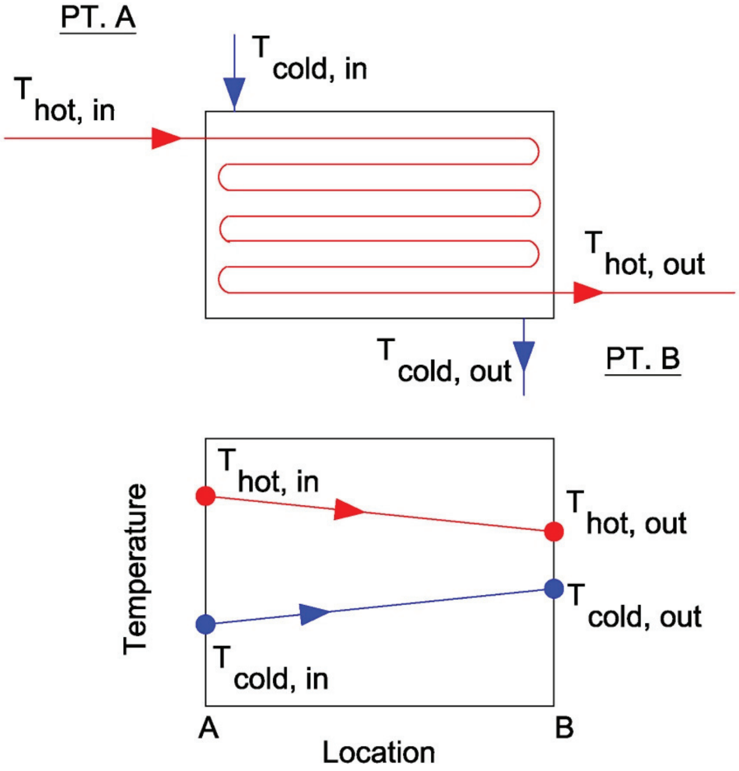

PT. A
T cold, in
T hot, in
T hot, out
T cold, out
PT. B
Temperature
T hot, in
T hot, out
T cold, out
T cold, in
A
B
Location

Figure 4: Parallel flow heat exchanger

(2) Counter-flow heat exchanger: The counter-flow heat exchanger is opposite of the parallel flow heat exchanger. The cold and hot fluids enter at opposite ends. The figure below shows the counter-flow heat exchanger, notice the opposing directional arrows.

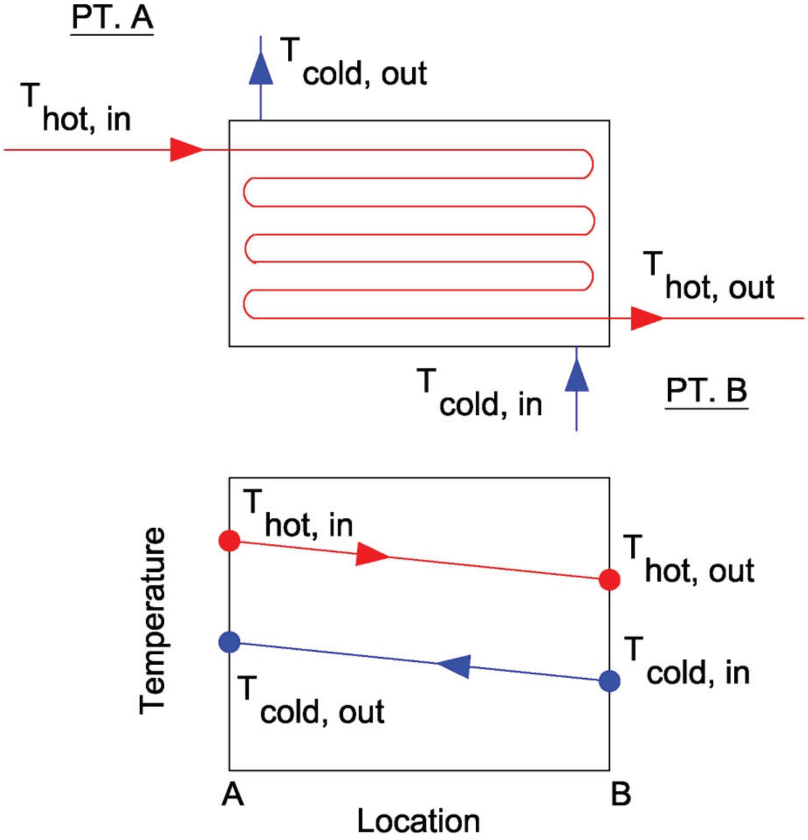


Figure 5: This figure shows a counterflow heat exchanger and diagram. If you get confused on the FE exam, it may help to draw this figure.

6.2 LOG MEAN TEMPERATURE DIFFERENCE

In heat exchangers that do not have a phase change, heat is transferred from the hot fluid to the cold fluid through the temperature difference between the cold and hot. However, in a heat exchanger as shown in the previous figures, the temperature difference between the cold and hot fluids is not always constant and depends on the location in the heat exchanger. Thus the log mean temperature difference is used. The LMTD describes the logarithmic average temperature difference between the cold and hot fluid through a generic heat exchanger (counter or parallel). LMTD cannot be used for heat exchangers with a phase change like a boiler or condenser. The equation for LMTD is shown below.

$$LMTD = \frac{\Delta T_a - \Delta T_b}{\ln(\frac{\Delta T_a}{\Delta T_b})}$$

$$\Delta T_a = temperature\ difference\ at\ side\ a$$

$$\Delta T_b = temperature\ difference\ at\ side\ b$$

The LMTD is then used to calculate the total heat exchanged by the heat exchanger through the following equation. The U-value is the heat transfer coefficient of the heat exchanger which is given by the heat exchanger manufacturer. The Area value is the total area where heat exchange occurs, which is also given by the heat exchanger manufacturer.

$$Q = U_{hx} * A_{hx} * LMTD$$

$$U_{hx} = overall\ heat\ transfer\ coefficient\ of\ the\ heat\ exchanger$$

$$A_{hx} = area\ of\ heat\ exchange$$

The overall heat transfer coefficient or U-value will be discussed later in this section.

6.2.1 Correction Factor

In practice, heat exchangers will rarely be a pure parallel or cross-flow type. There will be multiple passes and different types of arrangements such that the hot and cold fluids do not enter and leave at the exact locations required by the parallel and cross-flow arrangements. An example of the difference between an ideal parallel flow heat exchanger and non-ideal case is shown below.

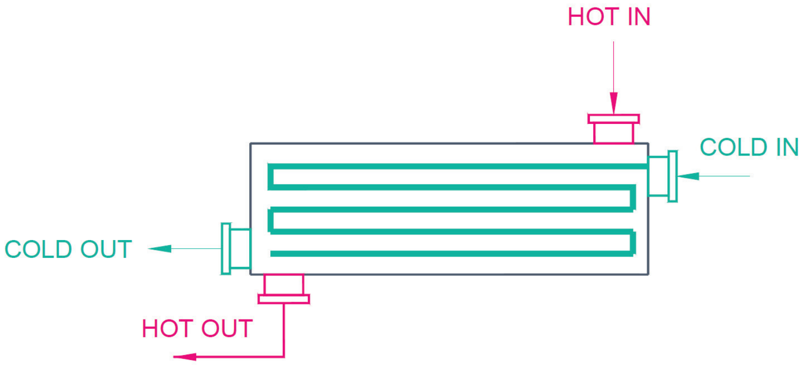


Figure 6: This figure shows an ideal parallel flow heat exchanger.

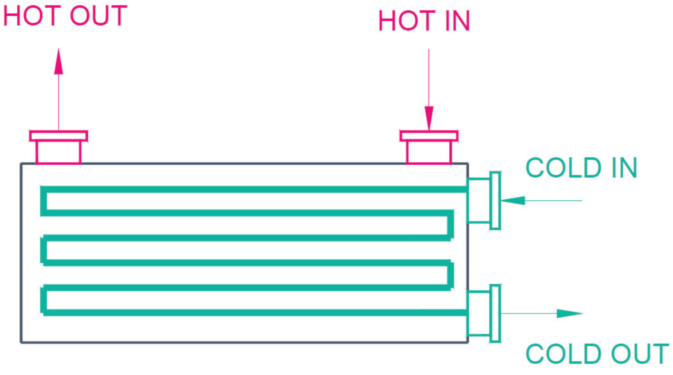


Figure 7: This figure shows a non-ideal case for a parallel flow heat exchanger.

When you encounter a non-ideal case on the exam, you must apply a correction factor to your equations. This correction factor is not solvable on the FE exam, so the question must give you this factor. You just need to know how to apply this factor as shown below.

$$Q = U_{hx} * A_{hx} * F * LMTD$$

$$F = correction\ factor$$

6.3 HEAT EXCHANGER U-VALUE

The amount of heat transfer that can be conducted by a heat exchanger can be characterized by its U-value, which was discussed earlier in this section. This is a material or in this case, an equipment's ability to transfer heat. There are 7 different components that make up the heat exchangers U-value. It is important to note that this discussion only applies to shell and tube heat exchangers and does not apply to plate-frame heat exchangers. Also as you go through each component, please note that the components are broken down into resistances or 1/U values, such that they can be added at the end of this discussion.

The first two components are the convective heat transfer between the inner fluid and the inner surface of the pipe and the conductive heat transfer through any fouling that may have built up on the inside of the pipes. Fouling occurs overtime and it is a result of mineral deposits and biological growth.

$$\frac{1}{h_i A_i} + \frac{R_{foul,i}}{A_i}$$

The second two components are the opposite of the first two. This is the convective heat transfer between the outer fluid and outer surface of the pipe and the fouling on the outer surface of the pipe.

$$\frac{1}{h_o A_o} + \frac{R_{foul,o}}{A_o}$$

The third component is the conductive heat transfer through the thickness of the pipe. This is similar to the radial heat transfer equation.

$$\frac{\ln(\frac{D_o}{D_i})}{2\pi k L}$$

All of these components can be combined to solve for the overall heat exchanger U-value as shown in the below equation. Notice, that the U-values are converted to the resistance format as shown

$$\frac{1}{U_{overall} A} = \frac{1}{h_i A_i} + \frac{R_{foul,i}}{A_i} + \frac{1}{h_o A_o} + \frac{R_{foul,o}}{A_o} + \frac{\ln(\frac{D_o}{D_i})}{2\pi k L}$$

The figure below shows how each of the components act as a layer of resistance to heat transfer.

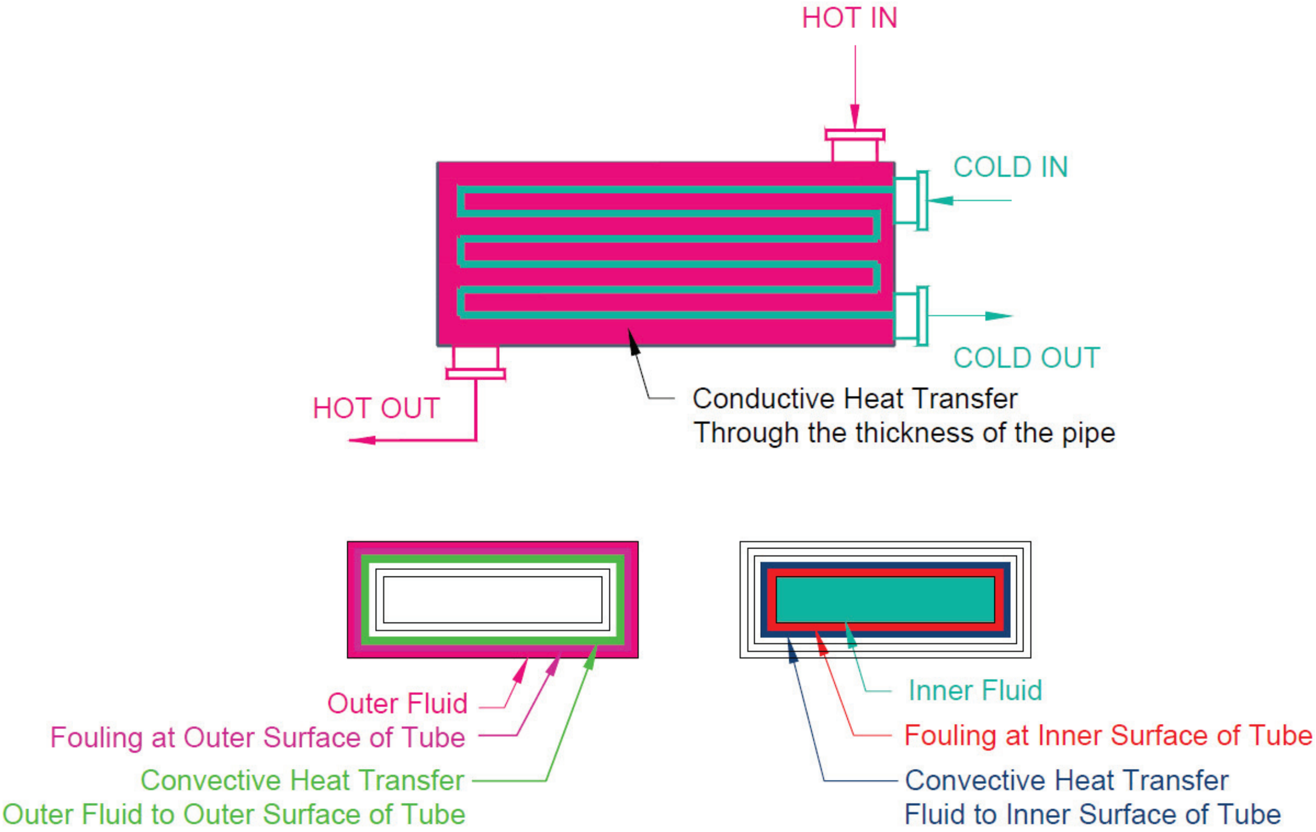

HOT IN
COLD IN
COLD OUT
HOT OUT
Conductive Heat Transfer
Through the thickness of the pipe
Outer Fluid
Fouling at Outer Surface of Tube
Convective Heat Transfer
Outer Fluid to Outer Surface of Tube
Inner Fluid
Fouling at Inner Surface of Tube
Convective Heat Transfer
Fluid to Inner Surface of Tube

Figure 8: This figure shows the seven components that sum up to equal the total resistance to heat transfer within a shell and tube heat exchanger.

6.4 HEAT BALANCE BETWEEN HOT AND COLD FLUIDS

A heat balance is conducted on a heat exchanger to show that a balance of heat loss from the hot fluid is shown as a heat gain to the cold fluid. For example, cooling coils are heat exchangers that transfer heat from air to water. The heat balance governing this heat transfer would be as shown below.

$$Q_{hx} = \dot{m}_{air} * c_{p,air} * (T_{air,in} - T_{air,out}) = \dot{m}_{water} * c_{p,water} * (T_{water,in} - T_{water,out})$$

If there is a phase change, then the following equation can be used.

$$Q_{hx} = \dot{m}_{fluid} * h_{phase} = \dot{m}_{fluid} * c_{p,fluid} * (T_{fluid,in} - T_{fluid,out})$$

6.5 HEAT EXCHANGER EFFECTIVENESS

The previous topic discussed the heat exchange between the hot and cold fluids in a heat exchanger, but in reality a heat exchanger will not be 100% effective to make the previous equation true. There will be losses in the system as heat is lost to the surroundings. Heat exchangers can be rated on their effectiveness to transfer heat between the hot and cold fluids. The effectiveness is the ratio of the actual heat transfer as compared to maximum possible of heat that could be transferred between the hot and cold fluids.

$$Effectiveness\ (\varepsilon) = \frac{Actual\ Heat\ Transfer}{Maximum\ Heat\ Transfer}$$

The maximum possible of heat that could be transferred between the hot and cold fluids would occur when all of the hot fluid is turned to the same temperature as the cold fluid, assuming that the mass flow rate of hot and cold fluids was the same. If the mass flow rates were not the same, then the maximum possible of heat that could be transferred would be the lower mass flow rate converted completely from hot to cold or from cold to hot. It doesn't matter which way, because only the change in temperature is important.

$$Maximum\ Possible\ Heat\ Transfer = \dot{m}_{min} * c_p * (T_{hot} - T_{cold})$$

The actual amount of heat transfer can be determined by finding the actual entering and leaving temperatures of the hot or cold fluid.

$$Actual\ Heat\ Transfer = \dot{m}_{hot} * c_{p,hot} * (T_{hot,enter} - T_{hot,leaving})$$

$$Actual\ Heat\ Transfer = \dot{m}_{cold} * c_{p,cold} * (T_{cold,enter} - T_{cold,leaving})$$

The effectiveness can then be found by dividing the actual heat transfer by the maximum possible heat transfer. There will be two possible effectiveness values, depending on which side you are rating your heat exchanger upon.

$$\varepsilon = \frac{\dot{m}_{cold} * c_{p,cold} * (T_{cold,enter} - T_{cold,leaving})}{\dot{m}_{min} * c_p * (T_{hot} - T_{cold})}\ or\ \frac{\dot{m}_{hot} * c_{p,hot} * (T_{hot,enter} - T_{hot,leaving})}{\dot{m}_{min} * c_p * (T_{hot} - T_{cold})}$$

6.6 NTU METHOD

Sometimes, you will not be given the temperatures entering and leaving the heat exchanger. If this is the case then you must use the NTU method to solve for the effectiveness of the heat exchanger. First, you start with the U-value, area of heat exchange, heat capacity and the minimum mass flow rate. This will give you the NTU or number of transfer units.

$$NTU = \frac{UA}{\dot{m}_{min} * c_{p,min}}$$

Next, you must solve for the ratio between the minimum mass flow rate and maximum flow rate multiplied by their specific heat values to get the heat capacity ratio.

$$Heat\ Capacity\ Ratio\ (C_r) = \frac{\dot{m}_{min} * c_{p,min}}{\dot{m}_{max} * c_{p,max}}$$

This heat capacity ratio and the type of heat exchanger arrangement (counter-flow concentric tube or parallel flow concentric tube) will determine which equation to use to solve for the effectiveness for the heat exchanger.

$$\varepsilon_{parallel\ concentric\ tube} = \frac{1 - e^{(-NTU(1+C_r)}}{1 + C_r}; for\ all\ values\ of\ C_r$$

$$\varepsilon_{counter-flow\ concentric\ tube} = \frac{1 - e^{(-NTU(1+C_r)}}{1 - C_r * e^{(-NTU(1+C_r)}}; for\ \ C_r < 1$$

$$\varepsilon_{counter-flow\ concentric\ tube} = \frac{NTU}{1 + NTU}; for\ \ C_r = 1$$

7.0 BOILING

One key principle that must be understood is the relationship between the boiling and condensing point of a fluid, in this case a refrigerant, and the temperature and pressure of the refrigerant. A refrigerant liquid's boiling point is a function of the vapor pressure of the refrigerant vapor that is in equilibrium with the refrigerant liquid. If the pressure is low, then there is a smaller force acting upon the refrigerant liquid, thus it will take a lower temperature to boil the refrigerant liquid. For example, water at a pressure of 1 atmosphere or 14.696 PSI will boil at 212 F (101.33 kPa at 100 C). However, if the water was at a pressure of 0.122 PSI, then the water will boil at 40 F (0.841 kPa at 4.44 C). You need to understand that low pressure liquids boil at a lower temperature, high pressure liquids condense at a higher temperature. For more detail refer to the following table and figure.

Water Temperature [F] Boiling Point	Pressure [PSI]	Water Temperature [C] Boiling Point	Pressure [kPa]
212	14.696	100	101.33
190	9.340	87.8	64.4
160	4.742	71.1	32.7
130	2.224	54.4	15.33
100	0.950	37.8	6.55
70	0.363	21.1	2.5
40	0.122	4.44	.841
10	0.031	-12.2	.214

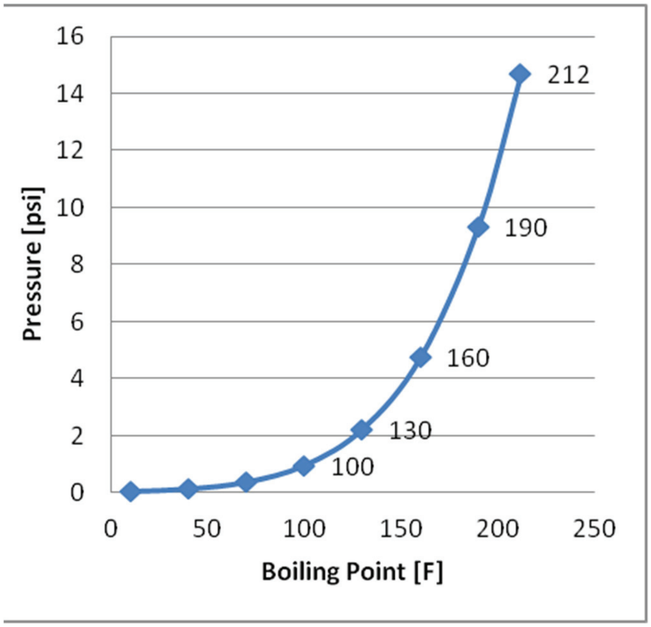


Figure 9: Relationship between boiling point and pressure

When a low pressure refrigerant changes from its liquid phase to a gas phase, it can absorb much more heat than if it were to simply increase in temperature. The same is also true when a high pressure refrigerant changes phase from its gas phase to a liquid phase; it releases much more heat than if it were to decrease in temperature. The energy required to change the phase of a liquid from a liquid to a gas is called the latent heat of evaporation. The energy released to change the phase of a gas to a liquid is called the latent heat of condensation.

Boiling is discussed in more detail in *Section 11.0 Thermodynamics* under boilers and steam power cycles. There are additional equations in the *NCEES FE Reference Handbook* on boiling, but these equations are much more straight forward and do not have a key skill or concept, besides being able to insert the variables with the correct units into the equation.

8.0 CONDENSATION

Condensers are pieces of equipment where a phase change in a fluid occurs. In condensers, gas is changed to liquid. In boilers and evaporators, energy is added into the system by the boiler or evaporator in order to heat the liquid to gas. In condensers, heat is removed from the system by the condenser in order to change the gas to a liquid. The energy balance equations governing these pieces of equipment take into account the phase change of the fluids.

$$Q_{boiler/evap} + \dot{m}_{liquid} * h_{liquid} = \dot{m}_{gas} * h_{gas}$$

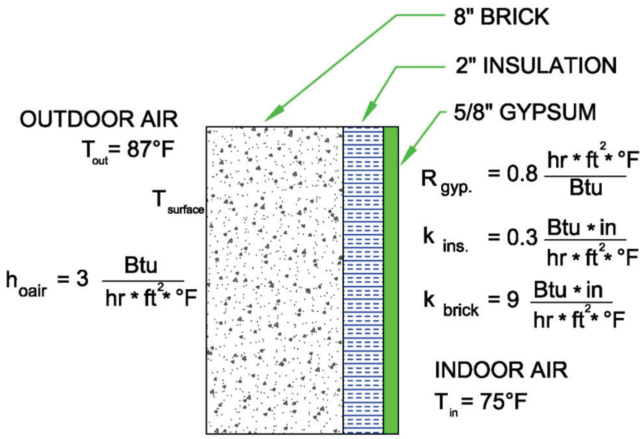
$$Q_{condenser} + \dot{m}_{liquid} * h_{liquid} = \dot{m}_{gas} * h_{gas}$$

Another important aspect about condensation is that moist air, the air around us contains water vapor. The water vapor in air can be condensed out of the air, if the air hits a surface that is below its dew point. This is covered in *Section 11.0 Thermodynamics*.

9.0 PRACTICE PROBLEMS

9.1 PROBLEM 1 - CALCULATE OVERALL HEAT TRANSFER COEFFICIENT

Calculate the overall heat transfer coefficient for the following wall conditions.


8" BRICK
2" INSULATION
5/8" GYPSUM
OUTDOOR AIR
T_{out} = 87°F
$T_{surface}$
$h_{oair} = 3 \frac{Btu}{hr * ft^2 * °F}$
$R_{gyp.} = 0.8 \frac{hr * ft^2 * °F}{Btu}$
$k_{ins.} = 0.3 \frac{Btu * in}{hr * ft^2 * °F}$
$k_{brick} = 9 \frac{Btu * in}{hr * ft^2 * °F}$
INDOOR AIR
T_{in} = 75°F

SI Unit Conversions

87 °F	=	30.556 °C	0.3 (Btu-in)/(h-ft^2-°F)	=	43.268 (W-mm)/(m^2-°C)
3 Btu/hr-ft^2-°F	=	17.035 W/(m^2-°C)	9 (Btu-in)/(h-ft^2-°F)	=	1298.05 (W-mm)/(m^2-°C)
8 in	=	203.2 mm	75 °F	=	23.889 °C
2 in	=	50.8 mm			
0.8 ft^2-°F/(Btu/h)	=	0.14088 (m^2-°C)/W			

(a) $0.12 \frac{Btu}{hr*ft^2*°F}$ $or\ 0.68 \frac{W}{m^2*°C}$

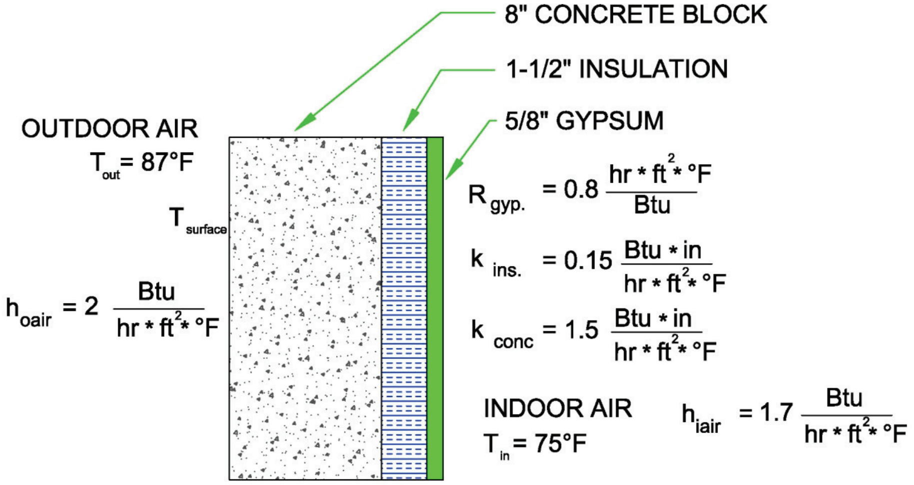
(b) $.25 \frac{Btu}{hr*ft^2*°F}$ $or\ 1.42 \frac{W}{m^2*°C}$

(c) $3.12 \frac{Btu}{hr*ft^2*°F}$ $or\ 17.72 \frac{W}{m^2*°C}$

(d) $8.7 \frac{Btu}{hr*ft^2*°F}$ $or\ 49.4 \frac{W}{m^2*°C}$

9.2 PROBLEM 2 - CALCULATE OVERALL HEAT TRANSFER COEFFICIENT

Calculate the overall heat transfer coefficient for the following wall conditions.


8" CONCRETE BLOCK
1-1/2" INSULATION
5/8" GYPSUM
OUTDOOR AIR
T_{out} = 87°F
$T_{surface}$
$h_{oair} = 2 \frac{Btu}{hr * ft^2 * °F}$
$R_{gyp.} = 0.8 \frac{hr * ft^2 * °F}{Btu}$
$k_{ins.} = 0.15 \frac{Btu * in}{hr * ft^2 * °F}$
$k_{conc} = 1.5 \frac{Btu * in}{hr * ft^2 * °F}$
INDOOR AIR
T_{in} = 75°F
$h_{iair} = 1.7 \frac{Btu}{hr * ft^2 * °F}$

SI Unit Conversions

87 °F	=	30.556 °C	0.15 (Btu-in)/(h-ft^2-°F)	=	21.634 (W-mm)/(m^2-°C)
2 Btu/hr-ft^2-°F	=	11.357 W/(m^2-°C)	1.5 (Btu-in)/(h-ft^2-°F)	=	216.34 (W-mm)/(m^2-°C)
8 in	=	203.2 mm	75 °F	=	23.889 °C
1-1/2 in	=	38.1 mm	0.8 ft^2-°F/(Btu/h)	=	0.14088 (m^2-°C)/W
5/8 in	=	15.875 mm	1.7 Btu/hr-ft^2-°F	=	9.653 W/(m^2-°C)

(a) $0.06 \frac{Btu}{hr*ft^2*°F}$ or $0.341 \frac{W}{m^2*°C}$

(b) $.11 \frac{Btu}{hr*ft^2*°F}$ or $.625 \frac{W}{m^2*°C}$

(c) $.21 \frac{Btu}{hr*ft^2*°F}$ or $1.192 \frac{W}{m^2*°C}$

(d) $15 \frac{Btu}{hr*ft^2*°F}$ or $85.174 \frac{W}{m^2*°C}$

9.3 PROBLEM 3 - CALCULATE THE REYNOLDS NUMBER

250 GPM (15.7725 L/s) of 120°F (48.9 °C) water flows through a 4” (101.6 mm) inner diameter pipe that is located in the ceiling of a building. Assume the water in the pipe is not under extreme pressure. Calculate the Reynolds number of the fluid inside the pipe.

(a) 57,734

(b) 295,941

(c) 355,745

(d) 136,836,344

9.4 PROBLEM 4 - CALCULATE THE CONVECTIVE HEAT TRANSFER COEFFICIENT

250 GPM of 120 °F water flows through a 4” diameter pipe that is located in the ceiling of a building. Calculate the convective heat transfer at the fluid inside the pipe. Assume the water in the pipe is not under extreme pressure.

120 °F, water properties: $\mu = 1.36\ \frac{lbm*ft}{hr}, c_p = 0.999\ \frac{Btu}{lbm*°\mathrm{F}}, k = 0.378\ \frac{Btu*ft}{h*ft^2*°\mathrm{F}}, thermal\ conductivity$

(a) $1{,}067\,\frac{Btu}{h*ft^2*°\mathrm{F}}$

(b) $5{,}182\,\frac{Btu}{h*ft^2*°\mathrm{F}}$

(c) $6{,}738\,\frac{Btu}{h*ft^2*°\mathrm{F}}$

(d) $12{,}784\,\frac{Btu}{h*ft^2*°\mathrm{F}}$

9.5 PROBLEM 5 – HEAT EXCHANGER

Air enters a heat exchanger at 10 lbs/hr and water enters a heat exchanger at an unknown mass flow rate. The entering temperature of the air is 200 °F and the air leaves at 150 °F. Water enters the heat exchanger at 75 °F and leaves at 100 °F. What is the mass flow rate of the water?

(a) 4.8 lbm/hr

(b) 12.1 lbm/hr

(c) 18.9 lbm/hr

(d) 24.0 lbm/hr

9.6 PROBLEM 6 – HEAT EXCHANGER

A new solar hot water heating system has two loops, (1) solar hot water loop and a (2) domestic hot water loop. The loops are separated by a plate frame heat exchanger. If the solar hot water fluid enters the heat exchanger at 160 °F and leaves at 130 °F and the domestic hot water enters at 80 °F and leaves at 120 °F, then what is the LMTD? Assume a parallel flow heat exchanger.

(a) 30 °F

(b) 31.3 °F

(c) 33.7 °F

(d) 35 °F

9.7 PROBLEM 7 – HEAT EXCHANGER

A new heat exchanger uses steam to heat water. Steam 50 lb/min (0.378 kg/s) enters the heat exchanger at 20 psia (137.895 KPa), fully saturated and leaves as a saturated liquid. What mass flow rate of water can be expected to provide a 40 °F (22.22 °F) change in temperature?

(a) 1,000 lb/min (7.56 kg/s)

(b) 1,200 lb/min (9.07 kg/s)

(c) 1,300 lb/min (9.83 kg/s)

(d) 1,400 lb/min (10.58 kg/s)

9.8 PROBLEM 8 - CONDENSATION

A closed feedwater heater is used to heat incoming 23.9 °C water. Saturated steam at 1,723 kPa is used to heat the incoming water. Saturated steam enters the feedwater heater at a mass flow rate of 10% of the incoming water mass flow rate. The condensate leaves as a saturated liquid. What is the outgoing temperature of the water?

$$c_{p,water} = 4.18 \frac{kJ}{kg - °K}$$

(A) 61 °C

(B) 70 °C

(C) 78 °C

(D) 83 °C

9.9 PROBLEM 9 – NTU METHOD

Water enters a heat exchanger at 1 kg/s and 200 °K. Air enters the heat exchanger at 0.5 kg/s at an unknown temperature. The overall heat transfer coefficient for the heat exchanger is 150 W/m^2-°K. If the NTU value is 1.2, then what is the area of heat exchange?

(a) 1 m^2

(b) 2 m^2

(c) 3 m^2

(d) 4 m^2

9.10 PROBLEM 10 - TRANSIENT

A motor is modeled as a box with dimensions, 3' x 5' x 1'. The motor is turned off and is at a temperature of 300 °K. Assume the properties for the motor as shown below. Assume the motor is raised with an open platform. The motor is located in a room that is at a temperature of 300 °K. The convective heat transfer coefficient is also shown below. What is the Biot number?

$$k_{motor} = 30\frac{W}{m - °K}; p_{motor} = 5{,}000\frac{kg}{m^3}; h_{air} = 15\frac{W}{m^2 - °K}$$

(a) 0.01

(b) 0.10

(c) 0.44

(d) 1.21

9.11 PROBLEM 11 - RADIATION

A motor is modeled as a box with dimensions, 1 m x 2 m x 0.5 m. The motor is turned off and is at a temperature of 300 °K. Assume the motor has an emissivity of 0.8. Assume the motor is raised with an open platform. The motor is located in a room that is at a temperature of 300 °K. What is the heat loss from the motor due to radiation?

$$\sigma = 0.1713\ x\ 10^{-8}\left(\frac{Btu}{h * ft^2 * R^4}\right)\ or\ \ 5.67\ x\ 10^{-8}\left(\frac{W}{m^2 * K^4}\right);$$

(a) 100 W

(b) 900 W

(c) 1,300 W

(d) 2,600 W

9.12 PROBLEM 12 – RADIATION & CONVECTION

A 4” diameter copper pipe carries hot water from the heater to the equipment. The surface temperature of the pipe is 200 °F. The ambient temperature is 75 °F. Assume that the convective heat transfer coefficient is 1.85 Btu/(hr-ft^2-°F). What is the total heat loss per 100 feet of pipe? Assume black body radiation.

$$\sigma = 0.1713\ x\ 10^{-8}\left(\frac{Btu}{h * ft^2 * R^4}\right)\ or\ \ 5.67\ x\ 10^{-8}\left(\frac{W}{m^2 * K^4}\right);$$

(a) 24,300 Btuh

(b) 31,900 Btuh

(c) 43,600 Btuh

(d) 67,100 Btuh

10.0 SOLUTIONS

10.1 SOLUTION 1 - CALCULATE OVERALL HEAT TRANSFER COEFFICIENT

The overall heat transfer coefficient is found by first converting all values to R-values.

$$R_{oair} = \frac{1}{h_{oair}} = \frac{1}{3} = 0.33 \frac{hr * ft^2 * °F}{Btu}$$

$$R_{ins} = \frac{t}{k_{ins}} = \frac{2}{0.3} = 6.67 \frac{hr * ft^2 * °F}{Btu}$$

$$R_{brick} = \frac{t}{k_{brick}} = \frac{8}{9} = 0.89 \frac{hr * ft^2 * °F}{Btu}$$

Sum up all R-values that are in series.

$$R_{total} = R_{oair} + R_{brick} + R_{ins} + R_{gyp}$$

$$R_{total} = 0.33 + 0.89 + 6.67 + 0.8$$

$$R_{total} = 8.69 \frac{hr * ft^2 * °F}{Btu}$$

The overall heat transfer coefficient is simply the inverse of the total resistance.

$$U_{overall} = \frac{1}{R_{total}} = \frac{1}{8.69}$$

$$U_{overall} = 0.115 \frac{Btu}{hr * ft^2 * °F}$$

The correct answer is most nearly, **(a) 0.12.**

SI Unit Conversions

87 °F	=	30.556 °C	0.3 (Btu-in)/(h-ft^2-°F)	**=**	43.268 (W-mm)/(m^2-°C)
3 Btu/hr-ft^2-°F	=	17.035 W/(m^2-°C)	9 (Btu-in)/(h-ft^2-°F)	=	1298.05 (W-mm)/(m^2-°C)
8 in	=	203.2 mm	75 °F	=	23.889 °C
2 in	=	50.8 mm			
0.8 ft^2-°F/(Btu/h)	=	0.14088 (m^2-°C)/W			

$$R_{oair} = \frac{1}{h_{oair}} = \frac{1}{17.035} = 0.0587 \frac{hr * ft^2 * °F}{Btu}$$

$$R_{ins} = \frac{t}{k_{ins}} = \frac{50.8\ mm}{43.268} = 1.17\frac{m^2 * °C}{W}$$

$$R_{brick} = \frac{t}{k_{brick}} = \frac{203.2\ mm}{1{,}298.05} = 0.1565\frac{m^2 * °C}{W}$$

Sum up all R-values that are in series.

$$R_{total} = R_{oair} + R_{brick} + R_{ins} + R_{gyp}$$

$$R_{total} = .0587 + .1565 + 1.17 + 0.14088$$

$$R_{total} = 1.526\frac{m^2 * °C}{W}$$

The overall heat transfer coefficient is simply the inverse of the total resistance.

$$U_{overall} = \frac{1}{R_{total}} = \frac{1}{1.526}$$

$$U_{overall} = 0.655\frac{W}{m^2 * °C}$$

10.2 SOLUTION 2 - CALCULATE OVERALL HEAT TRANSFER COEFFICIENT

The overall heat transfer coefficient is found by first converting all values to R-values.

$$R_{oair} = \frac{1}{h_{oair}} = \frac{1}{2} = 0.5\ \frac{hr * ft^2 * °F}{Btu}$$

$$R_{ins} = \frac{t}{k_{ins}} = \frac{1.5}{0.15} = 10\frac{hr * ft^2 * °F}{Btu}$$

$$R_{conc} = \frac{t}{k_{conc}} = \frac{8}{1.5} = 5.33\ \frac{hr * ft^2 * °F}{Btu}$$

$$R_{iair} = \frac{1}{h_{iair}} = \frac{1}{1.7} = 0.59\ \frac{hr * ft^2 * °F}{Btu}$$

Sum up all R-values that are in series.

$$R_{total} = R_{oair} + R_{conc} + R_{ins} + R_{gyp} + R_{iair}$$

$$R_{total} = 0.5 + 5.33 + 10 + 0.8 + 0.59$$

$$R_{total} = 17.22\ \frac{hr * ft^2 * °F}{Btu}$$

The overall heat transfer coefficient is simply the inverse of the total resistance.

$$U_{overall} = \frac{1}{R_{total}} = \frac{1}{17.22}$$

$$U_{overall} = 0.058 \frac{Btu}{hr * ft^2 * °\text{F}}$$

The correct answer is most nearly, **(a) 0.06.**

SI Unit Conversions

87 °F	=	30.556 °C	0.15 (Btu-in)/(h-ft²-°F)	=	21.634 (W-mm)/(m²-°C)
2 Btu/hr-ft²-°F	=	11.357 W/(m²-°C)	1.5 (Btu-in)/(h-ft²-°F)	=	216.34 (W-mm)/(m²-°C)
8 in	=	203.2 mm	75 °F	=	23.889 °C
1-1/2 in	=	38.1 mm	0.8 ft²-°F/(Btu/h)	=	0.14088 (m²-°C)/W
5/8 in	=	15.875 mm	1.7 Btu/hr-ft²-°F	=	9.653 W/(m²-°C)

$$R_{oair} = \frac{1}{h_{oair}} = \frac{1}{2} = 0.5 \frac{hr * ft^2 * °\text{F}}{Btu}$$

$$R_{ins} = \frac{t}{k_{ins}} = \frac{1.5}{0.15} = 10 \frac{hr * ft^2 * °\text{F}}{Btu}$$

$$R_{conc} = \frac{t}{k_{conc}} = \frac{8}{1.5} = 5.33 \frac{hr * ft^2 * °\text{F}}{Btu}$$

$$R_{iair} = \frac{1}{h_{iair}} = \frac{1}{1.7} = 0.59 \frac{hr * ft^2 * °\text{F}}{Btu}$$

Sum up all R-values that are in series.

$$R_{total} = R_{oair} + R_{conc} + R_{ins} + R_{gyp} + R_{iair}$$

$$R_{total} = 0.5 + 5.33 + 10 + 0.8 + 0.59$$

$$R_{total} = 17.22 \frac{hr * ft^2 * °\text{F}}{Btu}$$

The overall heat transfer coefficient is simply the inverse of the total resistance.

$$U_{overall} = \frac{1}{R_{total}} = \frac{1}{17.22}$$

$$U_{overall} = 0.058 \frac{Btu}{hr * ft^2 * °\text{F}}$$

10.3 SOLUTION 3 - CALCULATE THE REYNOLDS NUMBER

250 GPM (15.7725 L/s) of 120°F (48.9 °C) water flows through a 4" (101.6 mm) inner diameter pipe that is located in the ceiling of a building. Assume the water in the pipe is not under extreme pressure. Calculate the Reynolds number of the fluid inside the pipe.

The Reynolds number is found by the following equation

$$Re = \frac{D * v}{\nu}$$

Where,

$$D = 4\ in * \frac{1\ ft}{12\ in} = 0.33\ \text{ft}, \qquad Pipe\ Diameter$$

$$\dot{V} = 250\ gpm * \frac{8.33\ lbm}{gal} * \frac{ft^3}{61.3\ lbm} * \frac{1\ min}{60\ sec} = 0.566 \frac{ft^3}{sec}, \qquad volumetric\ flow$$

$$v = \frac{\dot{V}}{A} = \frac{0.566 \frac{ft^3}{sec}}{\pi * (\frac{0.33\ ft}{2})^2} = 6.5\ \frac{ft}{s}, \qquad velocity$$

Use the tables in the *NCEES FE Reference Handbook* to find the kinematic viscosity of water at 120°F. Because the properties of water do not change drastically under minor pressure differences, water at atmospheric pressure can be used as a close estimate.

$$\nu = 0.609 \text{ x } 10^{-5}\ \frac{ft^2}{s}, \quad kinematic\ viscosity$$

Solve for Reynolds number:

$$Re = \frac{D * v}{\nu} = \frac{(0.33\ ft) * 6.5\ \frac{ft}{s}}{0.609 \text{ x } 10^{-5}\ \frac{ft^2}{s}} = 355{,}775$$

The correct answer is most nearly, **(c) 355,745.**

10.4 SOLUTION 4 - CALCULATE THE CONVECTIVE HEAT TRANSFER COEFFICIENT

This is a forced convective heat transfer problem through a pipe and the objective is to solve for the convective heat transfer, h_{conv}. Use the Reynolds number in Problem 3 to determine whether the flow is laminar or turbulent. Because the Reynolds number is greater than 2,300, the flow is turbulent

The pipe is within the ceiling, exposed to ambient temperatures, meaning the temperature of the pipe wall temperature is lower than the fluid temperature.

Therefore, Equation 4.1.2 can be used to solve for the convective heat transfer coefficient, h_{conv}.

$$Nu = \frac{D * h_{conv}}{k} = .023 * Re^{.8} * Pr^{.3}$$

Solve for the Nusselt number, Nu, by first finding the Prandtl number. The Prandtl number, Pr, is calculated with the following equations.

$$Pr = \frac{\mu * c_p}{k}$$

Use the tables in the *NCEES FE Handbook* to find the properties of water. Because the properties of water do not change drastically under minor pressure differences, water at atmospheric pressure can be used as a close estimate.

At 120°F, water has the following properties:

$$\mu = 1.36\ \frac{lbm * ft}{hr}, dynamic\ viscosity$$

$$c_p = 0.999\ \frac{Btu}{lbm * °F}, specific\ heat$$

$$k = 0.378\ \frac{Btu * ft}{h * ft^2 * °F}, thermal\ conductivity$$

Therefore,

$$Pr = \frac{\mu * c_p}{k} = \frac{1.36\ \dfrac{lbm * ft}{hr} * 0.999\ \dfrac{Btu}{lbm * °F}}{0.378\ \dfrac{Btu * ft}{h * ft^2 * °F}} = 3.6$$

Solve for the convective heat transfer:

$$h_{conv} = \frac{.023 * Re^{.8} * Pr^{.3} * k}{D} = \frac{.023 * 355{,}745^{.8} * 3.6^{.3} * 0.378\ \dfrac{Btu * ft}{h * ft^2 * °F}}{0.33\ ft}$$

$$h_{conv} = 1{,}067 \frac{Btu}{h * ft^2 * °F}$$

The correct answer is most nearly, **(a) 1,067.**

10.5 SOLUTION 5 – HEAT EXCHANGER

Air enters a heat exchanger at 10 lbs/hr and water enters a heat exchanger at an unknown mass flow rate. The entering temperature of the air is 200 °F and the air leaves at 150 °F. Water enters the heat exchanger at 75 °F and leaves at 100 °F. What is the mass flow rate of the water?

In order to complete this problem, you must conduct an energy balance on the hot and cold sides of the heat exchanger.

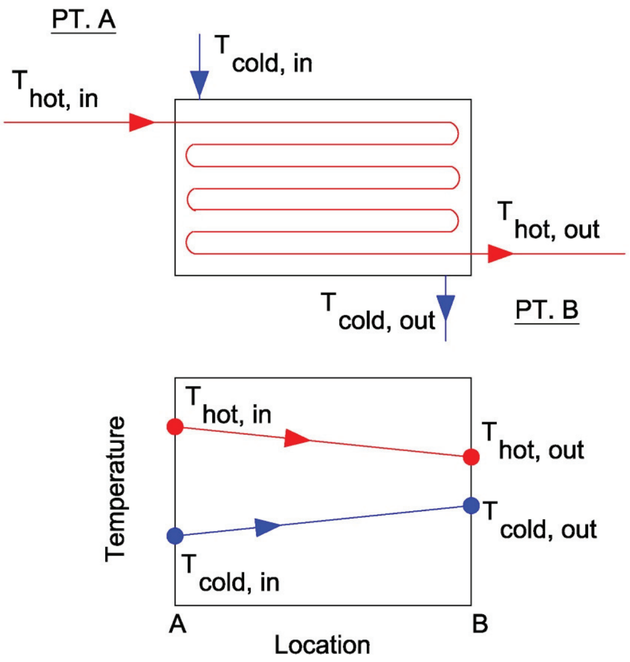
$$Hot \rightarrow Q_{hot} = \dot{m} * c_{p,air} * (T_{in} - T_{out}) = 10 \frac{lbm}{hr} * 0.24 \frac{Btu}{lbm\ °F} * (200 - 150) = \ 120 \frac{Btu}{hr}$$

$$Cold \rightarrow Q_{cold} = \dot{m} * c_{p,water} * (T_{out} - T_{in}) = x \frac{lbm}{hr} * 1.00 \frac{Btu}{lbm\ °F} * (100 - 75) = \ 120 \frac{Btu}{hr}$$

$$x = 4.8 \frac{lbm}{hr}$$

The correct answer is most nearly, **(a)** $\mathbf{4.8\ \frac{lbm}{hr}}$

10.6 SOLUTION 6 – HEAT EXCHANGER


PT. A
T hot, in
T cold, in
T hot, out
T cold, out
PT. B
Temperature
T hot, in
T hot, out
T cold, out
T cold, in
A
B
Location

$$LMTD = \frac{\Delta T_a - \Delta T_b}{\ln(\frac{\Delta T_a}{\Delta T_b})}$$

$$\Delta T_a = temperature\ difference\ at\ entrance = 160 - 80 = 80\ F$$

$$\Delta T_b = temperature\ difference\ at\ exit = 130 - 120 = 10\ F$$

$$LMTD = \frac{80 - 10}{\ln(\frac{80}{10})} = 33.7\ F$$

The correct answer is most nearly, **(c) 33.7 F.**

11.7 SOLUTION 7 – HEAT EXCHANGER

A new heat exchanger uses steam to heat water. Steam (50 lb/min) enters the heat exchanger at 20 psia, fully saturated and leaves as a saturated liquid. What mass flow rate of water can be expected to provide a 40 degree change in temperature?

This problem is an energy balance type problem. The energy from the steam is transferred to water.

Steam Available Energy

$$Q = \dot{m} * h_{fg}$$

From the steam tables, find the latent heat of vaporization of 20 psia steam.

$$h_{fg} = 960.05 \frac{Btu}{lb}$$

Energy Balance Equation

$$50 \frac{lb}{min} * 960.05 \frac{Btu}{lb} = \dot{m} * c_p * \Delta T$$

$$c_p = 1.0 \frac{Btu}{lbm * F}\ for\ water, refer\ to\ NCEES\ Mechanical\ PE\ Reference\ Handbook$$

$$50 \frac{lb}{min} * 960.05 \frac{Btu}{lb} = \dot{m} * 1.0 * 40$$

$$\dot{m} = 1200\ lb/min$$

The correct answer is most nearly**, (b) 1,200 lb/min.**

10.8 SOLUTION 8 - CONDENSATION

A closed feedwater heater is used to heat incoming 23.9 °C water. Saturated steam at 1,723 kPa is used to heat the incoming water. Saturated steam enters the feedwater heater at a mass flow rate of 10% of the incoming water mass flow rate. The condensate leaves as a saturated liquid. What is the outgoing temperature of the water?

$$c_{p,water} = 4.18 \frac{kJ}{kg - °K}$$

In this problem you should conduct an energy balance at the feedwater heater, where the energy lost by the steam is gained by the water.

$$Q_{steam} = Q_{water}$$

$$\dot{m}_{steam}(h_{in} - h_{out}) = \dot{m}_{water} * c_p * (T_{out} - T_{in})$$

Reorganizing the variables.

$$\frac{\dot{m}_{steam}(h_{in} - h_{out})}{\dot{m}_{water} * c_p} + T_{in} = T_{out}$$

But you also know that the ratio of the mass flow rate of steam to water is 0.1.

$$0.1 * \frac{\left(h_{in,steam} - h_{out,steam}\right)}{c_{p,water}} + T_{in} = T_{out}$$

Input your enthalpy values from the steam tables and then input your heat capacity and temperature values.

$$0.1 * \frac{(1{,}921\ kJ/kG)}{4.18 \frac{kJ}{kg - K}} + 23.9\ °C = T_{out}$$

$$T_{out} = 69.9\ °C$$

The correct answer is most nearly, **(b) 70 °C**.

10.9 SOLUTION 10 - NTU METHOD

Water enters a heat exchanger at 1 kg/s and 200 °K. Air enters the heat exchanger at 0.5 kg/s at an unknown temperature. The overall heat transfer coefficient for the heat exchanger is 150 W/m^2-°K. If the NTU value is 1.2, then what is the area of heat exchange?

$$NTU = \frac{UA}{\dot{m}_{min} * (c_{p,min})}$$

You need to use the heat capacity of air found in the *NCEES FE Reference Handbook*. The minimum value of the heat capacity multiplied by mass flow rate is for air. The heat capacity of water is larger than the heat capacity of air and the mass flow rate of water is also larger. You can also convert the Joules per second to watts, for use in the above equation.

$$\dot{m}_{air} * (c_{p,air}) = 0.5\frac{kg}{s} * 1.0\frac{kJ}{kg-K} = 0.5\ \frac{kJ}{K} = 500\frac{J}{s-K} = 500\frac{W}{K}$$

$$\dot{m}_{water} * (c_{p,water}) = 1.0\frac{kg}{s} * 4.8\frac{kJ}{kg-K} = 4.8\ \frac{kJ}{K} = 4{,}800\frac{J}{s-K} = 4{,}800\frac{W}{K}$$

Finally plug in the values for air to find the heat exchange area.

$$1.2 = \frac{150\frac{W}{m^2-°K} * A}{500\frac{W}{°K}}$$

$$A = 4\ m^2$$

The correct answer is most nearly, **(d) 4 m$^{2.}$**

10.10 SOLUTION 10 – TRANSIENT

A motor is modeled as a box with dimensions, 3' x 5' x 1'. The motor is turned off and is at a temperature of 300 °K. Assume the properties for the motor as shown below. Assume the motor is raised with an open platform. The motor is located in a room that is at a temperature of 300 °K. The convective heat transfer coefficient is also shown below. What is the Biot number?

$$k_{motor} = 30\frac{W}{m-K}; p_{motor} = 5{,}000\frac{kg}{m^3}; h_{air} = 15\frac{W}{m^2-K}$$

Use the Biot equation below and solve for the volume and surface area of the box. But remember to convert from feet to meters.

$$3' = 0.914\ m; 5' = 1.524\ m; 1' = 0.305\ m$$

$$Bi = \frac{hV}{kA}$$

$$Bi = \frac{15\frac{W}{m^2-K} * (0.914m * 1.524m * 0.305m)}{30\frac{W}{m-K} * (2 * (0.914m * 1.524m + 1.524m * 0.305m + 0.914m * 0.305m)}$$

$$Bi = 0.099$$

The correct answer is most nearly, **(a) 0.01.**

10.11 SOLUTION 11 - RADIATION

A motor is modeled as a box with dimensions, 1 m x 2 m x 0.5 m. The motor is turned off and is at a temperature of 300 °K. Assume the motor has an emissivity of 0.8. Assume the motor is raised with an open platform. The motor is located in a room that is at a temperature of 300 °K. What is the heat loss from the motor due to radiation?

For this question use the radiative heat transfer equation.

$$Q = \varepsilon\sigma A T^4$$

$$Q = radiation\ heat\ (Btuh\ or\ W); A = area\ (ft^2\ or\ m^2);$$

$$\sigma = 0.1713\ x\ 10^{-8}\left(\frac{Btu}{h * ft^2 * R^4}\right)\ or\ \ 5.67\ x\ 10^{-8}\left(\frac{W}{m^2 * K^4}\right);$$

$$T = \ surface\ temperature\ (°R\ or\ °K\ )$$

$$Q = 0.8 * \ 5.67\ x\ 10^{-8}\left(\frac{W}{m^2 * K^4}\right) * (2 * (1 * 2 + 2 * .5 + 1 * .5) * 300^4$$

$$Q = 2{,}572\ W$$

The correct answer is most nearly, **(d) 2,600 W.**

10.12 SOLUTION 12 – RADIATION & CONVECTION

A 4" diameter copper pipe carries hot water from the heater to the equipment. The surface temperature of the pipe is 200 °F. The ambient temperature is 75 °F. Assume that the convective heat transfer coefficient is 1.85 Btu/(hr-ft^2-°F). What is the total heat loss per 100 feet of pipe? Assume black body radiation.

The main concept to understand in this problem is that the total heat loss from a pipe includes both radiation and convection. Since the problem does not indicate that the pipe is physically touching anything, then we can assume that there is no conduction taking place.

$$Q_{conv} = h_{conv} * A_{surface} * (T_{surface} - T_{ambient})$$

$$Q_{conv} = 1.85\frac{Btu}{hr - ft^2 - °\mathrm{F}} * A_{surface} * (200 - 75°\mathrm{F})$$

$$A_{surface} = 100\ ft * \left(\pi * \frac{4"}{12"/ft}\right) = 104.72\ ft^2$$

$$Q_{conv} = 1.85\frac{Btu}{hr - ft^2 - °\mathrm{F}} * 104.72\ ft^2 * (200 - 75°\mathrm{F}) = 24{,}216\ Btuh$$

$$\varepsilon = 1\ (black\ body\ radiation)$$

$$Q_{rad} = \varepsilon\sigma * A_{surface}(T^4{}_{surface} - T^4{}_{ambient})$$

$$Q_{rad} = 1.714\ x\ 10^{-9} \frac{Btu}{hr - ft^2 - R^4} * 104.72\ ft^2 * ((200 + 460)^4 - (75 + 460)^4)$$

$$Q_{rad} = 19{,}353\ Btuh$$

$$Q_{total} = Q_{conv} + Q_{rad} = 24{,}216 + 19{,}353 = 43{,}569\ Btuh$$

The correct answer is most nearly, **(c) $\mathbf{43,600\ Btuh}$**

13 – Measurement, Instrumentations & Controls

MECHANICAL FE TEXTBOOK
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Section 13.0 – Measurement, Instrumentation & Controls

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1.0 INTRODUCTION

Measurement, Instrumentation and Controls accounts for approximately 5 to 8 questions on the Mechanical FE exam. This section covers the following topics, sensors, block diagrams, system response and measurement uncertainty. In the sensors topic, you must be familiar with the types of sensors that are used to measure, strain, temperature and pressure, since these are the properties that are most commonly used in mechanical engineering, unlike the pH and chemical sensors which are also shown in the *NCEES FE Reference Handbook.* Block diagrams are used to analyze a control system that consists of different functions in graphical format. You must be able to read and simplify these block diagrams for the FE exam. The system response topic focuses on how control systems respond to various inputs like step, ramp and parabolic inputs. This topic will teach you how to determine if a control system will be stable with the Routh test and how to determine the response error.

Section 13.0 Measurement, Instrumentation and Controls (5 to 8 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
13A	Section 2.0	Sensors and transducers
13B	Section 3.0	Control systems (e.g., feedback, block diagrams)
13C	Section 4.0	Dynamic system response
13D	Section 5.0	Measurement Uncertainty (e.g., error propagation, accuracy, precision, significant figures)
	Section 6.0	Practice Exam Problems

2.0 SENSORS & TRANSDUCERS

The sensors that you need to know for the FE exam are those sensors that measure the difference or change in a specific variable, like temperature. For example, a thermocouple responds to changes in temperature by expanding or contracting. A transducer converts that expansion or contraction into another format like degrees. Typically, the physical measurement, which is measured by the sensor will change a circuit's resistance, which the transducer will use to measure changes in voltage, if assuming that current remains constant.

For the FE exam, you just need to focus on the sensors and transducers used for measuring temperature, strain and pressure because these measurements are shown in the *NCEES FE Reference Handbook*.

2.1 TEMPERATURE SENSORS

There are two main types of temperature sensors that you need to know for the FE exam, (1) Thermocouple and (2) Resistance Temperature Detector.

The thermocouple uses a composite of two dissimilar metals that creates a voltage as a function of temperature. As the temperature increases, the thermoelectric effect occurs and this effect creates a voltage difference between the two sides of the composite. There are wires that are connected to the opposite sides of the composite that measure the voltage. The voltage increases as the temperature of the composite temperature increases.

The resistance temperature detector works off the basic concept that as a metal increases in temperature its resistance decreases. Thus if the current is maintained constant, then the voltage drop through a metal will decrease as the resistance increases, which increases when the temperature decreases. When the temperature increases, the resistance decreases, which decreases the voltage drop.

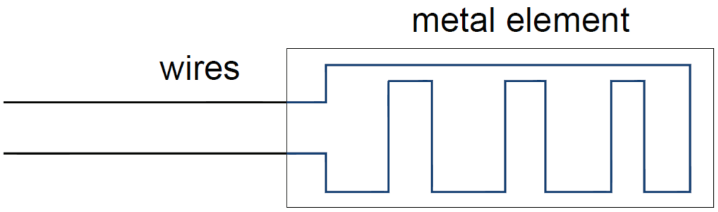


Figure 1: In a resistance temperature sensor, the metal element becomes more conductive as its temperature increases. This reduces the resistance which decreases the voltage drop.

2.2 STRAIN GAUGE

A strain gauge is similar to the resistance temperature detector. A strain gauge consists of a foil-like element with a circuit running through it. This element is placed on a component that will undergo strain. As the component lengthens, the strain gauge will lengthen as well. This will cause the circuit wires within the strain gauge to become narrow, which will increase the resistance of the strain gauge.

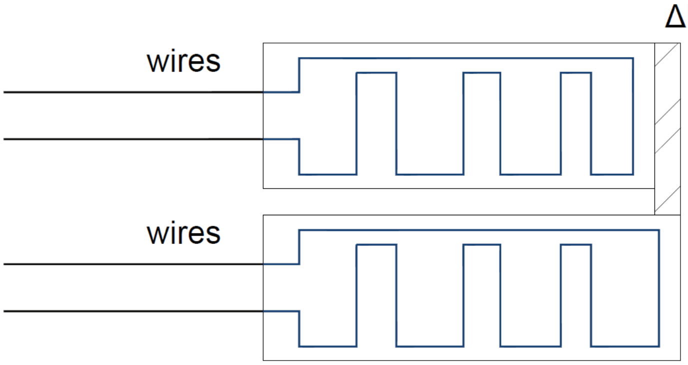


Figure 2: A strain gauge will measure the change in length of a component by measuring the change in resistance of the wires within the strain gauge.

The wires are connected to a Wheatstone bridge to measure the change in resistance. The change in resistance will correspond to a change in length of the component. This change in length can then be used to calculate strain. The ratio between the change in length of the strain gauge and the measured change in resistance is shown as the gauge factor. This gauge factor is pre-measured for each type of strain gauge. Typically the gauge factor is around 2.

$$Gauge\ Factor = G = \frac{\Delta R/_{R_0}}{\Delta L/_{L_0}} = \frac{\Delta R/_{R_0}}{\varepsilon}$$

$$\varepsilon = strain; \Delta R = change\ in\ resistance\ (\Omega); R_0 = original\ resistance\ (\Omega)$$

$$\Delta L = change\ in\ length\ (m); R_0 = original\ length\ (m); \varepsilon = strain$$

The strain can be solved by rearranging the gauge factor equation.

$$\varepsilon = G * {}^{\Delta R}/_{R_0}$$

Since, the gauge factor is known, you just need to solve for the difference in resistance of the strain gauge. The change in resistance of the strain gauge is indirectly measured through a Wheatstone bridge. The voltage across the Wheatstone bridge is measured as shown in the following diagram. A known voltage is input into the Wheatstone bridge and the output voltage is measured.

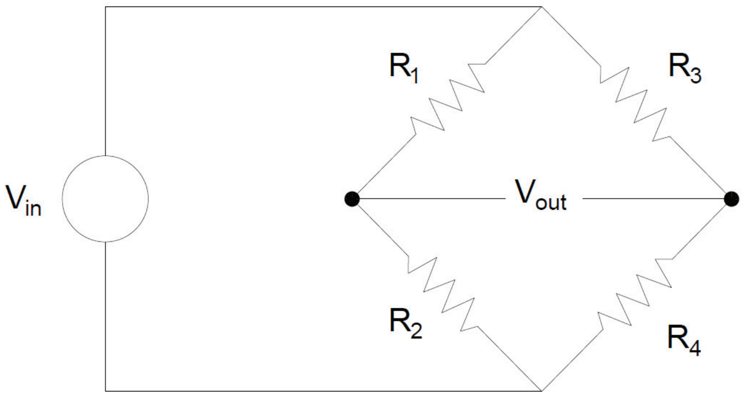


Figure 3: In a Wheatstone bridge all of the four locations have the same resistance value.

In a strain gauge, one of the resistors is replaced by the strain gauge. The strain gauge resistance changes as the strain in the component changes. Then the output voltage is measured and the change in resistance is found with the below relationship.

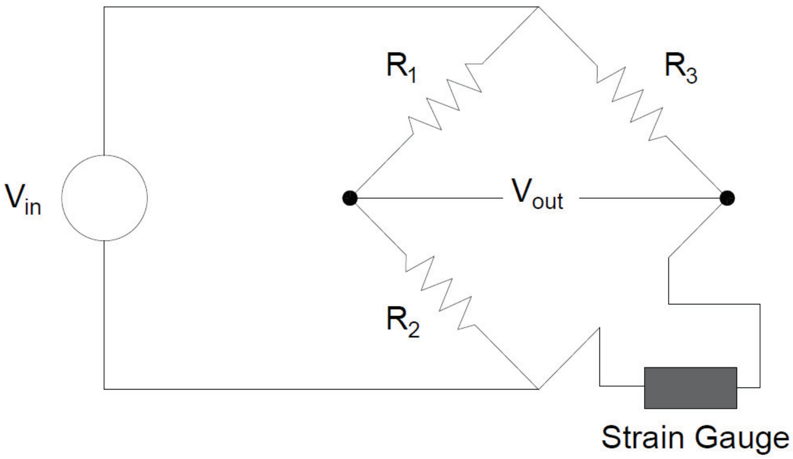


Figure 4: In this figure there is a strain gauge in one of the four locations within the Wheatstone bridge. This is called a quarter bridge. You will notice that in the following equations there is a constant "4".

$$Strain\ Gauge\ Resistance = \frac{\Delta R}{R_O} = 4\frac{V_{out}}{V_{in}}$$

$$R_O = strain\ gauge\ original\ resistance$$

All the resistances in a Wheatstone bridge are equal to one another.

$$R_O = R_1 = R_2 = R_3 = R_4$$

You can then plug in the values into the original equation to find the strain.

$$\varepsilon = G * {}^{\Delta R}\!/_{R_0} = G * 4\, {}^{V_{out}}\!/_{V_{in}}$$

Sometimes, you must apply multiple strain gauges to a single component to get a more accurate measurement of strain. When this is done, the equation changes as shown below for the different configurations.

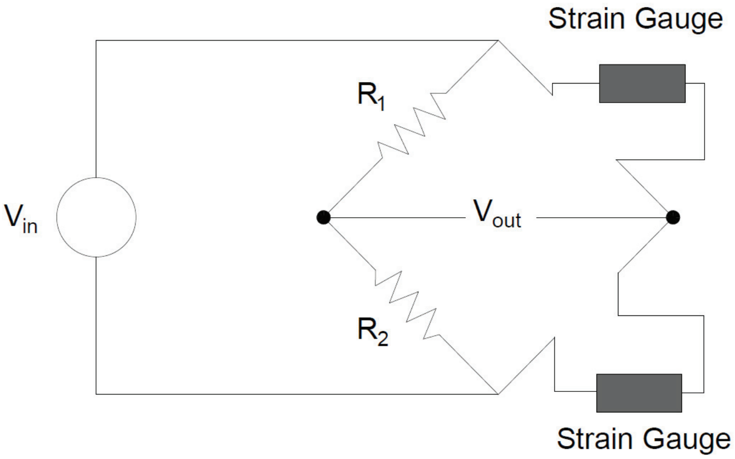

Strain Gauge
R1
Vin
Vout
R2
Strain Gauge

Figure 5: This figure shows strain gauges in two locations of the Wheatstone bridge. This is called a half bridge. You will notice that the constant in the equations below changes from "4" to "2".

$$Strain\ Gauge\ Resistance = \frac{\Delta R}{R_O} = 2\frac{V_{out}}{V_{in}}$$

$$\varepsilon = G * {}^{\Delta R}\!/_{R_0} = G * 2\, {}^{V_{out}}\!/_{V_{in}}$$

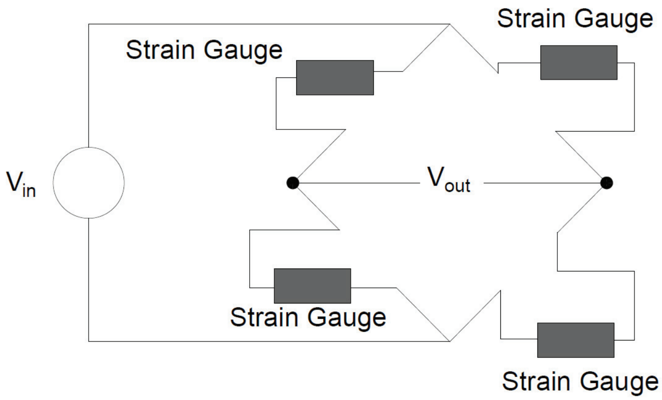

Strain Gauge
Strain Gauge
Vin
Vout
Strain Gauge
Strain Gauge

Figure 6: This figure shows a Wheatstone bridge that has strain gauges in each of the four locations. This is called a full bridge and you will notice that there is no constant in the below equations.

$$Strain\ Gauge\ Resistance = \frac{\Delta R}{R_O} = \frac{V_{out}}{V_{in}}$$

$$\varepsilon = G * {}^{\Delta R}/_{R_0} = G * {}^{V_{out}}/_{V_{in}}$$

2.3 PRESSURE SENSORS

There are two main types of pressure sensors that you should know, (1) Piezoelectric and (2) strain gauge. The strain gauge method uses the previous method to strain as a function of pressure. Then a calculator will convert the strain to a pressure value. The piezoelectric method uses a ceramic or quartz material that strains as a function of pressure. As the material changes shape, it generates an electric field that is measured via wiring and a voltmeter.

There are two different types of pressure scales, (1) absolute pressure and (2) gauge pressure. These two pressure scales differ by their 0 reference point. Gauge pressures have a 0-reference point as 1 atm. Thus 0 psig, where the g indicates gauge pressure, is equal to 1 atmospheric or 14.7 psia, where the "a" indicates absolute pressure. Most real world applications encountered by practicing engineers will have pressures indicated in gauge pressure. These include pressures measured at the discharge and intake of pumps and fans and the pressures measured at other pieces of equipment like heat exchangers, chillers and cooling towers. The relationship between gauge and atmospheric pressure is shown with the following equation and figure.

$$USCS\ Units \rightarrow P_{abs}[psi] = P_{gauge}[psi] + 14.7\ psi$$

$$SI\ Units \rightarrow P_{abs}[kPa] = P_{gauge}[kPa] + 101.325\ [kPa]$$

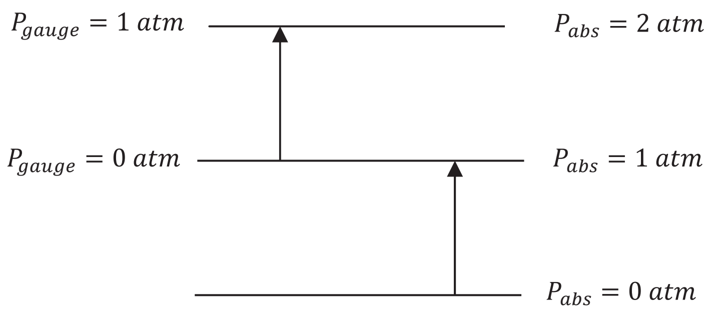


Figure 7: The relationship between gauge and absolute pressures

2.4 PH SENSORS

A typical pH sensor that will measure the acidity or basicity level of a liquid solution. The pH scale is a logarithmic scale that inversely indicates the amount of hydrogen ions in the liquid solution. The more hydrogen ions in a solution will result in a lower pH. A low pH is indicative of a basic solution. The less hydrogen ions in a solution will result in a higher pH. A high pH is indicative of an acidic solution.

$$Acidic \rightarrow ph > 7 \rightarrow More\ H^{+}\ \&\ Less\ OH^{-}$$

$$Basic \rightarrow ph < 7 \rightarrow Less\ H^{+}\ \&\ More\ OH^{-}$$

Another way to look at pH is to look at the hydroxide ions (OH^{-}), which is the counter ion to the hydrogen ions. The *NCEES FE Reference Handbook* has the pH equation shown in its Chemistry section, which is also shown below.

$$pH = \log_{10}\left(\frac{1}{[H^{+}]}\right) = -\log_{10}([H^{+}])$$

As you can see, pH is a function of the concentration hydrogen ions as opposed to the hydroxide ions. The concentration of these hydrogen ions will play a critical role in the pH sensor. A pH sensor will consist of two conductive electrodes that are connected on one side with a volt-meter. The electrodes will be placed in the liquid solution, with the solution serving as the completion of the circuit between the two conductive electrodes.

One of the electrodes is located in a glass and surrounded with a solution. The difference in hydrogen ions between the internal solution and the measured solution will cause hydrogen ions to either be attracted to the glass or repelled from the glass. This will create a difference in charge between one electrode and the other electrode. The difference in charge is also called a potential voltage. The volt-meter will measure the difference in voltage.

In order to get accurate measurements, this pH sensor is first calibrated with a buffer solution of a known pH, then the pH sensor is used to measure the desired solution's pH.

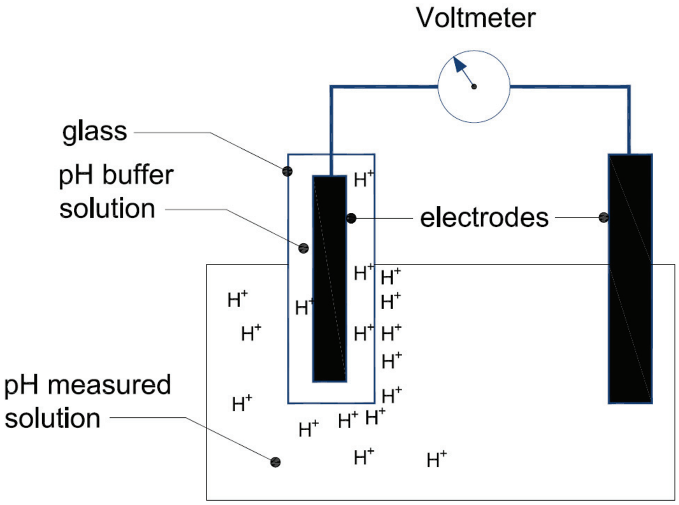


Figure 8: A pH sensor consists of two electrodes connected with a volt-meter. The volt-meter measures the difference in charge between the two electrodes. The pH sensor is first calibrated with a buffer solution on the left and then is used to measure the pH of the desired solution.

In the *NCEES FE Reference Handbook*, there is an equation that you may have to use to calculate the pH of a solution, given the measured voltage. The "S" value is a slope value that relates the difference in pH between the internal buffer pH and the pH of the measured solution. The E0 value is the pH of the internal buffer solution.

$$E_{el} = E^0 - S(pH_a - pH_i)$$

As an example, say a pH sensor measures 7.0 with a voltage of 10 mV and the pH sensor measures 4 with a voltage of 150 mV. The slope can be calculate that relates the voltage and pH.

$$Slope\ (S) = \frac{150\ mV - 10\ mV}{4 - 7} = -46.67 \frac{mV}{pH}$$

Now, you can calculate the pH for any voltage measurements. Let's say the voltage measured is -250 mV.

$$-250\ mV = 10\ mV + \left(-46.67 \frac{mV}{pH}\right)(pH_a - 7)$$

$$pH_a = 12.1$$

There is another equation that is not included in the handbook but it gives insight into the relationship between pH and temperature.

$$E_{el} = E^0 + 2.303\left(\frac{RT}{nF}\right)(\log[H^+])$$

This equation is called the Nernst equation. Since this equation is not given to you, you don't need to understand all the variables. You just need to know that this equation shows you the relationship between temperature and measured voltage. As the temperature increases, the same concentration of hydrogen ions will correspond to a larger voltage magnitude.

2.5 CHEMICAL CONSTITUENTS SENSORS

A chemical constituent sensor basically senses the concentration of a certain chemical. The sensor could measure the chemical concentration in air or liquid. The sensor will typically consist of a specific set of chemicals. These chemicals when exposed to the chemical that is being measured will have a chemical reaction. The chemical reaction will release energy in the form of an electric potential (electric field or potential). The sensor will be connected to a transducer which will take this voltage and connect it to a circuit, where the voltage signal can be used.

2.6 TRANSDUCERS

A transducer is used to convert the signal from the physical sensor into another signal. The transducer will take the elongation/contraction of the temperature sensor and convert that into an electrical signal. An electrical signal consists of a voltage & current. Either the voltage or current will react to changes in the sensor to produce a corresponding change in either voltage or current. A transducer will require a power source to provide voltage/current. The transducer could provide a constant current and the voltage through the sensor would change as a function of the changing resistance of the sensor. The voltage signal would be the signal that characterized the response of the physical sensor.

This voltage signal would then be sent to a filter, an amplifier and finally an analog or digital output that would provide the data to the user in a readable format.

3.0 CONTROL SYSTEMS

Block diagrams are used to diagrammatically show a control system. Each block contains either an input variable or a function that processes a variable.

3.1 BLOCK DIAGRAMS - FUNCTIONS IN SERIES

If two functions are in series, then the functions can be simplified by multiplying them together.

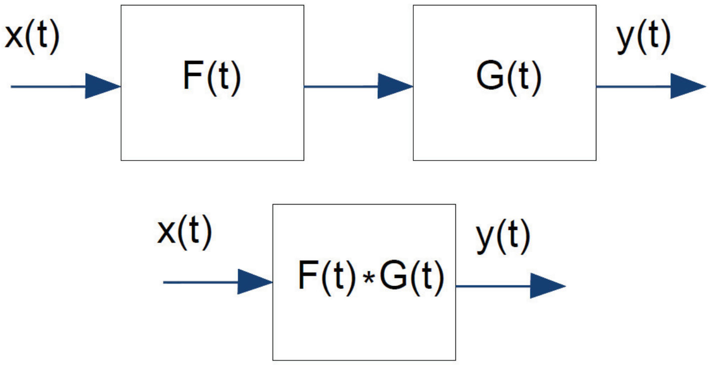


Figure 9: This block diagram shows an input function "x (t)" entering a control function called "F(t)". Then the output from F(t) enters another function called G (t). The final output is shown as y(t). The two functions, F(t) and G(t) can be simplified by multiplying the two functions together.

The block diagram of two functions in series can be simplified into the equation format shown below.

$$\frac{y(t)}{x(t)} = F(t) * G(t)$$

3.2 BLOCK DIAGRAMS - FUNCTIONS IN PARALLEL

If the functions are in parallel, then you can add the two functions.

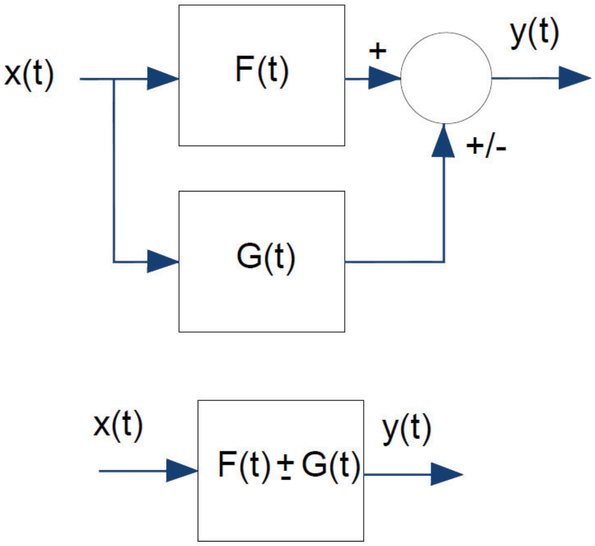


Figure 10: If two functions are arranged in parallel, then the two functions can be simplified by adding the two functions together.

The block diagram of two functions in parallel can be simplified into the equation format shown below.

$$\frac{y(t)}{x(t)} = F(t) \pm G(t)$$

3.3 BLOCK DIAGRAMS - FUNCTIONS WITH FEEDBACK LOOP

If the group of functions are arranged to have a feedback loop, then you can simplify the blocks as shown in the below process. In order to simplify the feedback loop, you need to assign an intermediary function B(t).

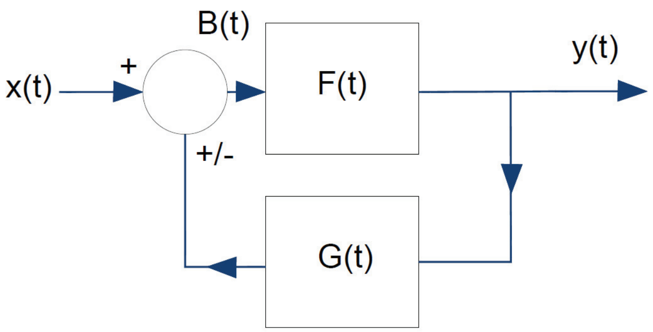


Figure 11: This figure shows a function x(t) that is processed by function F(t), but the output from F(t) is processed through G(t), which is fed back to the x(t). This is called a feedback loop.

The output is equal to B(t) multiplied by F(t). But you also know that B(t) is equal to y(t)x G(t) + x(t). Then substitute for B(t) into the first equation. The steps are shown below.

$$y(t) = B(t) * F(t);\ B(t) = x(t) \pm y(t)G(t)$$

$$y(t) = [x(t) \pm y(t)G(t)] * F(t);$$

$$y(t)(1 \pm F(t)G(t)) = x(t) * F(t)$$

This results in the simplification of the feedback loop as shown below.

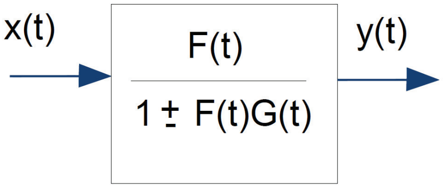


Figure 12: A feedback loop can be simplified by keeping the original function in the numerator and by adding the product of the original function and the feedback loop processing function.

$$\frac{y(t)}{x(t)} = \frac{F(t)}{(1 \pm F(t)G(t))}$$

4.0 DYNAMIC SYSTEM RESPONSE

There are two main system response skills that you need for the FE exam. First is determining whether or not the response of a function will be stable or unstable. This is accomplished through the Routh Test. The Routh test determines whether or not all of the roots of a function are negative and real and if they are all negative and real, then the function is stable. The second skill is calculating the steady state error response of a function given various inputs into that function.

4.1 ROUTH TEST

The Routh test is used to determine if all of the roots of a polynomial are negative and real. In order to complete this test, you must follow the below steps. ***The Routh Test was removed from the NCEES FE Handbook in the 10-01 version. You no longer need to understand this information, but has been kept in the book for completeness.***

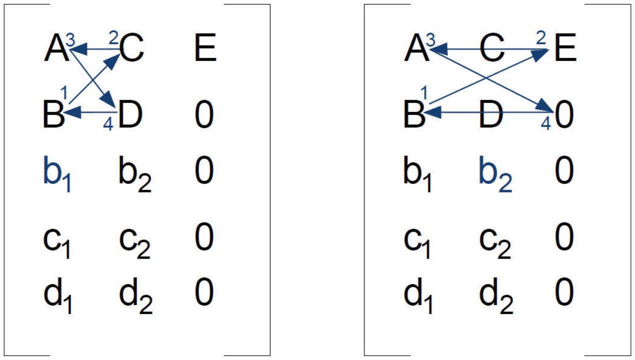
Step 1: Arrange the polynomial in descending order of power. In this example, x to the fourth is the highest power. If one of the powers is not available, still include it, but the coefficient should be 0.

$$Ax^4 + B_x^3 + Cx^2 + Dx + E = 0$$

Step 2: Fill out the top 2 rows of the Routh matrix. You fill out the matrix by starting with the top left cell and including the coefficient into the cell of the highest polynomial. Then input the next polynomial in descending order in the row below. Then go back up to the 1st row, but the second column. Then to the row below and so on and so on. If you have an odd number of polynomials, then the last value will be zero.

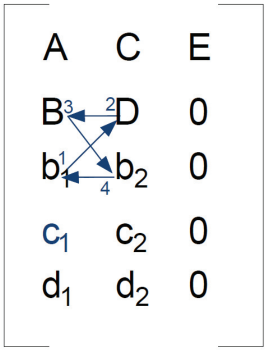
$$\begin{pmatrix} A & C & E \\ B & D & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \\ d_1 & d_2 & 0 \end{pmatrix}$$

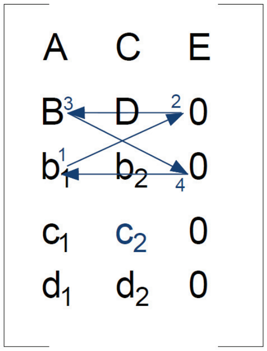
Step 3: Fill out the remaining rows. The total number of rows equals the highest polynomial plus 1. So in this example, the number of rows is 5. The remaining rows are filled out by following the pattern shown below. It is like a figure 8. The pattern is a figure 8 that starts at point 1 in the figure below and ends at point 4 below. This shows you how you solve for b1 and b2.



$$b_1 = \frac{B*C - A*D}{B}; b_2 = \frac{B*E - A*0}{B}$$

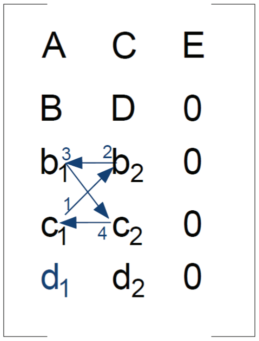
The next figure shows the same pattern but solving for c1 and c2.

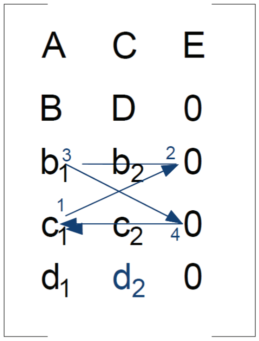

A C E
B D 0
b1 b2 0
c1 c2 0
d1 d2 0
1
2
3
4


A C E
B D 0
b1 b2 0
c1 c2 0
d1 d2 0
1
2
3
4

$$c_1 = \frac{b_1 * D - B * b_2}{b_1}; c_2 = \frac{b_1 * 0 - B * 0}{b_1}$$

The next figure shows the same pattern but solving for d1 and d2.


A C E
B D 0
b1 b2 0
c1 c2 0
d1 d2 0
1
2
3
4


A C E
B D 0
b1 b2 0
c1 c2 0
d1 d2 0
1
2
3
4

$$d_1 = \frac{c_1 * b_2 - b_1 * c_2}{c_1}; c_2 = \frac{c_1 * 0 - b_1 * 0}{c_1}$$

Step 4: Now that the matrix is complete. The last step is to check whether or not the first column does not change sign. If the first column is all negative or all positive, then the Routh Test has determined that the function is stable.

4.2 STEADY STATE ERROR

The steady state error is the error in a function as time approaches infinity. On the exam, you need to be sure that the functions are represented in the following format.

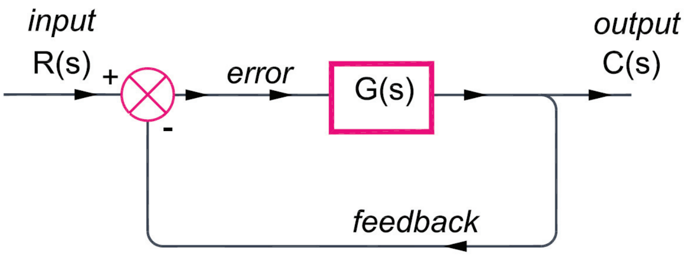


Figure 13: If you are given another block diagram, you need to transform that block diagram into this format before applying the steady state error equations.

Once you are in the correct format, then the steady state error can be found in the FE exam for systems that are between T = 0, 1 or 2. System types are defined by the variable "n" in the basic equation below. Again, you need to simplify G(s) and place it into this format, in order for the equations to work.

$$G(s) = \frac{K(s+P_1)(s+P_2)\ldots}{s^n(s+Q_1)(s+Q_2)\ldots}$$

For example, a system type of T = 0 is shown below.

$$G(s) = \frac{K(s+5)(s+10)}{s^0(s+4)(s+3)}$$

A system type of T = 1 is shown below.

$$G(s) = \frac{K(s+5)(s+10)}{s^1(s+4)(s+3)}$$

A system type of T = 2 is shown below.

$$G(s) = \frac{K(s+5)(s+10)}{s^2(s+4)(s+3)}$$

Most likely, you will be given the equation in the above format and that should tell you the system type.

The next thing you need to know is the input. The inputs can be step inputs (s), ramp inputs (s^2) or parabolic inputs (s^3).

You can then use the table to find the steady state error given the system type and step input.

$System\ Type \rightarrow s^0$	$System\ Type \rightarrow s^1$	$System\ Type \rightarrow s^2$
$G(s) = \frac{K(s+5)(s+10)}{s^0(s+4)(s+3)}$	$G(s) = \frac{K(s+5)(s+10)}{s^1(s+4)(s+3)}$	$G(s) = \frac{K(s+5)(s+10)}{s^2(s+4)(s+3)}$

Step Input $Error\ = \frac{1}{K_B + 1}$ $K_P = \lim_{s\to 0} G(s)$	*Step Input* $Error\ = \frac{1}{K_B + 1} = 0$ $K_P = \lim_{s\to 0} G(s)$	*Step Input* $Error\ = \frac{1}{K_B + 1} = 0$ $K_P = \lim_{s\to 0} G(s)$
Ramp Input $Error\ = \frac{1}{K_V} = \infty$ $K_V = \lim_{s\to 0} s * G(s)$	*Ramp Input* $Error\ (\infty) = \frac{1}{K_V}$ $K_V = \lim_{s\to 0} s * G(s)$	*Ramp Input* $Error\ = \frac{1}{K_V} = 0$ $K_V = \lim_{s\to 0} s * G(s)$
Parabolic Input $Error\ = \frac{1}{K_A} = \infty$ $K_A = \lim_{s\to 0} s^2 * G(s)$	*Parabolic Input* $Error\ = \frac{1}{K_A} = \infty$ $K_A = \lim_{s\to 0} s^2 * G(s)$	*Parabolic Input* $Error\ = \frac{1}{K_A}$ $K_A = \lim_{s\to 0} s^2 * G(s)$

4.2.2 Type 0 Example

For example, find the steady state errors for the following functions.

$$G(s) = \frac{50s + 20}{(s + 9)}$$

Now, apply the step input (s).

$$K_P(position\ constant) = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{50s + 20}{(s + 9)} = \frac{20}{9} = 2.22$$

$$Step\ Input \rightarrow Error\ (\infty) = \frac{1}{K_P + 1} = \frac{1}{2.22 + 1} = 0.31$$

You can also try the ramp input (s).

$$K_v(velocity\ constant) = \lim_{s\to 0} s * G(s) = \lim_{s\to 0} s \frac{50s + 20}{(s + 9)} = 0$$

$$Ramp\ Input \rightarrow Error\ (\infty) = \frac{1}{K_V} = \frac{1}{0} = \infty$$

Finally, try the parabolic input.

$$K_A(acceleration\ constant) = \lim_{s\to 0} s^2 * G(s) = \lim_{s\to 0} s^2 \frac{50s + 20}{(s + 9)} = 0$$

$$Parabolic\ Input \rightarrow Error\ (\infty) = \frac{1}{K_A} = \frac{1}{0} = \infty$$

4.2.2 Type 1 Example

For example, find the steady state errors for the following functions. You need to first put this equation into the correct format.

$$G(s) = \frac{50s + 20}{s^2 + 9s}$$

The correct format will show that this is a Type 1 system.

$$G(s) = \frac{50s + 20}{s(s + 9)}$$

Now, apply the step input (s).

$$K_P = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{50s + 20}{s(s + 9)} = \infty$$

$$Step\ Input \rightarrow Error\ (\infty) = \frac{1}{K_P + 1} = \frac{1}{\infty + 1} = 0$$

You can also try the ramp input (s).

$$K_V = \lim_{s \to 0} s * G(s) = \lim_{s \to 0} \cancel{s} \frac{50s + 20}{\cancel{s}(s + 9)} = \frac{20}{9} = 2.22$$

$$Ramp\ Input \rightarrow Error\ (\infty) = \frac{1}{K_V} = \frac{1}{2.22} = 0.45$$

Finally, try the parabolic input.

$$K_A = \lim_{s \to 0} s^2 * G(s) = \lim_{s \to 0} s * \cancel{s} \frac{50s + 20}{\cancel{s}(s + 9)} = 0$$

$$Parabolic\ Input \rightarrow Error\ (\infty) = \frac{1}{K_A} = \frac{1}{0} = \infty$$

4.2.3 Type 2 Example

For example, find the steady state errors for the following functions.

$$G(s) = \frac{50s + 20}{s^2(s + 9)}$$

Now, apply the step input (s).

$$K_P = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{50s + 20}{s^2(s + 9)} = \infty$$

$$Step\ Input \rightarrow Error\ (\infty) = \frac{1}{K_P + 1} = \frac{1}{\infty + 1} = 0$$

You can also try the ramp input (s).

$$K_V = \lim_{s \to 0} s * G(s) = \lim_{s \to 0} \cancel{s}\frac{50s + 20}{\cancel{s}\, s(s+9)} = \infty$$

$$Ramp\ Input \rightarrow Error\ (\infty) = \frac{1}{K_V} = \frac{1}{\infty} = 0$$

Finally, try the parabolic input.

$$K_A = \lim_{s \to 0} s^2 * G(s) = \lim_{s \to 0} \cancel{s^2}\frac{50s + 20}{\cancel{s^2}(s+9)} = \frac{20}{9}$$

$$Parabolic\ Input \rightarrow Error\ (\infty) = \frac{1}{K_A} = \frac{1}{2.22} = 0.45$$

5.0 MEASUREMENT UNCERTAINTY

There is one main skill you need to answer problems on measurement uncertainty, which is being able to calculate the propagation of error.

Measurement uncertainty arises from the crudeness of the measurement tools. Tools can only measure accurately to its nearest value. The uncertainty is typically taken as 1/2 the nearest value that a tool can measure. For example, a ruler can measure length to the nearest 1 millimeter, so its uncertainty would be ±0.5 mm. Uncertainty is defined as the value of possible error.

5.1 PROPAGATION OF ERROR

If a measured value has a certain uncertainty or possibility of error, then when that measured value is run through a certain function, then that error will be propagated through that function. The calculation of this propagation is found through the Kline-McClintock method. This method states that the propagation of error is found by taking the square root of the product of the square of the derivative of the function with respect to the measured variable and the square of the uncertainty of that measured variable.

$$y = f(x); x\ has\ uncertainty\ of \pm \delta$$

$$Uncertainty\ of\ y = \sqrt{\left(\frac{df(x)}{dx}\right)^2 * \delta^2}$$

An example is shown below.

$$y = x^2;\ measured\ value: x = 4 \pm 0.1; y = 16 \pm \delta_y$$

$$Uncertainty\ of\ y = \delta_y = \sqrt{(2x)^2 * 0.1^2}$$

$$\delta_y = \sqrt{(2 * 4)^2 * 0.1^2}$$

$$\delta_y = \pm 0.64$$

$$y = 16 \pm 0.64; \%\ error = \frac{0.64}{16} = 4\%$$

$$x = 4 \pm 0.1; \%\ error = \frac{0.1}{4} = 2.5\%$$

This method also works for functions with multiple variables.

$$y = f(x_1, x_2); x_1 \pm \delta_1; x_2 \pm \delta_2$$

$$Uncertainty\ of\ y = \sqrt{\left(\frac{df(x_1)}{dx_1}\right)^2 * \delta_1^2 + \left(\frac{df(x_2)}{d2}\right)^2 * \delta_2^2}$$

An example of this is shown below.

$$y = x_1^2 - x_2^2 * x_1;\ measured\ value: x_1 = 4 \pm 0.1; x_2 = 16 \pm 0.5$$

$$Uncertainty\ of\ y = \delta_y = \sqrt{(2x_1 - x_2^2)^2 * 0.1^2 + (0 - 2x_2 * x_1)^2 * 0.5^2}$$

$$\delta_y = \sqrt{(2 * (4) - 16^2)^2 * 0.1^2 + (0 - 2 * (16) * 4)^2 * 0.5^2}$$

$$\delta_y = \sqrt{615.04 + 4096} = 68.6$$

$$y = 4^2 - 16^2 * 4 = -1{,}008$$

$$y = -1{,}008 \pm 68.6$$

5.2 ACCURACY VS. PRECISION

Measurement accuracy describes how close the measurement is to the actual value. Precision describes how close the measurements are to one another. If you were to imagine taking a measurement with a caliper as opposed to a ruler, you can create more precise measurements with a caliper. You can measure more significant figures.

5.3 SIGNIFICANT FIGURES

Significant figures' rules are used in measurements to determine the number of digits you must report in your measurements. The rules are shown below.

1. All non-zero numbers are significant.

$$5 - Significant\ Figures \rightarrow 12345$$

2. Zeros between two non-zero numbers are significant.

$$5 - Significant\ Figures \rightarrow 10005$$

$$5 - Significant\ Figures \rightarrow 1.0005$$

3. Leading zeros are NOT significant.

$$5 - Significant\ Figures \rightarrow 00010005$$

$$4 - Significant\ Figures \rightarrow 0.005016$$

4. Trailing zeros to the right of a decimal point are significant.

$$5 - Significant\ Figures \rightarrow 0.0050160$$

$$3 - Significant\ Figures \rightarrow 0.00500$$

When you add or subtract measurements you also need to follow certain rules. All numbers are added together normally, but then the answer is reduced down to the least number of decimal points in the original measurements.

$$Add \rightarrow 38.1 + 504.123 + 0.19 = 542.413$$

Now reduce the answer to the least number of decimal points which is the tenths from 38.1.

$$Add \rightarrow 38.1 + 504.123 + 0.19 = 542.413 \rightarrow 542.4$$

When you multiply or divide, you round the final answer to the least number of significant figures.

$$Multiply \rightarrow 38\ x\ 123.4 = 4689.2$$

Now reduce the final answer to only 2 significant figures, which is the same as 38.

$$Multiply \rightarrow 38\ x\ 123.4 = 4689.2 \rightarrow 4700$$

6.0 PRACTICE EXAM PROBLEMS

6.1 PRACTICE PROBLEM 1 – ROUTH TEST

A characterizing equation is shown below. The variable "C" must be greater than what minimum value in order to ensure the equation stable?

$$t^3 + 1000t^2 + (C + 10)t + 100C = 0$$

(a) -9

(b) 0

(c) 9

(d) 1,000

6.2 PRACTICE PROBLEM 2 – THERMOCOUPLE

A Chromel/Alumel thermocouple is used in an industrial electric furnace. The device reads an initial voltage of 15mV. A second measurement is taken with a reading of 35mV. What is most nearly the temperature rise between the first and second readings?

(a) 750F

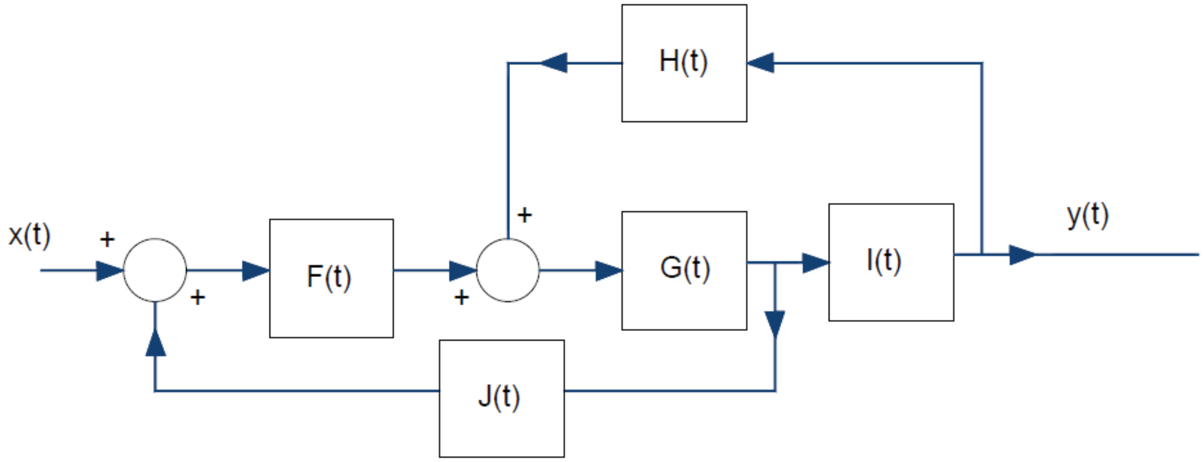
(b) 880F

(c) 1360F

(d) 1450F

6.3 PRACTICE PROBLEM 3 – BLOCK DIAGRAM

Simplify the below block diagram.



(a) $\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)H(t)}$

(b) $\frac{F(t)}{1+F(t)G(t)H(t)+G(t)I(t)H(t)}$

(c) $\frac{F(t)*G(t)*I(t)}{1-F(t)}$

(d) $\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)+H(t)G(t)I(t)}$

6.4 PRACTICE PROBLEM 4 – STRAIN GAUGE

A half bridge strain gauge with resistance of 120 ohms is used to measure the strain due to an axial stress. The measured output voltage from the wheatstone bridge is .002 V. The input voltage is 5V. What is the measured strain? Assume a gauge factor of 2.0.

(a) 0.0004

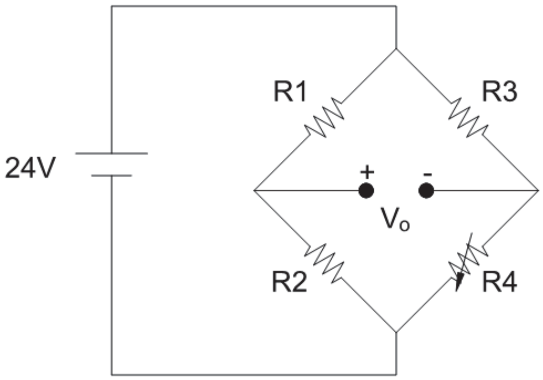
(b) 0.0008

(c) 0.002

(d) 0.2

6.5 PRACTICE PROBLEM 5 – WHEATSTONE BRIDGE

A strain gauge is installed on a beam and is connected as R4 in the Wheatstone bridge below. During normal conditions, R1=R2=R3=R4=100 Ω. If the change in output voltage is measured at 10mV, what is the change in resistance of R4?



(a) 0.10 Ω

(b) 0.08 Ω

(c) 0.17 Ω

(d) 0.20 Ω

6.6 PRACTICE PROBLEM 6 – PRESSURE SENSOR

A compressed air tank has a gauge pressure reading of 25 psig at sea level. If the compressed air tank is then relocated to a new elevation where the atmospheric pressure changes from 14.696 psia to 14.1 psia. What will the gauge pressure read on the tank?

(a) 11.0 psig

(b) 11.5 psig

(c) 25 psig

(d) 25.5 psig

6.7 PRACTICE PROBLEM 7 – STRAIN GAUGE

A resistive strain gauge is installed on a steel square rod. If the rod undergoes an axial compressive load and the gauge measures a change in resistance of 0.10 Ω, then what is the axial load experienced at the rod? The gauge is 5cm wide, 0.8cm thick, has a gauge factor of 2, and a nominal resistance of 50 Ω.

(a) 15 kN

(b) 80 kN

(c) 100 kN

(d) 200 kN

6.8 PRACTICE PROBLEM 8 – FITS & TOLERANCES

A nominal 2" shaft is designed to have a maximum value of 2.0025" and a minimum value of 1.995. The hole for the shaft is designed to have a minimum value of 2.00" and a maximum value of 2.0015". What is the largest clearance?

(a) .0035"

(b) .0040"

(c) .0055"

(d) .0065"

7.0 SOLUTIONS

7.1 SOLUTION 1 – ROUTH TEST

First, plug in the values to the first two rows of the matrix below.

$$t^3 + 1000t^2 + (C + 10)t + 100C + 0 = 0$$

$$\begin{pmatrix} A & C & E \\ B & D & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \\ d_1 & d_2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & C + 10 \\ 1000 & 100C \\ \dfrac{[1000 * (C + 10) - 100C]}{1000} & 0 \\ 100C & c_2 \end{pmatrix}$$

All values in the first column must be greater than 0. Check the third row.

$$\frac{[1000 * (C + 10) - 100C]}{1000} > 0$$

$$10C + 100 - C > 0$$

$$9C > -100$$

$$C > -11.1$$

Now check the fourth row.

$$100C > 0$$

$$C > 0$$

The correct answer is most nearly, (b) 0.

7.2 SOLUTION 2 - THERMOCOUPLE

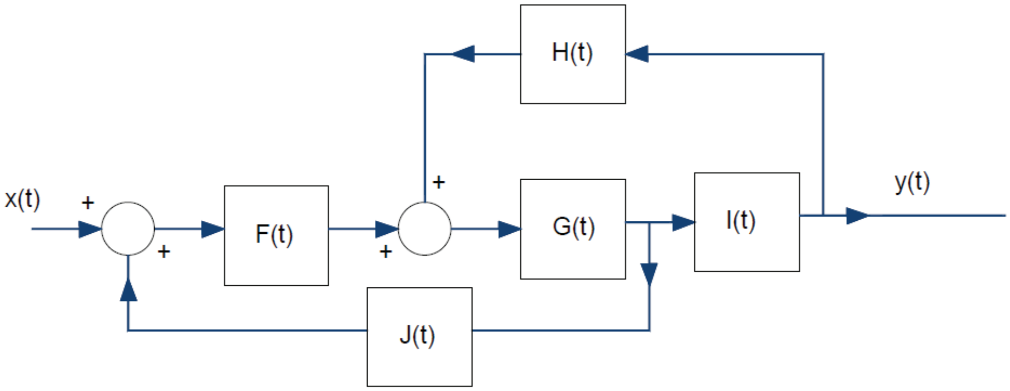
A thermocouple uses the change in voltages to measure the change in temperature. In this case, refer to the graph within the *NCEES FE Reference Handbook* which shows the correlation between the measured voltage and the relative temperature.

At 15mV, the temperature is approximately 750F. At 35mV, the temperature is approximately 1630F. The temperature difference is:

$$\Delta T = 1630F - 750F = 880F$$

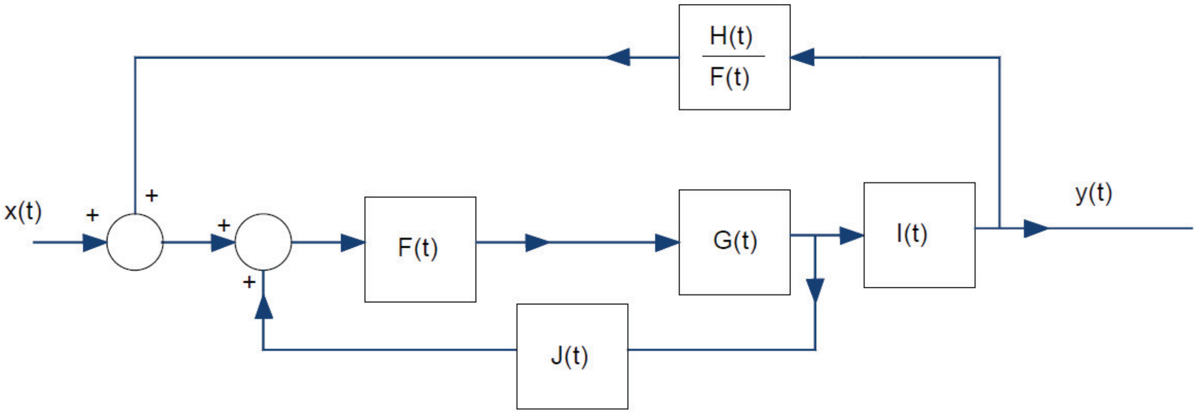
The answer is most nearly (B) 880F.

7.3 SOLUTION 3 – BLOCK DIAGRAM



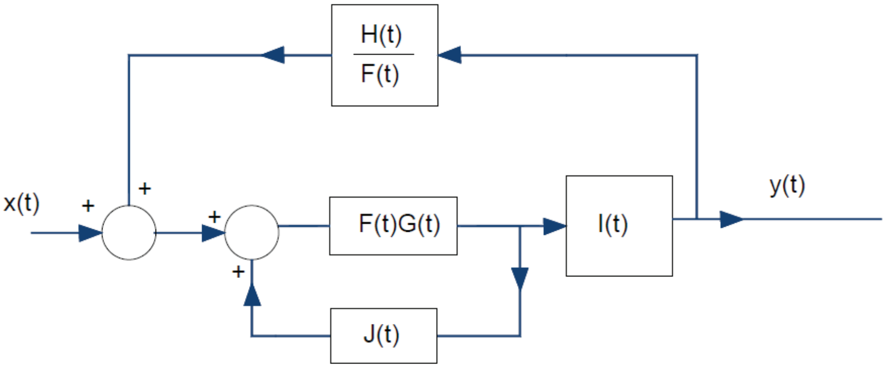
H(t)
x(t)
+
+
F(t)
+
+
G(t)
I(t)
y(t)
J(t)

First, move H(t) outside of F(t).



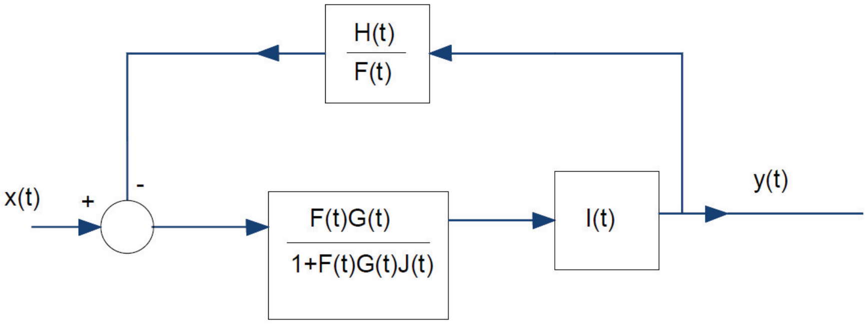
H(t)
F(t)
x(t)
+
+
+
+
F(t)
G(t)
I(t)
y(t)
J(t)

Then combine F(t) and G(t)



H(t)
F(t)
x(t)
+
+
+
+
F(t)G(t)
I(t)
y(t)
J(t)

Then combine the feedback loop J(t).



Combine the I(t) function then finally combine the feedback loop.

$$\frac{\left[\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)}\right]}{\left[1+\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)}*\frac{H(t)}{F(t)}\right]}=\frac{\left[\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)}\right]}{\left[1+\frac{H(t)*F(t)*G(t)*I(t)}{F(t)[1+F(t)G(t)J(t)]}\right]}$$

$$\frac{\left[\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)}\right]}{\left[\frac{F(t)[1+F(t)G(t)J(t)]+H(t)*F(t)*G(t)*I(t)}{F(t)[1+F(t)G(t)J(t)]}\right]}$$

$$\left[\frac{F(t)*G(t)*I(t)}{\cancel{1+F(t)G(t)J(t)}}\right]*\frac{[\cancel{F(t)[1+F(t)G(t)J(t)]}]}{[\cancel{F(t)}[1+F(t)G(t)J(t)]+H(t)\cancel{F(t)}G(t)I(t)]}$$

$$\left[\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)+H(t)G(t)I(t)}\right]$$

The correct answer is most nearly (d) $\frac{F(t)*G(t)*I(t)}{1+F(t)G(t)J(t)+H(t)G(t)I(t)}$

7.4 SOLUTION 4 – STRAIN GAUGE

For this problem you have a half bridge strain gauge. So the factor should be 1/2 as opposed to 1/4 which is used for a quarter bridge.

$$V_{out} = \left(\frac{1}{2}\right) * \frac{\Delta R}{R} V_{in}$$

$$\frac{2 * V_{out}}{V_{in}} = \frac{\Delta R}{R}$$

$$2 * \frac{0.002}{5} = \frac{\Delta R}{R}$$

$$\frac{\Delta R}{R} = 0.0008$$

Now plug in this ratio to the gauge factor equation.

$$GF = 2 = \frac{\Delta R}{R} / \varepsilon$$

$$\varepsilon = \frac{0.0008}{2} = 0.0004$$

The correct answer is most nearly (a) 0.0004

7.5 SOLUTION 5 - WHEATSTONE BRIDGE

For Wheatstone bridges, where the nominal resistances are the same, the following relationship applies when the change in the resistance is much smaller than the nominal resistance R.

$$V_{out} = \frac{\Delta R}{4R} * V_{in}$$

Where R=R1=R2=R3 and R4=R+ΔR. The change in resistance becomes:

$$\Delta R \approx \frac{V_{out}}{V_{in}} * 4R$$

$$\Delta R \approx \frac{10 * 10^{-3} V}{24V} * 4 * 100\Omega = 0.167\Omega$$

The answer is most nearly (C) 0.17 Ω

7.6 SOLUTION 6 – PRESSURE SENSOR

The gauge measurement compares the pressure at the measured location to the surrounding atmospheric pressure. At sea level, the compressed air tank is 25 psi above atmospheric pressure. At the new elevation, the compressed air tank will be 25 psi above the atmospheric pressure at that elevation.

The correct answer is (c) 25 psig.

7.7 SOLUTION 7 – STRAIN GAUGE

Axial force, stress and strain follow the following relationships.

$$\sigma = \frac{F}{A} \ \ and \ \ \sigma = \varepsilon E$$

Therefore,

$$F = A * \varepsilon * E$$

Use the gauge factor equation to solve for the normal strain experienced at the gauge.

$$GF = \frac{\Delta R}{R \cdot \varepsilon} \Longrightarrow$$

$$\varepsilon = \frac{\Delta R}{R \cdot GF} = \frac{0.10\Omega}{50\Omega * 2} = 0.001$$

From the *NCEES FE Reference Exam*, find that the modulus of elasticity for steel is 200*10^9 Pa. The area of the strain gauge used in the calculation will be the side of the gauge that is normal to the force.

Solve for the force.

$$F = A * \varepsilon * E = \frac{5cm * 0.8cm}{10^4 \frac{cm^2}{m^2}} * 0.001 * 200 * 10^9 \frac{N}{m^2} = 80kN$$

The answer is (B) 80 kN

7.8 SOLUTION 8 – FITS & TOLERANCES

A nominal 2” shaft is designed to have a maximum value of 2.0025” and a minimum value of 1.995. The hole for the shaft is designed to have a minimum value of 2.00” and a maximum value of 2.0015”. What is the largest clearance?

The maximum clearance occurs at the maximum hole value and the minimum shaft value.

$$Max\ Hole = 2.0015";\ Min\ Shaft\ =\ 1.995"$$

The difference between these two values is the clearance.

$$Max\ Clearance = 2.0015 \text{ - } 1.995. = 0065"$$

The correct answer is most nearly, **(d).0065”**.

14 – Mechanical Design & Analysis

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Section 14.0 – Mechanical Design and Analysis

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1.0 INTRODUCTION

Mechanical Design and Analysis accounts for approximately 10 to 15 questions on the Mechanical FE exam. This section primarily focuses on machine design mechanical components like springs, pressure vessels, beams, piping, bearings, power screws and transmissions. In addition, other topics that support machine design mechanical components are discussed in this section, like manufacturability, quality, and reliability. Then this section switches to topics that are common in Thermal & Fluids like hydraulics, pneumatics and electromechanical components. Finally, this section repeats topics like beams, piping, stress analysis, deformation, stiffness that are covered in previous sections. This section will point you to the correct section when these repeat topics are discussed.

A list of the topics are shown in the table below.

Section 15.0 Mechanical Design & Analysis (10 to 15 Problems)		
NCEES Outline Value	**Engineering Pro Guides**	
	Section 1.0	Introduction
15A	Section 2.0	Stress Analysis of Machine Elements *(repeat from previous Section 8.0 Mechanics of Materials)*
15B	Section 3.0	Failure Theories and Analysis
15C	Section 4.0	Deformation and Stiffness *(repeat from previous Section 8.0 Mechanics of Materials)*
15D	Section 5.0	Springs
15E	Section 6.0	Pressure Vessels and piping
15F	Section 7.0	Bearings
15G	Section 8.0	Power Screws
15H	Section 9.0	Power Transmission
15I	Section 10.0	Joining Methods (e.g., welding, adhesives, mechanical fasteners)
15J	Section 11.0	Manufacturability (e.g., limits, fits)
15K	Section 12.0	Quality and Reliability
15L	Section 13.0, 14.0 & 15.0	Components (e.g., hydraulic, pneumatic, electromechanical)
15M	Section 16.0	Engineering drawing interpretations and geometric dimensioning and tolerancing (GD&T)
	Section 17.0	Practice Exam Problems

2.0 STRESS ANALYSIS OF MACHINE ELEMENTS

Stress analysis of machine elements is discussed in *Section 8.0 Mechanics of Materials*, there are many topics within that section that discuss the stress for all possible types of loads.

3.0 FAILURE THEORIES AND ANALYSIS

Failure can mean a component has been completely fractured; permanently distorted or its function has been compromised. In *Section 9.0 Material Properties and Processing*, various strengths of material properties have been presented. Unfortunately, these strengths which can be used to determine the stress levels at which failure will occur apply only to simple loadings like tension, compression or shear that occurs in one axis. In real life situations and for most of the FE exam, the simple loadings can be assumed. However, there may be a couple of questions on the FE exam where there are complex loadings. For these questions you need to use one of the following failure theories, (1) Maximum Normal Stress Theory for Brittle Materials, (2) Mohr's Theory for Brittle Materials, (3) Maximum Shear Stress Theory for Ductile Materials and (4) Distortion Energy Theory for Ductile Materials also known as Von Mises Theory. Before you learn how to use each of the theories, you need to first understand factor of safety.

In failure theories, the term factor of safety is used to describe the ratio of the load at which the object will fail to the load at which the object will be allowed. The allowed loading can also be described as the design loading. For example, if you design a beam for a loading of 1,000 N then you could choose a beam that fails at 2,000 N. This will result in a factor of safety of 2.

$$Factor\ of\ safety\ (F.S.) = \frac{Failure\ loading}{Allowed\ or\ Design\ loading}$$

$$FS = 2 = \frac{2{,}000\ N\ beam\ failure}{1{,}000\ N\ beam\ design}$$

3.1 MAXIMUM NORMAL STRESS THEORY - BRITTLE

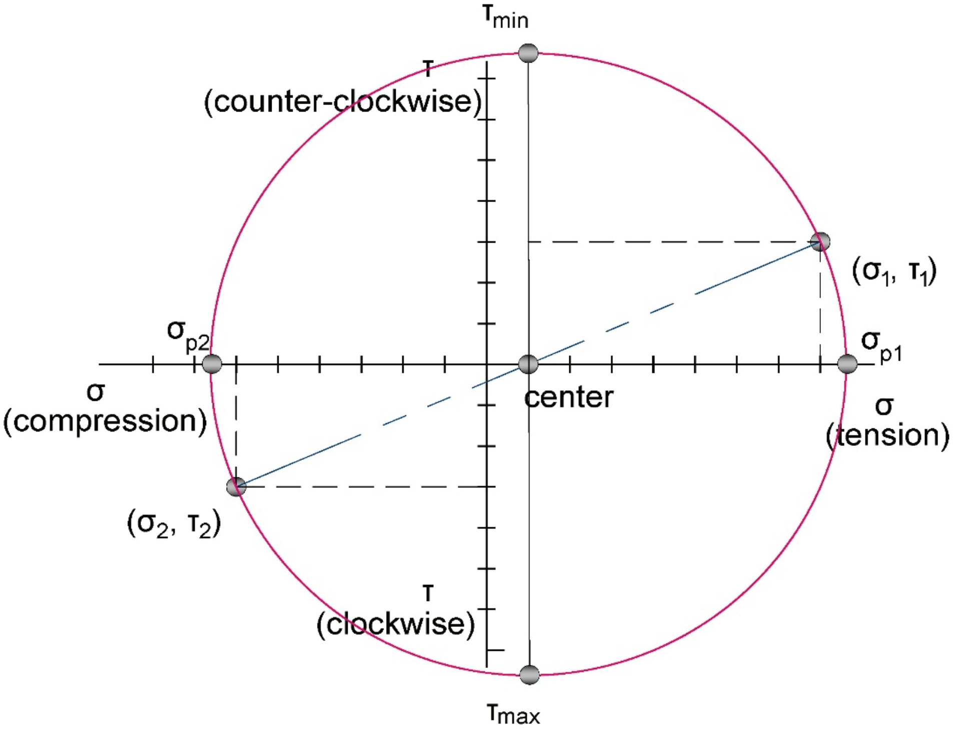
The maximum normal stress theory is similar to the methods presented in *Section 9.0 Material Properties and Processing.* The theory is that if the normal stress in tension or compression is greater than the ultimate strength of the material, then failure will occur.

$$Failure\ Occurs \rightarrow \sigma_{tensile} \geq S_{ut}; S_{ut} = ultimate\ tensile\ strength$$

$$Failure\ Occurs \rightarrow \sigma_{compression} \leq -S_{uc}; S_{uc} = ultimate\ compressive\ strength$$

3.2 MOHR THEORY - BRITTLE

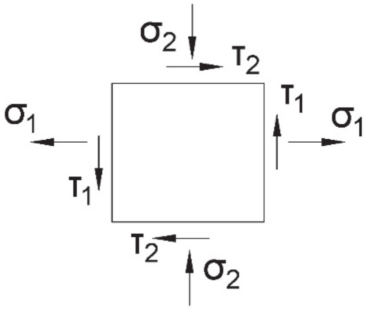
Mohr's theory is another theory used to assess when a material will fail. This theory is used when a material does not have similar compressive and tensile yield strengths. Mohr's theory can be best understood through the use of Mohr's Circle. This circle is a representation of 2-d stresses in the tensile/compressive direction and in shear. The circle is used to analyze complex stresses to determine the principle stresses that are applied to a component. The principle stresses can then be compared to the tensile, compressive and shear strengths.



τmin
τ
(counter-clockwise)
(σ1, τ1)
σp2
σp1
σ
(compression)
center
σ
(tension)
(σ2, τ2)
τ
(clockwise)
τmax

Figure 1: Mohr's circle is used to find the primary stresses in tension, compression and shear.

The complex stresses of (σ_1, τ_1) and (σ_2, τ_2) are typically given in a FE problem. Then you must use Mohr's circle to find the principal forces that act in tension, compression and shear. These principal forces are shown as the extreme points on the circle.



σ2
τ2
σ1
τ1
σ1
τ1
τ2
σ2

Figure 2: The given stresses that act upon an element within a component are shown in this figure. Notice how the clockwise and counter-clockwise shears correspond to the points shown on the previous figure. Also notice how the tension and compression stresses are shown in this figure correspond to the correct location in the previous figure.

Mohr's circle is created by first determining the center point of the circle from the complex stresses given in the problem. The center is found through this equation.

$$Center = \left(\frac{\sigma_1 + \sigma_2}{2}, \frac{\tau_1 + \tau_2}{2}\right)$$

Next you must find the radius of the circle. The radius is found through the Pythagorean theorem. First you take the difference in the x-directions, to find the x-length of the triangle. Then do the same for the y-direction. Finally, square root the sum of the squares to find the hypotenuse of the triangle, which is equal to the diameter of the circle. The radius is equal to ½ the diameter.

$$D = \sqrt{(\sigma_1 - \sigma_2)^2 + (\tau_1 - \tau_2)^2}; r = \frac{D}{2}$$

Finally, the principal stresses can be found by adding or subtracting the radius to the center values.

$$\sigma_{tensile} = \sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + r; \sigma_{compressive} = \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - r$$

$$\tau_{min} = \frac{\tau_1 + \tau_2}{2} + r; \tau_{max} = \frac{\tau_1 + \tau_2}{2} - r$$

The principal tensile and compressive stresses can then be compared to the ultimate tensile and compressive strength values to determine if failure will occur.

$$\frac{\sigma_{p1}}{S_{ut}} - \frac{\sigma_{p2}}{S_{uc}} \geq 1$$

3.3 MAXIMUM SHEAR STRESS THEORY - DUCTILE

This theory states that yielding occurs when the maximum shear stress in a ductile component is greater than the maximum shear stress that occurs when the same component is in tension and begins to yield. This is a very complicated statement, but you can break it down into two parts. First, the maximum shear stress when a material is at maximum tensile shear stress. This maximum shear strength is equal to the yield strength of the material divided by 2.

$$\tau_{strength\ at\ tensile\ yield} = \frac{\sigma_{tensile\ yield\ strength}}{2}$$

Secondly, this theory states that the shear stress in the component must not exceed this value. The maximum shear stress in a component can be calculated from the principal stresses that act upon a component during principal stresses that act in tension.

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2};\ \tau_{13} = \frac{\sigma_1 - \sigma_3}{2};\ \tau_{23} = \frac{\sigma_2 - \sigma_3}{2};$$

$$\tau_{\#\#} = shear\ stress\ due\ to\ principal\ stresses$$

If the forces act in pure shear, then the following is true.

$$\tau_{\#} = \sigma_{\#}$$

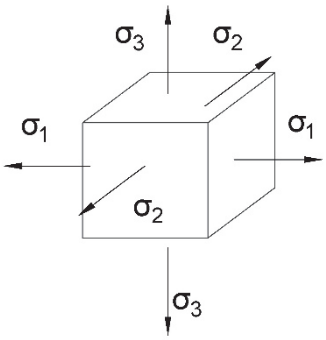


Figure 3: This figure shows the principal stresses acting upon a piece of a component.

In a uniaxial, tensile stress situation then there is only stresses in direction "1". Thus the previous equations are simplified as shown below.

$$\tau_{12} = \frac{\sigma_1 - 0}{2} = \frac{\sigma_1}{2};\ \ \tau_{13} = \frac{\sigma_1}{2};\ \tau_{23} = 0$$

The previous values then can be compared to the maximum shear stress criteria, which will simplify to compare the principal tensile stress with the yield strength of the material.

$$\tau_{12} \geq \ \tau_{strength\ at\ tensile\ yield};\ \tau_{13} \geq \ \tau_{strength\ at\ tensile\ yield};$$

$$\frac{\sigma_1}{2} \geq \frac{\sigma_{tensile\ yield\ strength}}{2};\ \frac{\sigma_1}{2} \geq \frac{\sigma_{tensile\ yield\ strength}}{2}$$

$$\sigma_1 \geq \sigma_{tensile\ yield\ strength}$$

3.4 VON MISES THEORY

The von Mises theory is also known as the shear-energy theory or the distortion energy theory. This theory predicts when a ductile material will yield under complicated loads. This theory calculates a von Mises stress value and says that if this value is greater than the yield strength of the material, then the material will fail.

$$Von\ Mises \rightarrow \sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

$$Failure\ occurs\ when, \sigma_v \geq \sigma_{yield}$$

The values of stresses "1", "2" and "3" are the stresses that occur in the three principal directions in a 3-dimensional object. The object below shows the direction of these stresses.

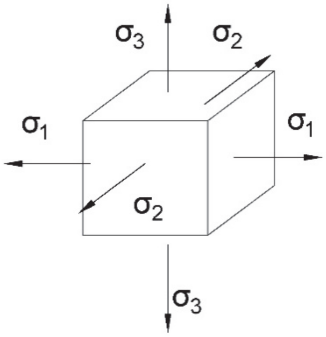


Figure 4: This figure shows the principal stresses acting upon a piece of a component.

In the event that there is uniaxial tension, then there will only be stresses in the 1st direction, thus von Mises will simplify as shown below.

$$Von\ Mises \rightarrow \sigma_v = \sqrt{\frac{(\sigma_1 - 0)^2 + (0 - 0)^2 + (0 - \sigma_1)^2}{2}} = \sigma_1$$

$$Failure\ occurs\ when, \sigma_1 \geq \sigma_{yield}$$

4.0 DEFORMATION AND STIFFNESS

Deformation and stiffness is covered in *Section 8.0 Mechanics of Materials*, there is another topic within that section called deformations. In addition, *Section 9.0 Material Properties and Processing* has a section on the physical properties of materials like stiffness.

5.0 SPRINGS

A spring provides an opposing force that is dependent on the stiffness of the spring and the deflection of the spring from its free length.

$$F = k\Delta L$$

$$F = spring\ force\ (N\ or\ lbf); k = spring\ stiffness\ coefficeint\ \left(\frac{N}{m}\ or\ \frac{lbf}{in}\right);$$
$$\Delta L = deflection\ (m\ or\ in)$$

If the spring is not deflected then it is at its free length. The spring force at a deflection of 0 will be equal to zero. When a spring is deflected at a design or operating load, then the force will be found using the previous equation. If the spring is fully deflected, then it is at its maximum load. This full deflection amount is often called solid length.

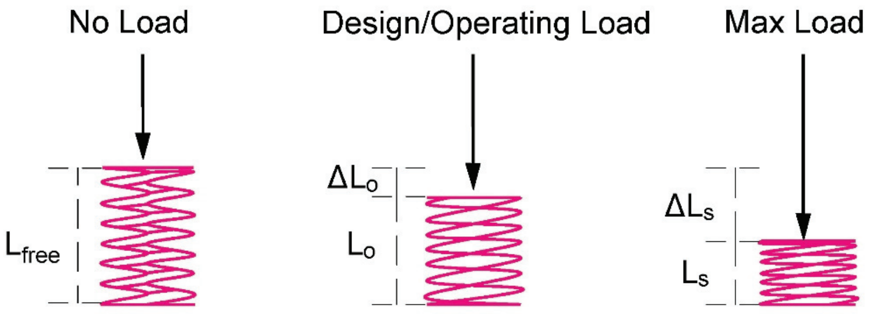

No Load
Design/Operating Load
Max Load
L_{free}
ΔL_o
L_o
ΔL_s
L_s

Figure 5: A spring's force is corresponds to the level of deflection.

5.1 TYPES OF SPRINGS

There are three main types of springs, (1) Push, (2) Pull and (3) Radial Torque. A push spring is one where the force is applied in compression and a pull spring is one where the force is applied in tension. Finally, a radial torque spring has force applied radially. The easiest example is a rubber band. When force is applied outwardly from the center of the rubber band then the rubber band expands. The spring force pushes inward towards the center.

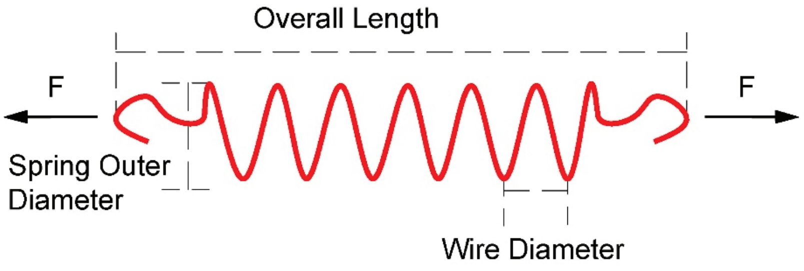

Overall Length
F
F
Spring Outer Diameter
Wire Diameter

Figure 6: A pull spring is used in tension. A vibrational hangar is an example of a pull spring.

Figure 7: A push spring is used in compression. A vibration isolator support is an example of a push spring.

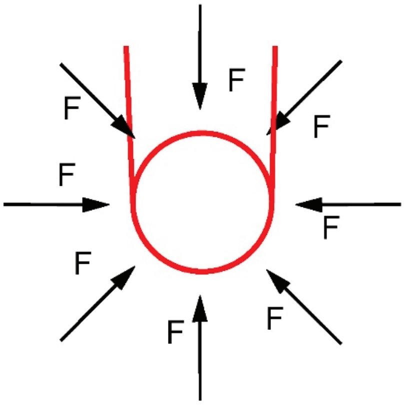


Figure 8: A radial torque spring is used to secure circular shaped objects. Pipe and hose clamps are examples of torsional springs.

5.1.1 Helical Springs

Another important type of spring that can be either push or pull is the helical spring. A helical spring is also known as a coil spring. This type of spring is important because the force analysis will be primarily focused on helical springs, so you must understand the geometry of a helical spring.

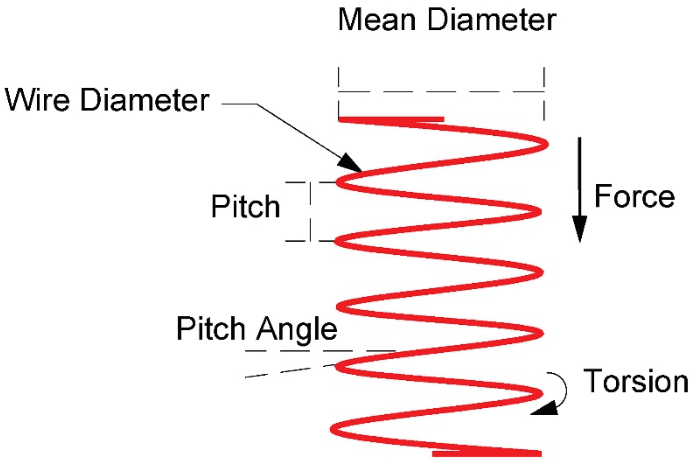


Figure 9: A helical spring is constructed by coiling a wire.

The pitch in the diagram above refers to the distance from one coil to the next coil. It is the axial distance that separates the start of one coil and the start of another coil. The pitch angle is created by the triangle with base equal to the circumference of the coil and the height equal to the pitch. Thus the pitch angle can be found by using the equation below.

$$Pitch\ Angle = tan^{-1}\left(\frac{Pitch}{\pi * Mean\ Diameter}\right)$$

$$P = pitch\ (mm\ or\ in); D_m = mean\ diameter\ (mm\ or\ in)$$

The typical pitch angles are around 10 to 12 degrees.

The ends of the helical spring will also affect the force analysis. Plain or open springs have a constant pitch throughout the length of the spring. The end of the plain spring can be grounded or cut at the end to give a flat appearance at the end.

Square or closed springs have a different pitch at the end. The pitch is small enough such that the last two coils are touching one another. Similar to plain springs, square springs can also be grounded or cut at the end. This will help the spring to stand by itself.

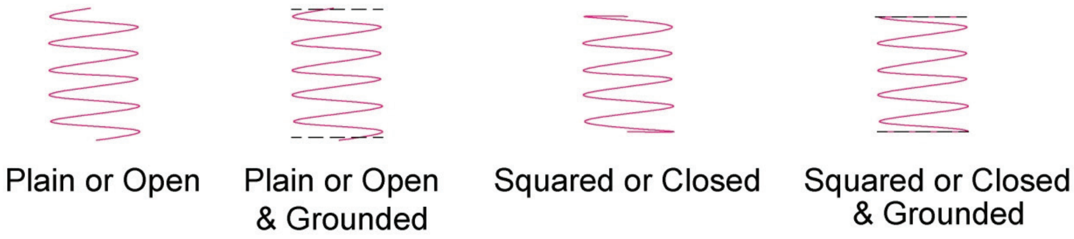


Figure 10: The various ends of the spring will affect its performance. The plain or open spring has the same shape at the end of the coil as the shape throughout the coil. The squared or closed spring changes its coil shape at the end, which is shown as a straight horizontal line. Grounded springs are simply cut at the end to give a flat appearance.

The spring ends will determine the number of active coils. This table will be given to you in the *NCEES FE Reference Handbook*, but if they give you a picture of a spring then you need to be able to determine which column to use.

	Spring Type			
	Plain	Plain & Ground	Square or Closed	Square and Ground
# End Coils	0	1	2	2
# Active Coils	N_t	$N_t - 1$	$N_t - 2$	$N_t - 2$
Free Length	$p(N_a + 1)$	$p(N_a + 1)$	$pN_a + 3d$	$pN_a + 2d$
Solid Length	$d(N_t + 1)$	dN_t	$d(N_t + 1)$	dN_t
Free Length Pitch	$(l_f - d)/N_a$	$l_f/(N_a + 1)$	$(l_f - 3d)/N_a$	$(l_f - 2d)/N_a$

5.2 FORCE ANALYSIS

5.2.1 Helical Springs

Once you understand helical springs, then you can use the following equations to determine the necessary spring rate for a given material, number of active coils, mean diameter, wire diameter and shear modulus. This equation is also given in the *NCEES FE Reference Handbook*, but the difficulty with using this equation is making sure you are using the correct units. All lengths must be the same units.

$$k = \frac{Gd^4}{8D^3N_a}$$

$$G = shear\ modulus\ (Pa\ or\ psi)$$

$$d = wire\ diameter\ (m\ or\ in);\ D = coil\ mean\ diameter (m\ or\ in);$$

$$N_a = number\ of\ active\ coils\ (unitless)$$

$$k = spring\ rate\ \left(\frac{N}{m}\ or\ \frac{lb}{in}\right)$$

The stress in the spring can be determined through the following equation, when a force "P" is applied to the spring. The force can be applied in either compression or tension.

$$\tau = Stress = \frac{K * 8 * P * D}{\pi d^3}$$

$$\tau = Stress\ (Pa\ or\ psi);\ P = Force\ (N\ or\ lbf);$$

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}\ or\ K = 1 + \frac{0.5}{C}$$

$$C = spring\ index\ (unitless) = \frac{D}{d} \rightarrow ratio\ of\ coil\ mean\ diameter\ to\ wire\ diameter$$

Big "K" is called the Wahl correction factor. This takes into account additional stresses that are present in a spring. There are various formulas for this correction factor based on the situation and test results.

It is important to understand that when a helical linear spring is stretched to its new length, the spring wire material will undergo a shear stress. As opposed to when a helical torsion spring is stretched to its new diameter, the spring wire material will undergo a tensile stress. So when you compare the strength of the wire material, you must make sure you compare the strength in torsion, which is shown in the *NCEES FE Reference Handbook* and below for certain steels.

$$Cold - Drawn\ Carbon\ Steel\ \tau = 0.45\ S_{ut};$$

$$Hardened - Tempered\ Carbon\ Steel\ \tau = 0.50\ S_{ut};$$

$$S_{ut} = ultimate\ tensile\ strength\ of\ the\ material\ (Pa\ or\ psi)$$

Recall that since $F = k\Delta L$ the deflection, ΔL, sometimes written as "δ," of a helical spring can be found with the following equation, where "F" is the design load.

$$\delta = \frac{F}{k} = \frac{8FD^3N_a}{d^4G}$$

$$\delta = deflection\ (m\ or\ in); F = design\ load\ (N\ or\ lbf)$$

5.2.2 Helical Torsion Springs

Similar to helical compression and extension springs, helical torsion springs have equations for "k" and stress.

$$k = \frac{Ed^4}{64DN_a}\left[\frac{N-m}{rad}\right]$$

$$E = Young's\ modulus\ (Pa)$$

$$d = wire\ diameter\ (m); D = coil\ mean\ diameter(m);$$

$$N_a = \#\ of\ active\ coils$$

The stress in the spring can be determined through the following equation, when a force "P" is applied to the spring. The force can be applied in either compression or tension.

$$\sigma = Stress = K\frac{32 * F * r}{\pi d^3}$$

$$\sigma = Stress\ (Pa);\ r = radius, center\ of\ coil\ to\ load\ (m\ or\ in)$$

$$K = \frac{4C^2 - C - 1}{4C^2 - 4C}$$

It is important to understand that when a helical torsion spring is stretched to its new diameter, the spring wire material will undergo a tensile stress. As opposed to when a linear spring is stretched to its new length, the spring wire material will undergo a shear stress. So when you compare the strength of the wire material, you must make sure you compare the strength in tension, which is shown in the *NCEES FE Reference Handbook* and below for certain steels.

$$Cold-Drawn\ Carbon\ Steel\ \ \sigma = 0.78S_{ut}\ (Pa\ or\ psi)$$

$$Hardened-Tempered\ Carbon\ Steel\ \ \sigma = 0.87S_{ut}\ (Pa\ or\ psi)$$

5.3 FATIGUE ANALYSIS

Cyclical loading of springs will cause fatigue. The fatigue analysis equations for springs are similar to the fatigue analysis presented in *Section 8.0 Mechanics of Materials*. First, the alternating force acting and mean force acting on the spring must be found.

$$F_a = \frac{F_{max} - F_{min}}{2};\ F_{mean} = \frac{F_{max} + F_{min}}{2};$$

Then the alternating shear stress and midrange (mean) shear stresses can be found for springs.

$$\tau_a = K\frac{8F_aD}{\pi d^3}; \tau_{mean} = K\frac{8F_{mean}D}{\pi d^3};$$

The previous equations determine the actual alternating and mean stress/force. These values need to be compared to the actual endurance and mean strengths for springs. One important thing with springs is that the size, material and tensile strength of the wires that make up a helical spring does not affect the endurance limit for wires under 3/8 in. The endurance strength for springs is shown below for peened and unpeened conditions. Peened refers to a material that has been cold-worked to become stronger. The exact process is not important for this section.

$$Unpeened \rightarrow S_{alternating\ strength} = 35\ kpsi;\ S_{mean\ strength} = 55\ kpsi;$$

$$Peened \rightarrow S_{alternating\ strength} = 57.5\ kpsi;\ S_{mean\ strength} = 77.5\ kpsi;$$

The next term that you will need is the ultimate tensile strength of the spring. The ultimate tensile strength will typically be given. Then with the alternating, mean and ultimate tensile strength, you can find the fatigue endurance stress with the Goodman fatigue criteria equation.

$$\sigma_{fatigue\ endurance} = \sigma_{alternating\ strength}/(1 - \frac{\sigma_{mean\ strength}}{\sigma_{ultimate\ tensile\ strength}})$$

Fatigue endurance for the Goodman fatigue criteria can also be shown graphically.

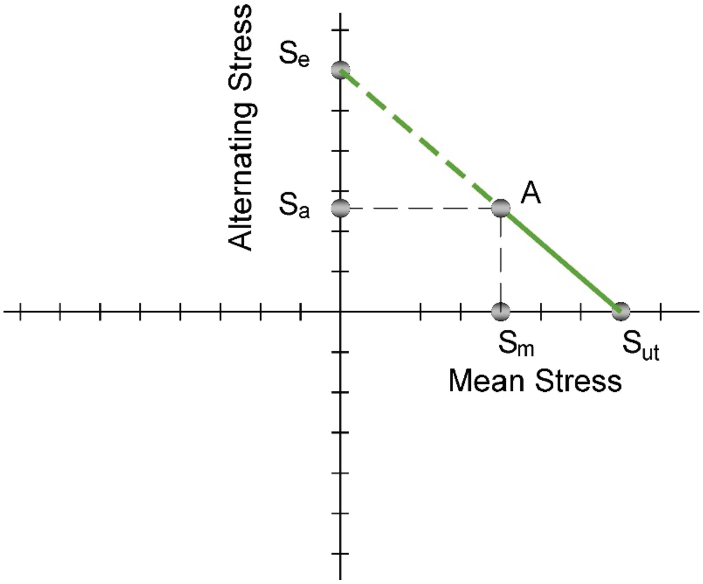


Figure 11: The fatigue stress in a spring can be found through the Goodman failure line as shown in the graph above. The ultimate tensile strength is shown as the maximum mean stress value, when the alternating stress is equal to zero. This is shown as S_{ut}. Point A is shown as having coordinates equal to (mean strength, alternating strength) (S_m, S_a.) These values were given for both peened and unpeened springs. Finally draw the line that connects (S_{ut}, 0) and (S_m, S_a.). The intersection of this line and the alternating stress axis will equal the fatigue endurance strength (S_e.)

This fatigue endurance strength component cannot be compared to the actual alternating stress component yet. Because this fatigue endurance strength component assumes that there is no mean stress. You need to use the fatigue endurance strength (S_e.) and ultimate tensile strength

(S_{ut}) values, along with the actual mean and alternating stress values to find out the actual fatigue endurance strength (S_{sa}) that is available at the actual mean stress values.

$$S_{sa} = \frac{rS_{se}S_{ut}}{rS_{ut} + S_{se}}$$

The value, "r" is equal to the ratio of the actual mean and alternating stress values.

$$r = \frac{\tau_a}{\tau_m}$$

Once you find the actual fatigue endurance strength, S_{sa}, then this value can be compared to actual alternating shear stress. If the strength is greater than the stress, then the spring will not fatigue.

6.0 PRESSURE VESSELS & PIPING

A pressure vessel is a container designed to hold gases or liquids at a pressure substantially higher than ambient pressure. In mechanical engineering, pressure vessels are used to hold compressed air or hydraulic fluid.

Pressure vessels are used to store both gases and liquids. Pressure vessels can come in a mixture of spherical and cylindrical shapes and these vessels can have thick or thin walls. The primary pressure vessels that you should know are the thin wall pressure vessels, but thick wall pressure vessels are also in the *NCEES FE Reference Handbook*.

Typical questions on the FE exam may include finding stresses. The stresses within the pressure vessel can be one of three types, (1) hoop stress, (2) longitudinal stress or (3) radial stress.

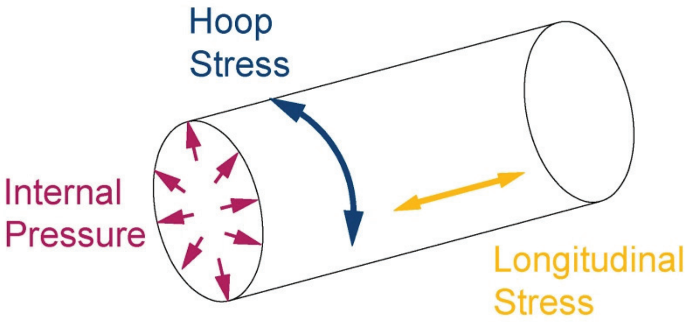


Figure 12: A pressure vessel will experience internal pressure due to the compressed fluid. This will result in stresses in the material in both the longitudinal direction and also in the tangential direction.

(1) Hoop Stress: Hoop stress can also be called tangential stress. This is the stress that is tangent to the surface of the pressure vessel.

(2) Longitudinal Stress: The longitudinal stress is sometimes called axial stress. This stress is parallel with the axis of the pressure vessel. This stress creates tension in the longitudinal direction.

(3) Radial Stress: Radial stress is the stress that pushes perpendicular against the surface of the pressure vessel from the inside. This would be equivalent to the internal pressure of the liquid or gas within the pressure vessel.

6.1 THIN VS. THICK WALL

The thin walled pressure vessels make calculations easier. A thin walled pressure vessel is defined as having a ratio of radius to thickness as greater than 10.

$$\frac{R}{t} > 10 \rightarrow thin\ walled\ pressure\ vessel\ assumption$$

$$\frac{R}{t} < 10 \rightarrow thick\ walled\ pressure\ vessel\ assumption$$

A thin walled pressure vessel uses membrane theory, which assumes that the hoop stress in the thickness of the wall does not vary. The hoop stress within the thickness of the wall is constant from the outside of the wall to the inside of the wall.

6.2 DESIGN FACTOR

A design factor is a factor of safety that is applied to the design of the pressure vessel. In engineering practice, you do not design a pressure vessel at its design pressure, you typically provide a factor of safety to ensure that the pressure vessel is designed to withstand much more than the expected stresses in the pressure vessel.

6.3 MATERIALS

The pressure vessel stresses should be designed well below the maximum allowable stresses of the pressure vessel material. For the exam, you may need to compare the material properties with the calculated stress within the pressure vessel. These material properties correspond to various failure conditions and you should be familiar with the meaning of each condition.

- Ultimate tensile strength or tensile strength: The maximum stress before failure.
- Yield strength: The maximum stress before permanent deformation occurs.

If the problem gives you the material properties and the calculated stress within the pressure vessel, then you can calculate the safety factor/design factor. This is the factor or multiple of the actual stress acting upon the pressure vessel can handle before a specific failure condition.

$$If\ the\ question\ asks\ to\ compare\ ultimate\ tensilte\ stress, then$$

$$Safety\ factor = \frac{\sigma_{ultimate\ tensile\ stress}}{\sigma_{actual}}$$

$$If\ the\ question\ asks\ to\ compare\ yield\ strength, then$$

$$Safety\ factor = \frac{\sigma_{yield\ strength}}{\sigma_{actual}}$$

6.4 SPHERICAL VS. CYLINDRICAL

The two main pressure vessel shapes in Mechanical engineering are spherical and cylindrical. The hoop stress for these pressure vessels will be dependent on the internal pressure (assuming no external pressure), radius and thickness. These equations only work if the pressure vessel qualifies as a thin walled pressure vessel.

$$for\ cylindrical\ thin\ walled\ pressure\ vessels \rightarrow\ \sigma_{hoop} = \frac{PR}{t}; \sigma_{longtiudinal} = \frac{\sigma_{hoop}}{2}$$

$$for\ spherical\ thin\ walled\ pressure\ vessels \rightarrow \sigma_{hoop} = \frac{PR}{2T}; \sigma_{longtiudinal} = \sigma_{hoop}$$

$$where\ \sigma_{hoop} = hoop\ stress\ (psi); P = internal\ pressure\ (psi); R = radius\ (in)$$

$$t = thickness\ (in)$$

The hoop stress is typically the design criteria for pressure vessels, since this stress is much higher than the internal pressure and is twice the longitudinal stress for cylindrical pressure vessels. The hoop stress and the longitudinal stress are equal to one another because stresses are the same in all direction for spheres. There is no longitudinal axis in a sphere.

6.5 THICK WALL THEORY

If you encounter a thick wall pressure vessel, then you cannot use the previous equations. You must use the following Thick Wall Theory equations. These equations show that the stress in the thickness of the wall of the pressure vessel will vary based on the location that stress is measured within the wall thickness. These equations are shown in the *NCEES FE Reference Handbook*. The key is just knowing when to use the thick wall and thin wall equations.

The hoop stress and radial stress will vary based on location “r”.

$$Tangential(Hoop)\ Stress \rightarrow \sigma_T = \sigma_\theta = \frac{p_{in} * r_{in}^2 - p_{ext} * r_{out}^2 - r_{in}^2 * r_{out}^2 (p_{ext} - p_{in})/r^2}{r_{out}^2 - r_{in}^2}$$

$$Radial\ Stress \rightarrow \sigma_r = \frac{p_{in}*r_{in}^2 - p_{ext}*r_{out}^2 + r_{in}^2*r_{out}^2(p_{ext}-p_{in})/r^2}{r_{out}^2 - r_{in}^2}$$

$$Longitudinal\ Stress \rightarrow \sigma_l = \frac{p_{in}*r_{in}^2 - p_{ext}*r_{out}^2}{r_{out}^2 - r_{in}^2}$$

The longitudinal stress is independent of location “r”.

$$p_{in} = internal\ pressure;\ p_{ext} = external\ pressure;$$

$$r_{in} = internal\ radius; r_{out} = outer\ radius;\ r = location\ of\ stress\ measurement\ (r_{in} \leq r \leq r_{out})$$

If you substitute the external pressure as zero, then the hoop stress equation reduces to the following.

At inside surface, $r = r_{in}$. Internal pressure only.

$$Hoop\ Stress \rightarrow \sigma_T = p_{in}(\frac{r_{in}^2 + r_{out}^2}{r_{out}^2 - r_{in}^2})$$

$$Radial\ Stress \rightarrow \sigma_R = -p_{in}$$

$$Longitudinal\ Stress \rightarrow \sigma_l = \frac{p_{in} * r_{in}^2 - p_{ext} * r_{out}^2}{r_{out}^2 - r_{in}^2}$$

At outside surface, $r = r_{out}$. Internal pressure only.

$$Hoop\ Stress \rightarrow \sigma_T = (\frac{2p_{in}r_{in}^2}{r_{out}^2 - r_{in}^2})$$

$$Radial\ Stress \rightarrow \sigma_R = 0$$

$$Longitudinal\ Stress \rightarrow \sigma_l = \frac{p_{in} * r_{in}^2 - p_{ext} * r_{out}^2}{r_{out}^2 - r_{in}^2}$$

Typically pressure vessels are only subject to internal pressures. However, you should be able to deduce the equations for external pressure only, with the critical condition occurring at the outer or inner radius.

6.5 PIPING

Piping is used in mechanical engineering to transport fluids (gases or liquids). The typical FE questions that can result around piping are calculating the pressure drop within the pipe, calculating the Reynolds number of a fluid flowing through a pipe and calculating the velocity within a pipe. These skills are located under the Hydraulic Components Heading and the 15.1 Pumps Sub-Heading.

7.0 BEARINGS

Bearings are used to separate two components, while permitting relative motion between the two components. Bearings are designed to have very little friction and often have a lubricant between the bearing and the components.

There are two main types of bearings, plain surface bearings and rolling contact bearings. A rolling contact bearing consists of either a ball or roller between the two components. A plain surface or sliding bearing or journal bearing consists of a bearing surface and no rolling elements. This is common of a shaft rotating within a bushing. This is a housing that sits around a shaft that provides a bearing surface. The FE exam will most likely focus on the rolling contact bearings and the lubrication of plain surface bearings. Each type of bearing has sub-types as shown in the figure below.

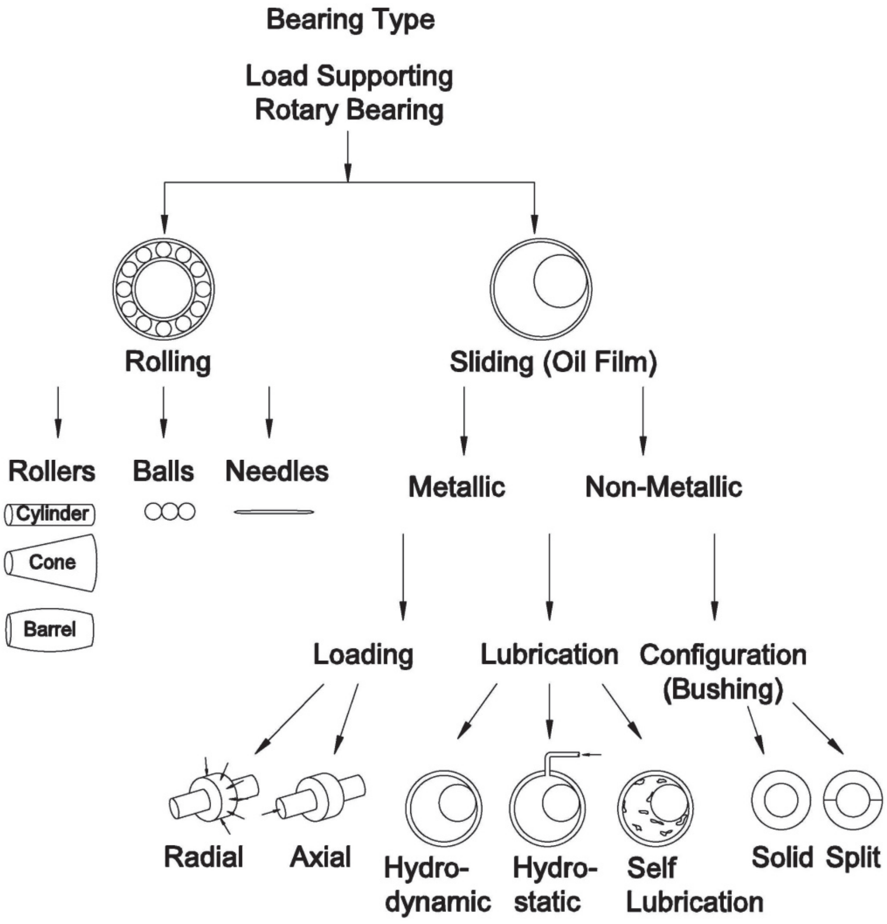

Bearing Type
Load Supporting
Rotary Bearing
Rolling
Sliding (Oil Film)
Rollers
Cylinder
Cone
Barrel
Balls
Needles
Metallic
Non-Metallic
Loading
Lubrication
Configuration
(Bushing)
Radial
Axial
Hydro-
dynamic
Hydro-
static
Self
Lubrication
Solid
Split

Figure 13: A rolling bearing consists of rolling elements like rollers, balls or needles. A plain surface bearing or sliding bearing consists of a bearing surface with lubrication. Rolling

bearings can also have lubrication but lubrication type problems are more consistent with plain surface bearings.

In the Machine Design field, the two types of bearings each have their own typical design questions that can occur on the FE exam. For roller contact bearings, typical questions focus on the load and life of the bearing. In application, you will encounter bearings that have a rated load and rated life. You will have to interpret this information and apply it to your situation. This typical process will be discussed in the life-load analysis.

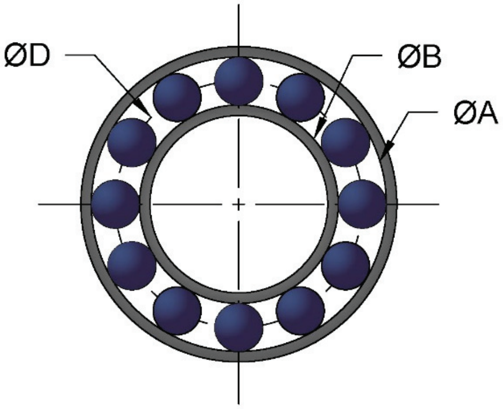


Figure 14: This figure shows ball bearings that separate the rotating shaft from the housing. The housing has a diameter, "A", the shaft has an outer diameter of "B" and the center line of the ball bearings occurs on the circle with diameter "D".

7.1 LIFE-LOAD ANALYSIS

The life load analysis topic in the *NCEES FE Reference Handbook* primarily refers to roller contact bearings, since these bearings will fail due to fatigue and their life is dependent on the load. The basic equation that you must use for life-load analysis type questions for ball bearings is as shown below. The life of a ball bearing varies by the cubed of the load. This means that if the load on a ball bearing is doubled, then the life of the ball bearing compared to the original life is cut by 1/8th.

$$Ball\ Bearings \rightarrow \frac{Life_{condition\ 2}}{Life_{condition\ 1}} = \left(\frac{Load_{condition\ 1}}{Load_{condition\ 2}}\right)^{3}$$

$$Roller\ Bearings \rightarrow \frac{Life_{condition\ 2}}{Life_{condition\ 1}} = \left(\frac{Load_{condition\ 1}}{Load_{condition\ 2}}\right)^{3.33}$$

Roller bearings have a different ratio from ball bearings. These roller bearings are cylindrical or needle type bearings.

Roller contact bearing manufacturers provide data for the load, rating life, life, median life, basic load rating, etc. The rating life is the life at which 10% of bearings have failed and is shown as the variable, L_{10}. Life is determined as the number of revolutions the roller bearing can withstand prior to showing signs of fatigue. The median life is the life at which 50% of the bearing have failed and is shown as the variable, L_{50}. These life values are given as a function of the rated load, C. If your actual load is different from the rated load, then you must use the previous equations to find the actual life of the roller bearing under your actual load.

It is important to note that the load shown is the dynamic load. This means that the bearings are rotating and there is no static concentrated load on the bearings. If the bearing is not moving, then there will be a static concentrated load on the bearing. The bearing must be strong enough to withstand this load. The capacity given by manufacturers is shown as the variable, P_{st}. This is the maximum load that the bearing can withstand when at rest. If this value is exceeded then the bearing will be indented and would lead to the failure of the bearing through crack propagation.

8.0 POWER SCREWS

The purpose of a power screw is to convert rotary motion to linear motion. The simplest example is a metal screw and a nut. When installing the metal screw into a nut, you must rotate the screw in order to cause the screw to move linearly past the nut. The threading on the screw causes it to move downward or upward based on whether or not the rotation is clockwise or counterclockwise. In this example, the screw is moved and the nut is at rest, but the opposite can also be true. The screw can be at rest and the nut can move up and down the screw.

In Machine Design, power screws are used to exert large directional forces, like with presses and to lock objects in place, like with C-clamps. Power screws are able to translate rotational force from motors to exert a large directional force.

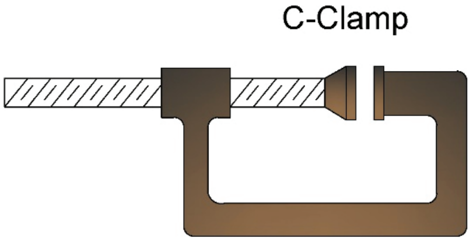


Figure 15: A C-clamp is an example of a power screw

8.1 TYPES

There are many different types of power screws that vary based on the shape of the thread.

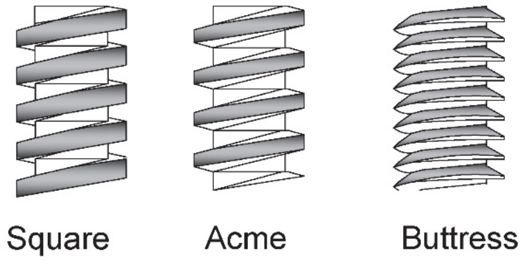


Figure 16: There are three main types of power screws.

Square threads are the most efficient type of power screws at transforming rotational energy into linear energy. The thread angle or angle between the threads is equal to zero degrees. The angle from one thread to the next does not change.

Acme threads are the most common type of power screw, because it is easy to construct. This type of power screws has a thread angle of 19 degrees. The angle between the threads causes the efficiency of the acme thread power screw to be less than the square thread power screw.

Buttress threads are of a triangular shape. These are used where the load force on the screw is only applied in one direction.

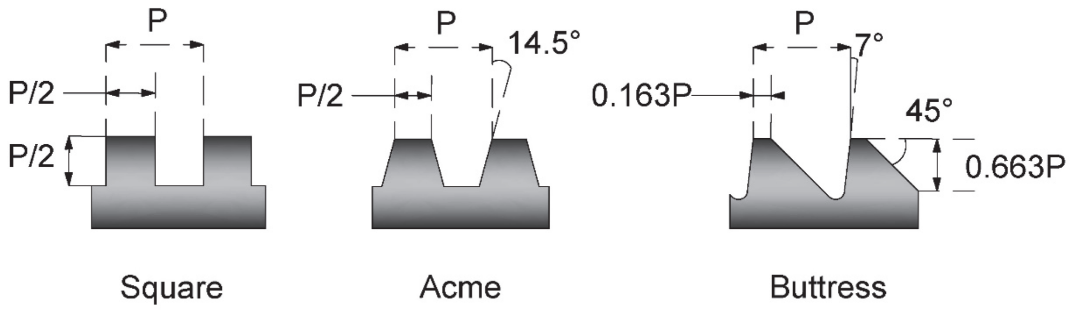
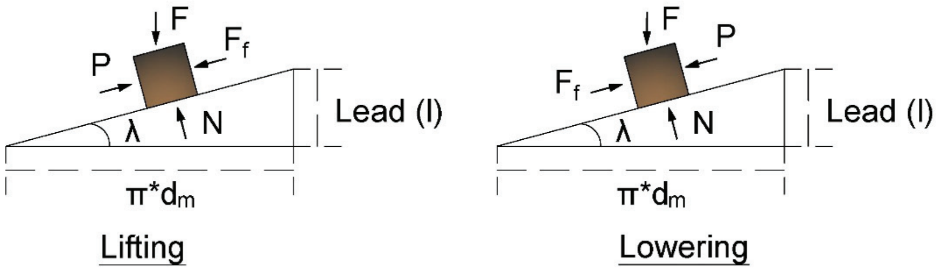


Figure 17: This figure shows the difference in construction between the different types of power screws.

8.2 LIFTING AND LOWERING TORQUE

Power screws are often used to lift and lower objects. For the purposes of the exam, you should be able to calculate the amount of torque to exert a force that can be used to lift or lower an object.



Lifting

Lowering

Figure 18: This figure shows a single thread. The base of the triangle is equal to the circumference of the power screw. Imagine that the circumference is unwound. The lead is the distance to the next thread, also known as the pitch. The angle shown assumes the thread is an acme thread. If the thread was square then the angle would be 90 degrees. In the lifting situation a downward force "F" is equal to the load that needs to be moved. The frictional force acts against the applied force, "P". In the lowering situation, the applied force is downward and the frictional force opposes the applied force. The load, "P" and the normal force "N" are the same in both situations.

In order to complete the lifting and lowering torque type problems on the FE exam, you need to be giving the pitch, lead and diameters for the specific power screw. This information is typically given in a power screw table.

In these tables the following terms will be introduced, Major Diameter, Minor Diameter and Pitch Diameter. The major diameter is high extreme of the thread and is typically the outside diameter of the power screw. The minor diameter is the low extreme diameter of the thread. The pitch diameter is the diameter that is used in the torque lowering and lifting calculations. Sometimes the pitch diameter can also be called the mean or nominal diameter.

The lead angle will also be given in these tables and is typically in the range of 1 to 5.7 degrees for ACME threads.

The lead distance is the distance the screw moves in one revolution and pitch is the distance from one thread to the next thread. If one thread is advanced in a single revolution, then the power screw is a single start. If the thread is a double start, then the screw moves two threads in one revolution and if the thread is a quad start, then the screw moves four threads in one revolution. This is important because the tables often only give the pitch distance, but not the lead distance. The relationship between pitch and lead are shown below.

Start	Pitch	Lead Distance
Single	P	L = P
Double	P	L = 2*P
Quad	P	L + 4*P

The next value that you need to complete a lifting/lowering torque problem is the coefficient of friction. This will depend on the material and most likely will be given to you in the exam, since the coefficient of friction will vary widely. The following equations for lifting and raising for square threads is shown in the *NCEES FE Reference Handbook*.

Once you have all these values, you can use the following equation for square threads to calculate the lifting and lowering torque.

$$Square\ Threads \rightarrow T_{lift} = \frac{Fd_m}{2} * \left(\frac{\pi\mu d_m + l}{\pi d_m - \mu l}\right)$$

$$Square\ Threads \rightarrow T_{lower} = \frac{Fd_m}{2} * \left(\frac{\pi\mu d_m - l}{\pi d_m + \mu l}\right)$$

$$Torque = (N \cdot m\ or\ lb \cdot ft); F = screw\ load\ (N\ or\ lbf); d_m = mean\ or\ pitch\ diameter (m\ or\ ft)$$

$$\mu = coefficient\ of\ friction\ (unitless); l = lead\ (m\ or\ ft);$$

Another version of the same equations above uses the thread or lead angle. The lead angle is found with the equation below and in some tables, the lead angle is already calculated for you.

$$\tan(\lambda) = \left(\frac{L}{\pi * d_m}\right)$$

$$Square\ Threads \rightarrow T_{lift} = \frac{Fd_m}{2} * \left(\frac{f + \tan(\lambda)}{1 - \tan(\lambda)}\right)$$

$$Square\ Threads \rightarrow T_{lower} = \frac{Fd_m}{2} * \left(\frac{f - \tan(\lambda)}{1 + \tan(\lambda)}\right)$$

$$f = coefficient\ of\ friction$$

The following equations are applicable for ACME threads. Since this equation is not shown in the handbook you can assume that it won't be on the exam. So this information is just for your general knowledge.

$$Acme\ Threads \rightarrow T_{lift} = \frac{Fd_m}{2} * \left(\frac{f + \cos(\phi) * \tan(\lambda)}{\cos(\phi) - \tan(\lambda) * f}\right)$$

$$Acme\ Threads \rightarrow T_{lower} = \frac{Fd_m}{2} * \left(\frac{f - \cos(\phi) * \tan(\lambda)}{\cos(\phi) + \tan(\lambda) * f}\right)$$

The variables are defined below.

$$Torque = (N \cdot m\ or\ lb \cdot ft); F = screw\ load; d_m = mean\ diameter\ or\ pitch\ diameter$$

$$\mu = coefficient\ of\ friction\ (unitless); l = lead;$$

$$\phi = angle\ of\ friction = 14.5° \ for\ acme\ threads;\ \lambda = lead\ angle;$$

Sometimes, a power screw has a collar. The collar will require additional torque due to the frictional forces between the collar and the load. The calculation for this additional torque can be found with the below equation. This is also shown in the *NCEES FE Reference Handbook* as part of the square thread power screw total torque equation. The handbook adds this value to the previously shown square thread power screw torque equation.

$$T_{collar} = \frac{F * f * d_c}{2}$$

$$f = collar\ coefficient\ of\ friction;\ d_c = mean\ diameter\ of\ collar;$$

The torque for lifting and lowering can then be translated to required power, based on the speed required.

$$Torque = \frac{Power * 5{,}252}{\omega}$$

$$Torque = (lb \cdot ft); Power = (HP); \omega = speed\ (revolutions\ per\ minute)$$

8.3 LOCKING CONDITIONS

Locking occurs when the lead angle is not steep. If the lead angle is too steep, then the force acting upon the thread will be greater than the frictional force that opposes the movement of the object. The exact angle can be found through the equation below.

$$f > \tan(\lambda)$$

8.4 POWER & EFFICIENCY

Power transmitted by a power screw can be found by using the torque and the power equation. The efficiency is found by dividing the output power by the input power. The input power is found through the below equation which takes the torque multiplied by the rotational speed. This assumes that the rotational speed is constant.

$$Power\ Input\ (Horsepower) = Torque(lb \cdot ft) * RPM/5{,}252$$

The output power is equal to the input power minus the power lost due to friction. This output power is also equal to the power required to move the screw without friction. You can substitute a value of f = 0, for the previous equations to find the torque without friction. For acme threads, the equation is reduced to the following.

$$Acme\ Threads\ (no\ friction) \rightarrow T_{lift} = \frac{FL}{2\pi}$$

$$Acme\ Threads\ (no\ friction) \rightarrow T_{lower} = -\frac{FL}{2\pi}$$

The efficiency for an acme thread screw can be found with the below equation.

$$Efficiency\ (acme\ thread) = \frac{T_{no\ friction}}{T_{with\ friction}}$$

$$Efficiency\ (acme\ thread) = \frac{\frac{FL}{2\pi}}{T_{with\ friction}}$$

9.0 POWER TRANSMISSION

All mechanical power for mechanical components is provided by motors, but motors only provide rotational energy to shaft and mechanical components need a way to transmit this power from rotational to other forms as required. The power transmission system consists of shafts, axles, gears, cams, crankshafts, belts, pulleys, etc. But for the purposes of the exam you need to be knowledgeable about shafts, belts, pulleys and gears, since these are the only components of the power transmission system that is shown in the *NCEES FE Reference Handbook* and the other components require equations of their own.

9.1 GEARS

Gears are used to transmit rotational energy from one shaft to another shaft. Gears can also be used to adjust the speed to match a certain requirement. For the purposes of the FE exam you should have a basic understanding of the construction of gears, the different types of gears, how speeds can be changed with gear ratios and the forces present in a gear.

Gear ratios are defined as the ratio of teeth, N, between two mated gears. The number of teeth for gears can be used since the spacing between the teeth is the same, in order for the gears to properly mate.

$$Gear\ Ratio = \frac{N_1}{N_2}$$

An important thing to understand is the relationship between the rotational speeds, n, of two gears is the inverse of the quantity of teeth, N, in each gear and that the ratio of torque, T, is the inverse of the rotational speeds.

$$\frac{N_1}{N_2} = \frac{n_2}{n_1} = \frac{T_1}{T_2}$$

Typically the gear ratio is given in terms of the driven over the driving or the output over the input gear. In the previous equations, n_1 is the driven gear speed and n_2 is the driving gear speed.

The various gear types are identified by the geometry of their teeth. Common gear types include spur, helical, bevel, and worm gears.

9.1.1 Spur Gears

The spur gear is the most commonly seen gear type, where the teeth are straight and are used to transfer rotation between parallel shafts.

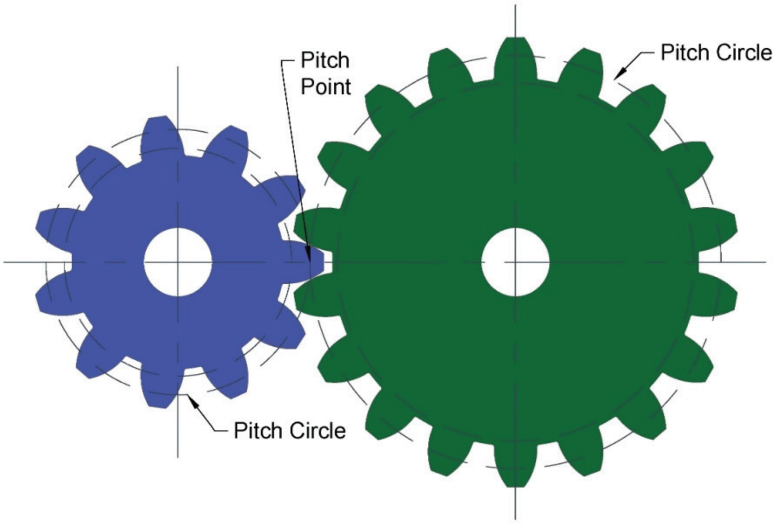


Figure 19: Spur Gears

9.1.2 Helical Gears

Helical gears on the other hand have teeth that are angled, as shown in the figure below. The shafts between helical gears are also generally parallel to each other, but perpendicular arrangements are also possible. The advantage of a helical gear is lower stresses due to the larger contact ratios. The contact is also smoother since the teeth roll on each other. The disadvantage is an additional axial thrust is generated for helical gears because of the angled teeth. Therefore, the bearings must be designed for this additional force. Helical gears are characterized by the angle of their teeth, also known as the helix angle, and direction of the angle, left or right, when lying flat and facing up.

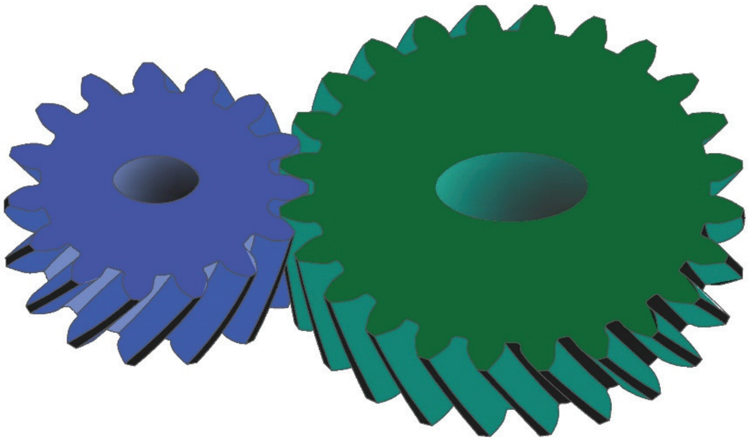


Figure 20: Helical gears in parallel arrangement

9.1.3 Bevel Gears

Bevel gears have teeth oriented on the top face of the gear in a cone-like manner. Bevel gears are commonly used for perpendicular shaft orientations. The teeth on bevel gears can be straight like spur gears or angled like helical gears. Bevel gears can also be oriented where the perpendicular shafts are offset, where the axis are no longer in the same plane. This is called a hypoid gear.

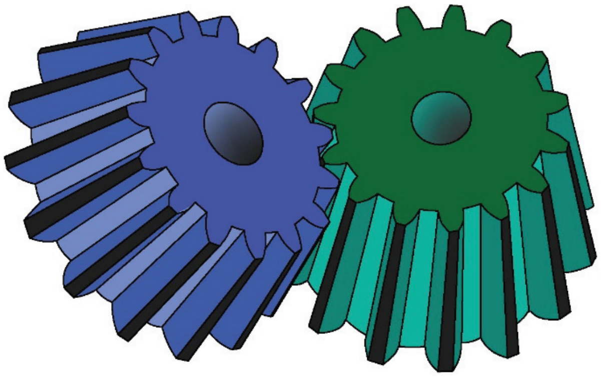


Figure 21: Bevel gears

9.1.4 Worm Gears

Worm gears are perpendicular in shaft orientation, where there is a worm that looks like a screw, and a worm gear that looks like the more typical gear. The worm is used to rotate the worm gear and is used where large speed reductions are required. The major disadvantage of the worm gear is the larger frictional losses at the contact of the gears.

9.1.5 Gear Construction

Gears must have similar shape and size of teeth in order to mate with each other. It is the difference in the quantity of teeth between two gears that creates the transfer of rotational speed or power. To perform calculations for gears, the following terminology of a gear's construction is defined.

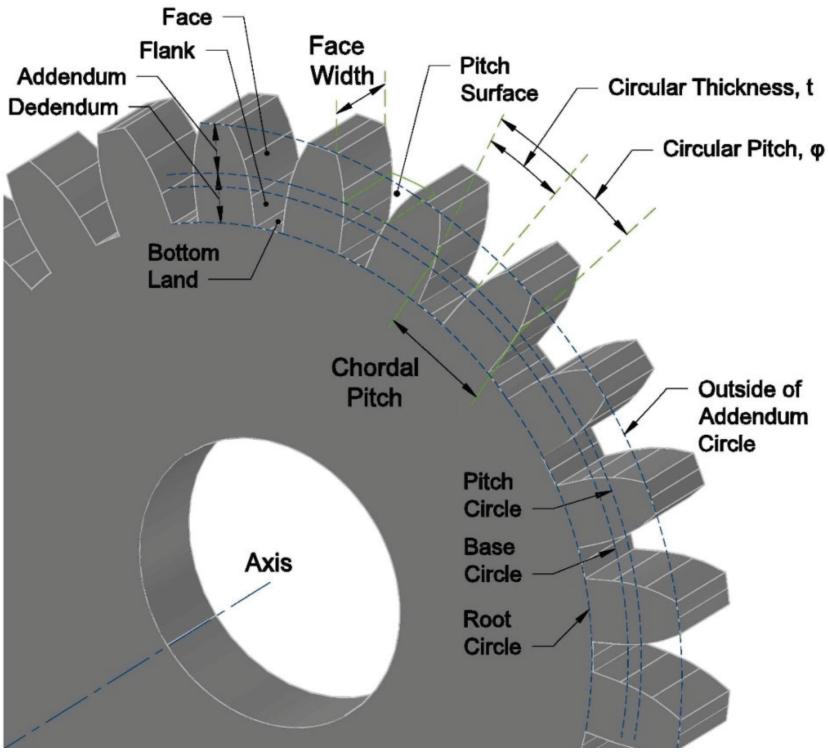

Face
Flank
Addendum
Dedendum
Face Width
Pitch Surface
Circular Thickness, t
Circular Pitch, φ
Bottom Land
Chordal Pitch
Outside of Addendum Circle
Pitch Circle
Base Circle
Root Circle
Axis

Figure 22: Gear Terminology – Spur Gear

9.1.5.1 Pitch Circle

The *pitch circle* is used to represent the diameter, d, of a gear. The pitch circle is sized such that the diameter of two gears would meet at a single, tangent point (i.e., not overlapping, and not too small where they never meet). See the figure in the Spur Gears section above for a diagram of the pitch circle with two mated gears.

9.1.5.2 Circular Pitch

The *circular pitch*, p, is the arc distance from on gear tooth to the adjacent tooth, measured along the pitch circle. This distance is found by the following equation, where d is the diameter of a gear and N is the number of teeth. A smaller pitch means there are more teeth for a given diameter.

$$p\ [in] = \frac{d\pi}{N}\ (spur\ gear)$$

Because the circular pitches of two gears must be the same for a perfect mate, the following relationship applies for gears 1 and 2.

$$p = \frac{d_1\pi}{N_1} = \frac{d_2\pi}{N_2}$$

9.1.5.3 DIAMETRAL PITCH

The *diametral pitch,* P_d, is the most common way in industry to describe the pitch of a gear. It is basically the number of teeth per diameter. A diametral pitch less than 20 in^{-1} is considered course, and anything above that is fine.

$$P_d\ [in^{-1}] = \frac{N_1}{d_1} = \frac{N_2}{d_2}$$

9.1.5.4 HELICAL GEAR PITCHES

It is unlikely that there will be a question on helical gears because it is more in depth into a specific gear type, and the list of Mechanical Components topics is broad. However, helical pitches are mentioned briefly here. As discussed previously, helical teeth are angled at an angle, ψ, which creates an additional axial force.

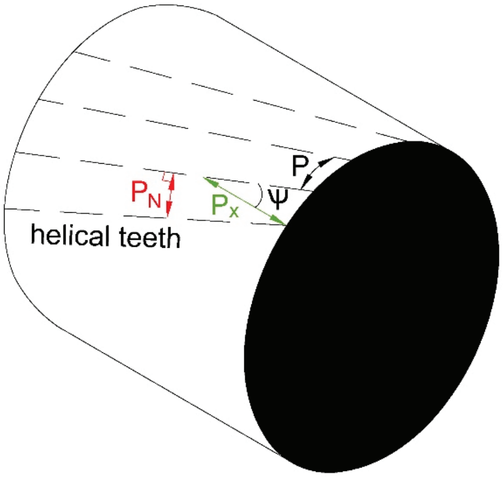


Figure 23: A helical gear has different pitches given its unique geometry.

The equation for the circular pitch remains the same as for spur gears. However, there is an additional pitch to account for, called the *normal circular pitch,* p_n. Whereas the circular pitch measures the distance between adjacent teeth when looking at the flat face of the gear, the normal circular pitch is the distance between teeth measured perpendicular from teeth to teeth.

$$Normal\ Circular\ Pitch: p_n\ [in] = pcos(\psi) = \frac{d\pi}{N}cos(\psi) \quad (helical\ gear)$$

The *normal diametral pitch* is found in a similar manner.

$$Normal\ Diametral\ Pitch: P_{nd}\ [in^{-1}] = \frac{P_d}{cos(\psi)} = \frac{N}{d * cos(\psi)} \quad (helical\ gear)$$

Finally, the axial pitch is the distance from teeth to teeth, measured parallel to the gear axis.

$$Axial\ Pitch: P_x\ [in^{-1}] = \frac{p}{tan(\psi)} = \frac{\pi}{P_d * tan(\psi)} \quad (helical\ gear)$$

9.1.5.4 TOOTH THICKNESS

The *tooth thickness*, t, is a measure of how wide the tooth is. The tooth thickness is equal to half the pitch.

$$t\ [in] = \frac{p}{2} = \frac{d\pi}{2N}$$

9.1.5.5 ADDENDUM AND DEDENDUM

The *addendum* is the distance between the pitch circle and the top of the teeth. The *dedendum* is the distance between the pitch circle and the bottom of the teeth. The significance of the addendum and the dedendum is that it is used to find the *clearance* between two gears. The clearance is the space between the bottom landing of one gear and the top of the tooth of another gear when they mate.

$$Clearance\ [in] = Dedendum_{Gear\ 1} - Addendum_{Gear\ 2}$$

9.1.5.6 LINE OF ACTION/PRESSURE ANGLE

The image below is a simplified version of the spur gear figure in the previous section, with the teeth removed. The *pressure line*, also known as the *line of action*, is a line drawn between the base circles of two mated gears, where the line is tangent to each base circle (i.e., the line of action is perpendicular to the line drawn from the center of each gear to the point at which the line of action intersects the base circle).

The *pressure angle*, δ, is the angle between the pressure line and the line tangent to both pitch circles. The force from the driving gear is applied onto the second gear along the line of action. The pressure angle is used to find the direction of this force.

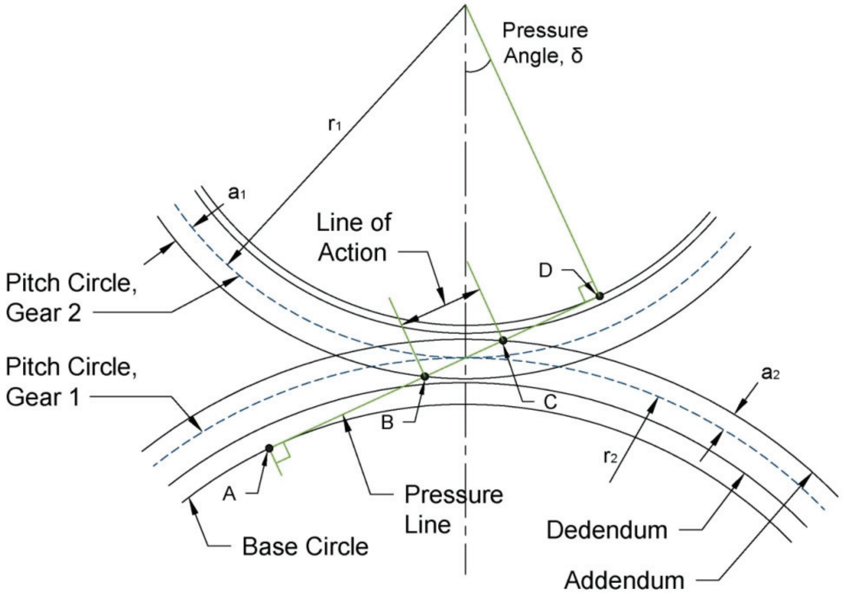


Figure 24: Line of Action

9.1.6 Speed Analysis

When two gears are mated together the gear that is driving the rotation will cause a rotational speed that is in inverse proportion to the ratio of their diameters or gear ratio.

$$w_{output} = \omega_{input} * (Gear\ Ratio)^{-1}\ or$$

$$w_{output} = \omega_{input} * \left(\frac{D_{ouput}}{D_{input}}\right)^{-1}\quad or$$

$$w_{output} * D_{ouput} = \omega_{input} * D_{input}$$

9.1.7 Force Analysis

When two gears are mated together the gear that is driving the rotation will cause a force that varies based on the gear ratio. The important thing to remember is that the torque in the driving (input) gear will equal the torque in the driven (output) gear. If you are given the horsepower and rotational speed of the driving gear then you can determine the torque.

$$Torque_{input} = Power * Rotational\ Speed$$

The difficulty with these equations is that the question may not give you the terms in the correct units, so please be careful when using this equation and make sure you convert to the necessary terms.

$$Torque_{input}(lb \cdot in) = 63{,}000 * \frac{Power\ (HP)}{N\ (RPM)}$$

$$Torque_{input} = Torque_{output}$$

The torque on the output gear will cause a force that is tangential to the rotation of the gear. This tangential force can be computed with the following equation.

$$F_{input\ gear,tangential} * r_{input\ gear} = F_{output\ gear,tangential} * r_{output\ gear}$$

$$r = gear\ radius$$

9.2 SHAFTS

In the *NCEES FE Reference Handbook*, there is a reference to shafts and axles. Shafts are rods that are connected to a motor that is used to translate the rotational energy from the motor to another mechanical component like a gear or camshaft. An axle is a specific type of shaft that translates the rotational energy to a set of wheel(s).

There are three stresses that act upon the shaft, (1) stress due to torsion, stress due to (2) bending and stress due to (3) an axial load. The stress due to torsion is a function of the torque, radius and the polar moment of inertia. This equation was shown previously in *Section 8.0 Mechanics of Materials*, but is shown again

$$\tau = \frac{T * \left(\frac{D}{2}\right)}{J}; J = \frac{\pi D^4}{32} (for\ solid\ circular\ shafts)$$

When you plug in the equation for the polar moment of inertia into the shear stress equation you get the following equation.

$$\tau = \frac{16 * T}{\pi * D^3}$$

The next stress is due to the bending moment. The stress due to a bending moment is shown here and it was also shown in the *Section 8.0 Mechanics of Materials*.

$$\sigma = \frac{M * \left(\frac{D}{2}\right)}{I}; I = \frac{\pi D^4}{64} (for\ solid\ circular\ shafts)$$

When you plug in the equation for the moment of inertia into the stress equation you get the following equation.

$$\sigma_M = \frac{32 * M}{\pi * D^3}$$

The next stress is due to any axial forces acting on the shaft. The stress due to an axial force is simply the force divided by the area, which is shown here and it was also shown in the *Section 8.0 Mechanics of Materials*.

$$\sigma_A = \frac{F}{\frac{\pi * D^2}{4}} = \frac{4 * F}{\pi * D^2}$$

9.2.1 Von Mises Stress

The von mises stress was discussed in *Section 8.0 Mechanics of Materials*, but it is reiterated here to show you the von mises stress for a circular, solid shaft.

$$\sigma_V = \sqrt{(\sigma_M + \sigma_A)^2 + 3\tau^2}$$

The von mises stress is a function of the stress due to the bending moment, axial load and the torsion. These three stresses are combined with the above equation. If you substitute the previously calculated stress equations in to the above equation, the result will be as shown below.

$$\sigma_V = \sqrt{\left[\left(\frac{4 * F}{\pi * D^2}\right) + \frac{32 * M}{\pi * D^3}\right]^2 + 3\left(\frac{16 * T}{\pi * D^3}\right)^2}$$

You can simplify this equation by removing the π and D terms.

$$\sigma_V = \frac{1}{\pi * D^3}\sqrt{[(4 * F * D) + (32 * M)]^2 \ + 3 * (16 * T)^2}$$

Another common element that you can remove is the “4” value.

$$\sigma_V = \frac{4}{\pi * D^3}\sqrt{[(FD) + (8M)]^2 \ + 3 * (4T)^2}$$

Finally, you can simplify the value with the “T” term and you will get the final equation shown in the *NCEES FE Reference Handbook*. Now that you understand the origin of this equation, you should know that this equation only applies to solid shafts with an axial, bending moment and torsion load.

$$\sigma_V = \frac{4}{\pi * D^3}\sqrt{[(FD) + (8M)]^2 \ + 48 * T^2}$$

The maximum shear stress equation is found using a similar method, with the maximum shear stress equation from *Section 8.0 Mechanics of Materials*.

$$\tau_{max} = \sqrt{\frac{(\sigma_M + \sigma_A)^2}{2} + \tau^2}$$

$$\tau_{max} = \sqrt{[\left(\frac{2 * F}{\pi * D^2}\right) + \frac{16 * M}{\pi * D^3}]^2 \ + \left(\frac{16 * T}{\pi * D^3}\right)^2}$$

$$\tau_{max} = \frac{1}{\pi * D^3}\sqrt{[(2 * F * D) + (16 * M)]^2 \ + (16 * T)^2}$$

The same process leads you to the equation for the maximum shear stress of a solid circular shaft. This equation is what is shown in the *NCEES FE Reference Handbook*.

$$\sigma_V = \frac{2}{\pi * D^3}\sqrt{[(FD) + (8M)]^2 \ + (8T)^2}$$

10.0 JOINING METHODS

There are three main methods for joining mechanical components, (1) welding & brazing, (2) fasteners and (3) adhesives. There are no equations for (1) welding & brazing or (3) adhesives in the *NCEES FE Reference Handbook*, so you can assume that any questions on joining methods will be on fasteners. However, this section will briefly go over welding, brazing and adhesives just for your information.

10.1 WELDING AND BRAZING

Welding and brazing are methods of joining two metal pieces together. Welding joins the metals together through fusing, where the two metal pieces are melted and mixed together along with a filler material. Brazing is different because the two metal pieces are not melted together. In brazing, a filler material is melted and creates a bond with itself and the two metal pieces, thus joining them together. Brazing does not melt the two metals because a lower temperature is used to selectively only melt the filler material.

10.2 FASTENERS

Bolts, screws and rivets are types of fasteners. These types of fasteners are not as strong as the joints created with welding, but are stronger than adhesives. Fasteners are defined as a device that connects two or more components. There are a lot of different types of fasteners but the FE exam luckily focuses only on bolts, screws and rivets.

In preparation for the FE exam, you should be able to calculate the forces and stresses that act upon bolts, screws and rivets. This section will guide you through these calculations.

Finally, this section will cover forces in a group of fasteners (bolts, screws and rivets).

10.3 BOLTS

A bolt is a type of fastener that passes through a hole in two or more components and is then tightened by a nut on its threaded end. A screw has similar construction, with a head and a threaded end. The difference between a screw and a bolt is in their intended use. Bolts are used to fasten together two unthreaded components with a nut. Screws are used to join components, where at least one of the components is threaded. For example, a screw that is drilled into two pieces of wood joins two threaded components. The screw creates the threads within the wood. A bolt that connects two pieces of wood would require a non-threaded hole. The bolt would be set into the hole, connecting the two pieces of wood. A nut would then be used to tighten the two pieces together.

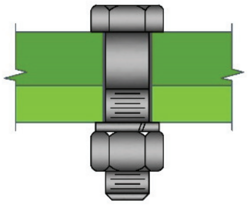


Figure 25: A bolt connects two non-threaded components. A bolt is placed at the threaded end of the bolt to secure and tighten the two components together.

The nut is threaded and matched to the bolt. The lifting and lowering torque required to tighten or loosen the nut is similar to the equation shown for power screws. If you encounter a question with a threaded bolt or screw, you may either have to check this section or the power screw section.

The last aspect of bolts is that bolts are removable, similar to screws. Rivets on the other hand are permanent fasteners.

10.3.1 Tension or Clamping Force

The main FE questions will revolve around the purpose of the bolt, which is to clamp two components together. The clamping force causes the bolt to stretch, which means the bolt is undergoing tension. The bolt can also undergo more tension as the joined components undergo more loading. The bolt must be strong enough to resist this tension. The bolt will experience other external loads like moment and shear loads, which will be discussed later.

When a bolt is secured to two components, it produces an initial tension also known as pretension or bolt preload. This is the first force that acts upon the bolt.

The next force is the force due to an external tension load. When a tension load acts upon the two components that are joined together, the amount of external load applied will be divided between the two components and the bolt(s). The equation that determines how much of the load acts upon the bolt and how much of the load acts upon the two components is shown below. In this equation, the initial tension load is added to a fraction of the external tension load or separating force. This fraction is determined by the ratio of the bolt stiffness to the sum of the bolt and clamping stiffness.

$$F_b = F_i + \frac{k_b}{k_b + k_c} F_e$$

$$F_i = initial\ tension\ (N\ or\ lbf); F_b = bolt\ force; F_e = external\ separating\ force$$

$$k_b = bolt\ stiffness\ \left(\frac{N}{m}\ or \frac{lbf}{in}\right);\ k_c = clamping\ stiffness\ \left(\frac{N}{m}\ or \frac{lbf}{in}\right);$$

The bolt stiffness and clamping stiffness values are very difficult to calculate. It is highly unlikely that you will have to calculate these values and it is more likely that you will be given the external force that acts upon the bolt. However, just for completeness, the bolt stiffness equation is shown below. The clamping stiffness will vary greatly on the material of the two components selected. The calculation is also much more involved due to the geometry of the bolt in relation to the two joined components.

$$k_b = \frac{k_t * k_d}{k_t + k_d}$$

$$k_t = thread\ stiffness\ \left(\frac{N}{m}\ or \frac{lbf}{in}\right);\ k_d = unthreaded\ stiffness\ \left(\frac{N}{m}\ or \frac{lbf}{in}\right)$$

$$k_t = \frac{A_t * E}{l_t}; k_d = \frac{A_d * E}{l_d};$$

$$A = cross\ sectional\ area\ (m^2\ or\ in^2); E = modulus\ of\ elasticity\ (Pa\ or\ psi);$$

$$l = length\ (m\ or\ in)$$

The stiffness of the bolt is a function of the stiffness of the threaded and unthreaded parts of the bolt. The stiffness for each of those parts can be found by multiplying the area of that part by the modulus of elasticity divided by the length of that part.

On the FE exam, you may be asked to calculate the maximum tensile force that a bolt can withstand. This can be found by multiplying the tensile area by the tensile strength of the bolt.

$$F_b = A_t * S_t$$

$$A_t = tensile\ area\ (m^2\ or\ in^2);\ S_t = tensile\ strength\ of\ bolt (Pa\ or\ psi);$$

$$F_b = max\ tensile\ force (N\ or\ lbf)$$

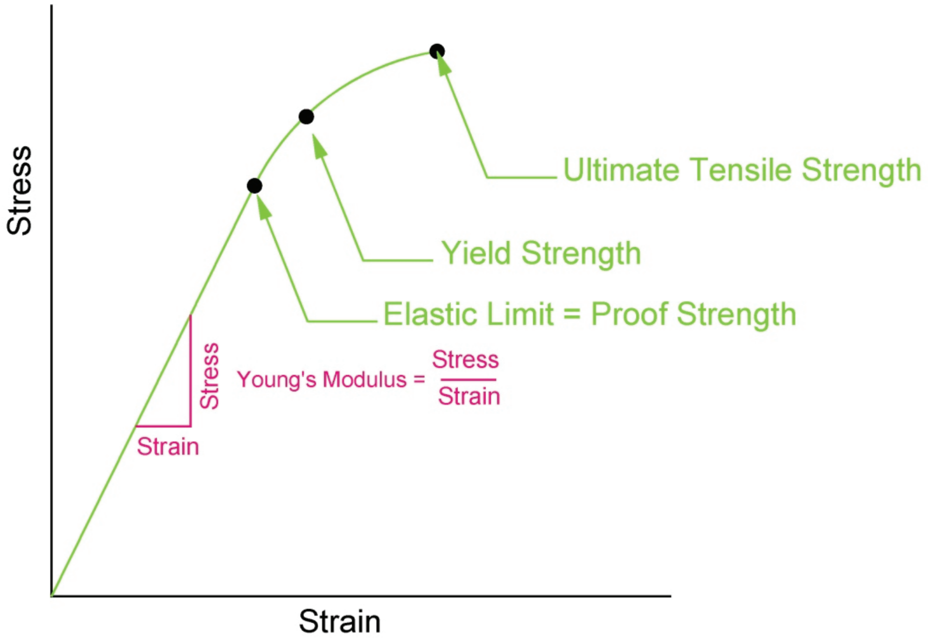


Figure 26: The question will need to give you the proof strength of the bolt, since there is no data on typical bolts.

If the tensile stress area is not provided in the tables and the FE exam asks you to calculate the tensile stress area, then you can use the following equation for threaded areas. For non-threaded areas it is simply pi multiplied by the radius squared.

$$A_{tensile\ stress\ area-threaded} = \frac{1}{4} * \pi \left[\frac{(d_{minor} + d_{pitch})}{2}\right]^2$$

$$A_{tensile\ stress\ area-non\ threaded} = \frac{1}{4} * \pi[d]^2$$

10.3.2 Thread Shear

Bolts can also be loaded in shear. Shear strength of a bolt is the maximum load that can be supported prior to the bolt fracturing. This is different from the proof stress and is more equivalent to the ultimate stress.

Unlike the properties of tensile stress and tensile area, shear strength of bolts is not readily available for this reason it is highly unlikely that you will need typical shear strength values for bolts, screws and rivets. In practice, engineers often use 60% of the tensile stress value or engineers seek information from fastener manufacturers directly.

A more typical problem will be determining the shear stress imposed by a load. The simplest calculation involves taking the force and dividing by the shear area which will be equivalent to the tensile stress area, if the stress occurs in the threaded region.

10.3.3 Fatigue

On the FE exam, there may be a question with a bolt under repeated stress. In order to calculate the alternating stress and midrange stress acting upon a bolt, you can use the following equations.

$$\sigma_a = \frac{F_{b,max} - F_{b,min}}{2 * A_T}; \sigma_m = \frac{F_{b,max} + F_{b,min}}{2 * A_T};$$

$$F_{b,max}/F_{b,min} = the\ maximum\ or\ minimum\ force\ acting\ upon\ the\ bolt$$

$$A_T = tensile\ area, found\ in\ bolt\ property\ tables$$

$$\sigma_a = alternating\ stress\ acting\ upon\ bolt;\ \sigma_m = midrange\ stress\ acting\ upon\ bolt$$

Once you find the actual alternating and midrange stresses, then you need to find the fatigue endurance strength of the bolt. This method will also apply to screws and rivets. First, you need to find the fatigue endurance strength for the desired life from the question.

This fatigue endurance strength component cannot be compared to the actual alternating stress component yet. Because this fatigue endurance strength component assumes that there is no mean stress. You need to use the fatigue endurance strength (S_e.) and ultimate tensile strength (S_{ut}) values, along with the actual mean and alternating stress values to find out the actual fatigue endurance strength (S_{sa}) that is available at the actual mean stress values.

$$S_{sa} = \frac{rS_{se}S_{ut}}{rS_{ut} + S_{se}}$$

The value, “r” is equal to the ratio of the actual mean and alternating stress values.

$$r = \frac{\sigma_a}{\sigma_m}$$

Once you find the actual fatigue endurance strength, S_{sa}, then this value can be compared to actual alternating shear stress. If the strength is greater than the stress, then the fastener will not fatigue.

10.4 SCREWS

The screws used in joints and fasteners are similar in construction to the Power Screws that were previously discussed. Screws have a threaded end that fits into a hole of one component and into a threaded hole of another component. Even though these screw properties were discussed in the Power Screw topic, for completeness, here is a brief overview of the necessary terms required for these types of screws.

Pitch: The distance between threads.

Major Diameter: The major diameter is the largest diameter of a screw thread.

Minor Diameter: The minor diameter is the diameter at the root of the thread.

Pitch Diameter: This is the mid-point between the top and bottom of the thread.

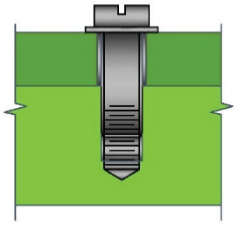


Figure 27: A screw is used to join two components, where at least one of the components is threaded.

10.4.1 Forces in Screw

The calculations for forces and stresses in screws are the same as bolts. The only differences are in finding the properties for tensile area and strength for screws.

10.5 RIVETS

Rivets are non-threaded fasteners and are semi-permanent fasteners. A rivet consists of a head and a smooth, cylindrical shaft. When rivets are installed, the shaft enters a hole and once through the hole, the end of the shaft is deformed. This creates a head at the other end of the hole, which secures the rivet in place and connects the two components. Basically, you insert the rivet into a snug fit hole between two components, then you hit the tail end with a special tool and cause the tail to deform and to create a head to secure the components together.

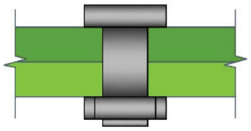


Figure 28: A rivet is used to permanently join two components

10.5.1 Forces in Rivets

The calculations for forces and stresses in rivets are the same as bolts. The only differences are in finding the properties for tensile area and strength for rivets.

10.6 FASTENER GROUP ANALYSIS

Often times in practice, multiple fasteners are used to secure components together. When this occurs, you need to calculate how an external force affects each individual fastener (bolt, screw or rivet). When you encounter a problem with a group of fasteners, you should follow this process for each specific type of situation.

10.6.1 Concentric Tensile Force

When an external force is applied in parallel to the direction of the axis of the fastener group, then the fasteners will be in tension. The figures below show bolts, but the same the concept can be applied to screws and rivets.   If the tensile force is applied concentrically, meaning that there is no moment applied to the bolts, then the external tensile force will cause an equal tensile stress in each bolt.

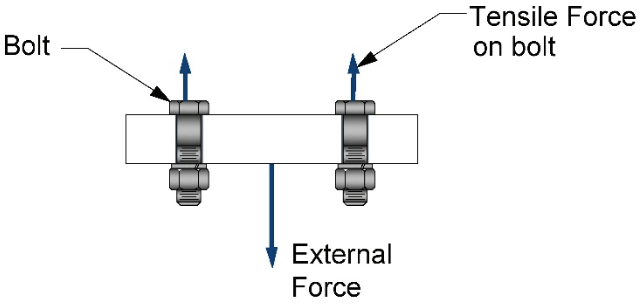


Figure 29: When a concentric external force is applied to a fastener group solely in tension, then the bolts will share the tensile force based on their area.

The external force divided by the sum of the bolt cross sectional areas will equal the stress applied in each bolt.

$$\sigma_{tensile\ bolt} = \frac{External\ Tensile\ Force}{\sum_1^n Cross\ sectional\ area\ of\ bolt\ 'x'};$$

The force in each bolt can then be found by dividing the tensile stress in an individual bolt by the cross sectional area of the individual bolt.

$$\sigma_{tensile\ bolt} = \frac{F_{bolt\ x}}{Area_{bolt\ x}}$$

10.6.2 Concentric Shear Force

This situation is when an external load is applied in shear with the fastener group. The shear force is also applied in shear, thus there is no moment produced.

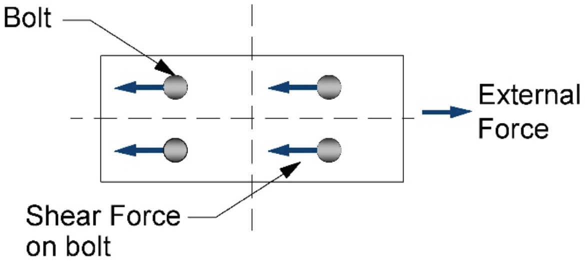


Figure 30: This figure shows an external force applied to a fastener group. The external force is applied in shear to the bolts. Since the fasteners do not move, the shear force in the fastener will oppose the direction of the external force.

The shear stress acting upon each bolt is the same, because all the components are connected together and experience the same shear strain. In order to calculate the shear stress acting on each bolt, you must divide the external force by the total area that the force is acting upon. The total area is the sum of the cross sectional area of each bolt.

$$\sigma_{shear\ bolt} = \frac{External\ Shear\ Force}{\sum_i^n Cross\ sectional\ area\ of\ bolt\ 'i'};$$

Next in order to calculate the shear force acting upon each individual bolt, you can use the following equation. The shear bolt acting upon each bolt multiplied by the area

$$\sigma_{shear\ bolt} = \frac{F_{bolt\ x}}{Area_{bolt\ x}}$$

10.6.3 Concentric Tensile & Shear Force

Sometimes, there may a situation where the tensile force and shear force are acting simultaneously. When this occurs, simply use the previous two equations to calculate the shear and tensile stress in each bolt and then use those stress values to find the tensile and shear forces acting on each bolt.

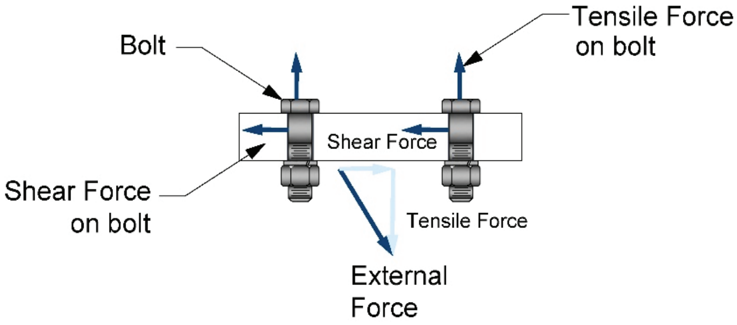


Figure 31: This figure shows concentric tensile and shear forces acting upon a fastener group. Since the fastener remains in place the force in the fastener will oppose the direction of the external force.

10.6.4 Eccentric In-Plane Force

An in-plane force is similar to a shear force acting upon a fastener group, however since the force is eccentric, the fastener group will be caused to rotate. This force due to rotation must be added to the shear force that was calculated in an earlier section. This figure only shows the shear force acting upon the bolts.

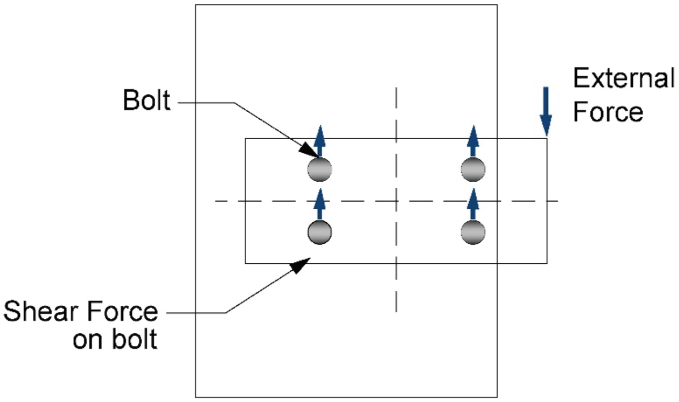


Figure 32: This guide shows an external force applied eccentrically and in-plane to a fastener group. Only the shear forces acting on the fasteners are shown. The next figure will show the effect of the external force on the applied moment.

The next step is to add the rotational effect of the external force on the fastener group. But, first, determine the centroid of the pattern. The centroid is defined as the center of mass of the bolt pattern. Do not find the centroid of the component, only the bolt pattern. On the exam, you should only encounter simple patterns, like finding the center of a square, rectangular or straight line pattern. But if you encounter a difficult pattern, then you need to be able to visualize the force vectors and moments. The examples below will teach you how to attain this skill.

Once you have the centroid, you know that the applied external force is translated to this point. In this situation, the external force will cause a rotation about the centroid as shown in the figure below.

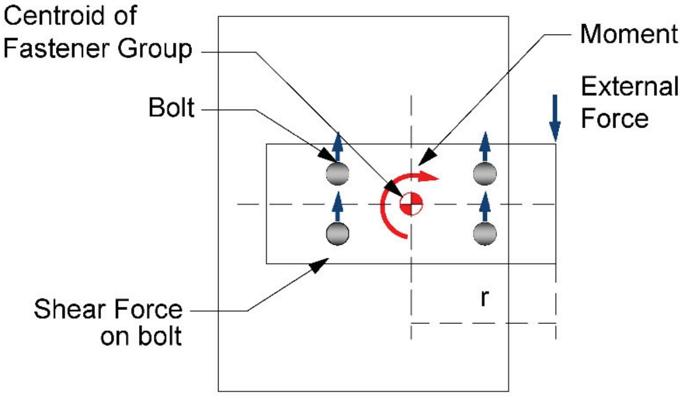

Centroid of Fastener Group
Bolt
Shear Force on bolt
Moment
External Force
r

Figure 33: This figure shows that the external force will cause a moment at the centroid of the fastener group pattern.

The moment can be calculated by multiplying the external force by the distance between the centroid and the external force. Remember the external force and the radius must be perpendicular to each other.

$$Moment = External\ Force * r$$

Now, you must apply this moment to each fastener with the same equation in order to find the force acting on each fastener. The direction of the force will be perpendicular to the imaginary circle from the centroid to the fastener. The value will be equal to the moment divided by the radial distance from the centroid to the fastener.

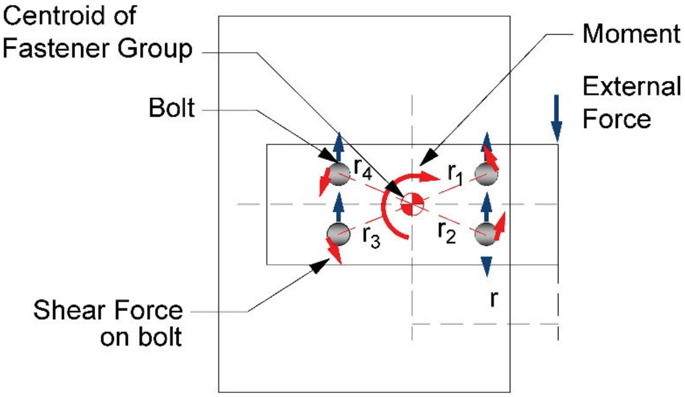

Centroid of Fastener Group
Bolt
Shear Force on bolt
Moment
External Force
r_4
r_1
r_3
r_2
r

Figure 34: This figure shows additional force acting upon each fastener in red. This additional force is due to the moment. The direction of the force in the fastener will oppose the direction of the moment.

$$Moment = I_{total} \sum \frac{F_{fastener\ i\ due\ to\ moment}}{r_i}$$

$$r_i = radius\ from\ centroid\ to\ fastener\ "i"$$

The moment of inertia of a bolt is the moment of inertia of a point, which is simply the distance from the centroid squared.

$$I_i = r_i^2$$

The two forces, (1) force due to shear and (2) force due to moment can then be added. Be sure to add the vectors and not simply the values.

$$F_{fastener\ "i"} = F_{fastener\ \text{"i"},shear}\angle angle + F_{fastener\ \text{"i"},due\ to\ moment}\angle angle$$

10.6.5 Eccentric Out of Plane Force

The final situation is an eccentric, out-of-plane force. This situation is solved similar to the previous eccentric, in-plane force. You must follow the three steps that were outlined in the previous example.

First, find the tensile force that opposes the external force. This is shown in the next figure as the upward force opposing the downward external force. This is similar to the first concentric tensile external force.

Next, find the moment that is created by the eccentric application of the external force. In the next figure, the moment is found to apply at the centroid of the fastener group. The value of this moment is by multiplying the external force by the radial distance between the centroid and the external force.

Finally, the moment must be applied to each fastener. Each fastener will experience the same moment. The force of this moment at each fastener will be equal to the moment divided by the radial distance between the centroid and the fastener.

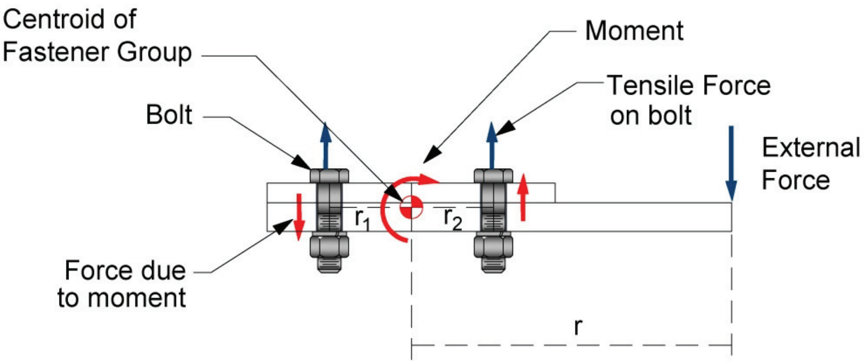


Figure 35: This figure shows an external, eccentric, out-of-plane force applied to a fastener group. Two forces on each bolt are created. One force is tensile and the other is due to the created moment.

10.7 ADHESIVES

Adhesive bonding is another method of joining two or more components. Adhesives are advantageous when weight is an issue and a lesser fastening strength is required. Welding and brazing most often have the largest fastening strength, then bolts, screws and rivets and finally followed by adhesives. On the FE exam, you should be familiar with adhesive property tables which contain shear strength and peel strength values.

11.0 MANUFACTURABILITY

The manufacturability section in the NCEES FE handbook focuses on the fits and tolerances for shafts, holes and other mechanical components. The fits and tolerances topic is one that is easily tested on the FE exam. You should expect at least two problems in this topic. The problems may be disguised with a lot of extraneous information, but as long as you understand the main concepts in this topic, then you should be able to navigate your way through the extraneous information and arrive at the correct solution.

During the manufacturing process, *tolerances* are provided to indicate the minimum precision required to allow for a part to operate properly. The *fit* refers to the clearances between two mating parts. The higher the precision, the more costly a part becomes to manufacture. Therefore, the tolerances and fits are usually given as large a value as possible.

11.1 TOLERANCES

Tolerances are presented as a part's dimension limits. It can either be shown with +/- values or as a range. The following are examples of tolerance nomenclature which all mean the same thing.

Bilateral Tolerance: $\mathbf{10 \pm 0.05\ \boldsymbol{in}}$, varies 0.05" in either direction away from 10"

Unilateral Tolerance (positive): $\mathbf{9.95\ \substack{+0.10 \\ -0.00}\ \boldsymbol{in},}$ varies 0.10" in the positive direction.

Unilateral Tolerance (negative): $\mathbf{10.05\ \substack{+0.00 \\ -0.10}\ \boldsymbol{in}}$*,* varies 0.10" in the negative direction

Limits: $\mathbf{9.95\boldsymbol{in} - 10.05\boldsymbol{in}}$, the minimum and maximum dimension.

Tolerance: $\mathbf{0.10\boldsymbol{in}}$, the span between the two limits. The difference between the maximum and minimum size limits of the parts.

The tolerances are often labeled along with nominal sizes on design drawings for the fabricator. The following figure depicts examples of dimensioning various shapes.

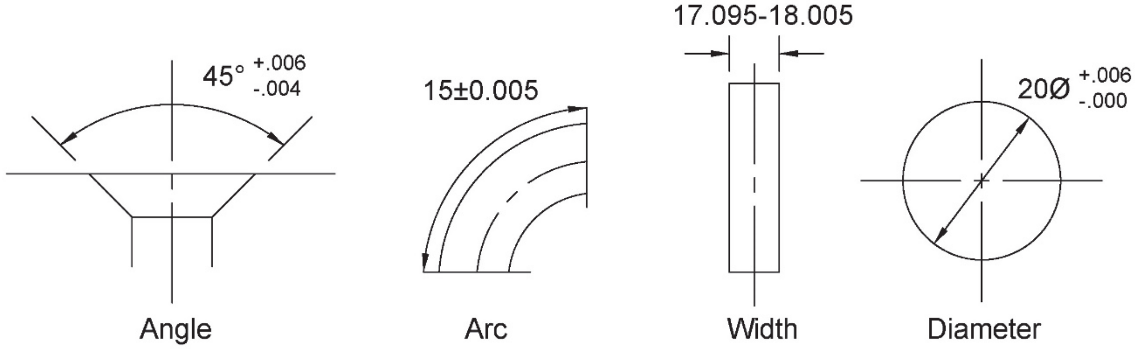

45° +.006 -.004
15±0.005
17.095-18.005
20Ø +.006 -.000
Angle
Arc
Width
Diameter

Figure 36: Dimensioning can occur on angles, arcs, lengths, diameters and more.

11.2 FITS

Mating two parts together will either require a clearance for smooth movement of one part within another or an interference for a snug, pressure tight fit. Each part is provided with a tolerance. The exam will likely ask you to use a limit dimension from both parts to find a clearance or an allowable dimension for proper operation.

The following image illustrates clearance requirements for a shaft that requires rotation within a hole. If the tolerances were too large, the shaft could potentially be too small to cause unstable rotation or too large causing excessive friction, preventing proper movement.

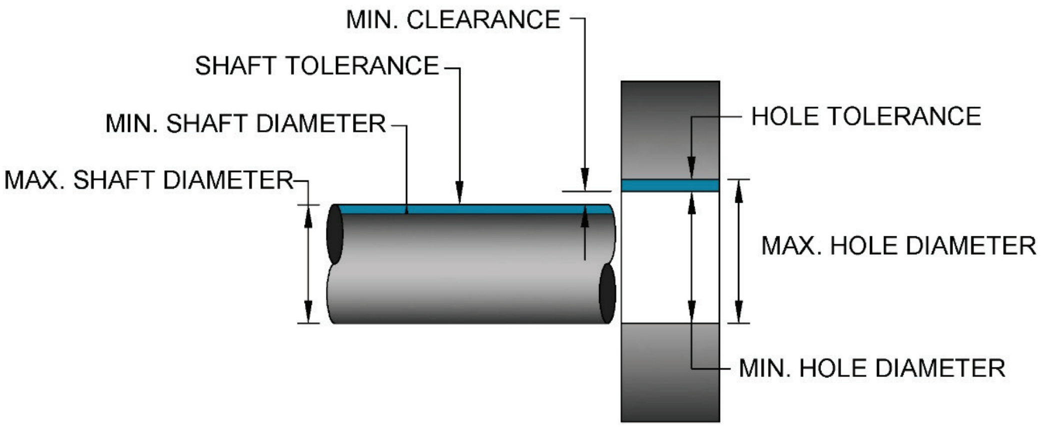

MIN. CLEARANCE
SHAFT TOLERANCE
MIN. SHAFT DIAMETER
MAX. SHAFT DIAMETER
HOLE TOLERANCE
MAX. HOLE DIAMETER
MIN. HOLE DIAMETER

Figure 37: A clearance fit occurs when there is positive clearance between the hole and the shaft.

The following example of interference is a shaft that has a larger diameter than the hole. In this situation, the rod is forced into the hole for a pressure fit, causing one part to be constrained to the other.

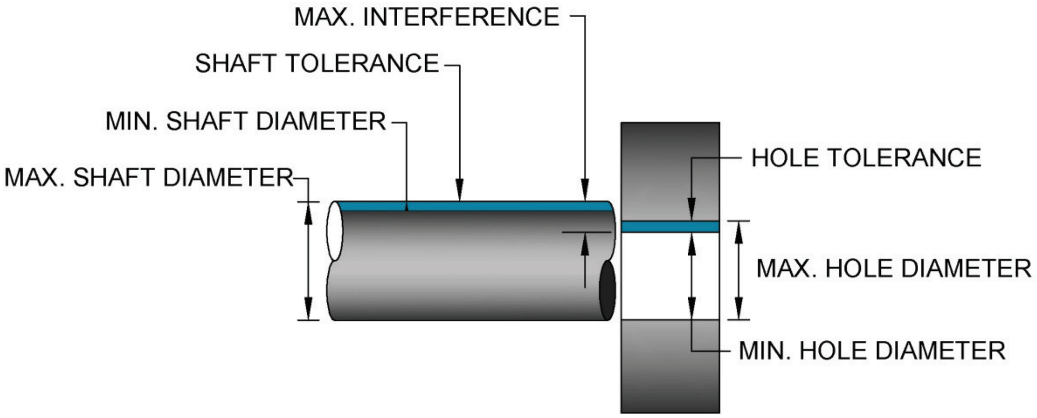


Figure 38: Interference fit in holes occurs when there is a fit with negative clearance between the hole and shaft.

The final type of fit is a transition fit. This type of fit can either have a positive or negative clearance.

11.3 HOLE BASIS AND SHAFT BASIS

In a hole-based system, the size of the hole is kept constant and the shaft size is varied to get a different fit (clearance, transition or interference). The basic size of the hole is taken as the low limit. The high limit of the hole, the low limit of the shaft and the high limit of the shaft are selected to give the required fit. The term basic size is the reference point for the upper and lower limits. Thus if a hole has a minimum value of 25.00, then the basic size is 25.00. If the maximum value is 25.130, then the shaft must be selected to fit into these requirements.

In a shaft basis system, the size of the shaft is constant and the hole-size is varied to achieve the desired fit. The basic size of the shaft is taken as the maximum size of the shaft. The other limit of the shaft is adjusted, along with the limits of the hole to achieve the required fit.

11.4 STANDARD TOLERANCES

There are various standards for keeping uniform tolerances, one standard is ANSI Standard B4.1, *Preferred Limits and Fits for Cylindrical Parts*. This is a standard of tolerances for mating parts used in typical applications. The tables below are a sample of tables that could be encountered on the exam and a version of this table is shown in the *NCEES FE Reference Handbook*, although the version in the handbook is in millimeters. For the purposes of the FE exam, you need to be able to quickly decipher this nomenclature.

Nominal Size		Hole (H11)	Shaft (c11)	Fit
25	Max	25.130	24.890	0.370
	Min	25.000	24.760	0.110

Nominal Size		Hole (H9)	Shaft (d9)	Fit
30	Max	30.052	29.935	0.169
	Min	30.000	29.883	0.065

The tables above show various fits, H11/c11 and H9/d9. You can see that the fit on the H9/d9 is tighter than the fit for the H11/c11. The following table shows how the various fits match up to the descriptions. The fits get tighter as you move down. But before you get to the table below you also be aware of how the fit is determined. The maximum fit is found by finding the difference between the maximum hole and the minimum shaft size. The minimum fit value is found by finding the difference between the minimum hole and the maximum shaft size.

	ISO Symbol		
	Hole Basis	**Shaft Basis**	**Description of Fit**
Clearance Fits	H11/c11	C11/h11	Loose running
	H9/d9	D9/h9	Free running
	H8/f7	F8/h7	Close running
	H7/g6	G7/h6	Sliding fit
	H7/h6	H7/h6	Locational clearance
Transition Fits	H7/k6	K7/h6	Locational transition
	H7/n6	N7/h6	Locational transition
Interference Fits	H7/p6	P7/h6	Locational interference
	H7/s6	S7/h6	Medium drive
	H7/u6	U7/h6	Force

The clearance fits will always have positive fit values or clearance values. The transition fits can have both positive and negative fit values. The interference fits will have negative fit values.

11.5 GEOMETRIC DIMENSIONING AND TOLERANCING

Geometric dimensioning and tolerancing refers to the process of controlling all the aspects of a component to ensure proper fitting and assembly with other components. The important skill within this topic is being able to interpret drawings and symbols.

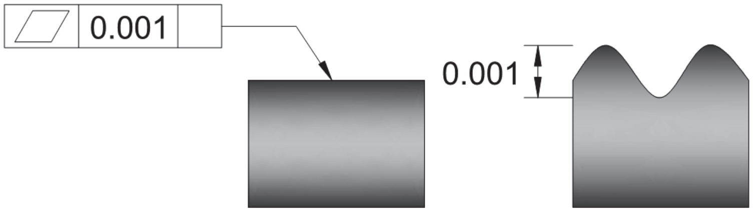


Figure 39: This figure is an example of the geometric dimensioning and tolerancing process.

In the figure above, geometry is first stated, followed by the tolerance and in some cases the reference plane or datum. The geometry presented in this example is straightness. This means that the component must be straight (no slope), but there is a tolerance of 0.001. As long as the difference between the minimum and maximum value of this "straight" surface is within 0.001 of each other, then the component meets the requirements of the specification.

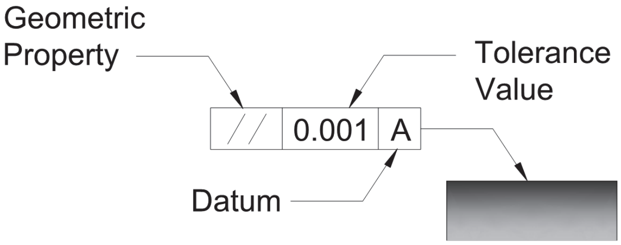


Figure 40: This figure shows the basic layout of the geometric and tolerancing method.

12.0 QUALITY AND RELIABILITY

In Mechanical Engineering, quality assurance and quality control are very important. This topic is generally broad, but given the restrictions of the FE exam, this topic will most likely focus on the conformity of products. In order to produce quality control in the most efficient manner, quality control charts and techniques were created. Since it would be very inefficient to individually inspect and measure each part and product for perfect quality, the following statistical approaches are used. Calculations for these statistical approaches are time consuming and typically require complex calculators or software. However there are some aspects of quality control that can be tested on the FE exam.

12.1 CONTROL CHARTS

On the FE exam you may be presented with a control chart or a table. Both forms are used to present the level of quality in a production batch. The control chart will measure a certain aspect of the product that needs to be quality controlled. For example, a bolt's diameter must be measured in order for it to fit within a specific application. Once the bolt diameter is measured within a set of tolerances, it can be labeled as a certain size. The allowable tolerance values are represented by the upper and lower control limits in the equations below.

$$Centerline = Mean\ value = x$$

$$Upper\ control\ limit(UCL) = x + k\sigma$$

$$Lower\ control\ limit(LCL) = x - k\sigma$$

$$where\ \sigma = standard\ deviation\ and\ k = multiple\ factor\ (typically\ 3)$$

An example will be shown to walk you through the calculations for mean value and the UCL/LCL values.

Sample	QC Characteristic	Sample	QC Characteristic
1	40	8	38
2	44	9	31
3	45	10	31
4	48	11	42
5	38	12	35
6	35	13	41
7	35	14	36

In the above chart, the QC characteristic is a measurement for a specific sample. The mean or average value of the samples is found by adding up all the QC Characteristic values and dividing by the total number of samples which is 14. In this example, the mean or average value is 38.5. The standard deviation is found by adding up the square of the differences between each QC characteristic value and the mean value, then dividing this number by the number of samples. Finally take the square root to find the standard deviation.

$$Standard\ Deviation = \sigma = \sqrt{\frac{\sum |x - \bar{x}|^2}{n}}$$

$$n = number\ of\ samples; x = sample\ value; \bar{x} = \ mean;$$

The mean and standard deviation can then be used to find the UCL and the LCL. For this example, assume k = 3.

$$\bar{x} = 38.5; \sigma = \ 4.9$$

$$UCL = \bar{x} + 3\sigma = 53.2$$

$$LCL = \bar{x} - 3\sigma = 23.8$$

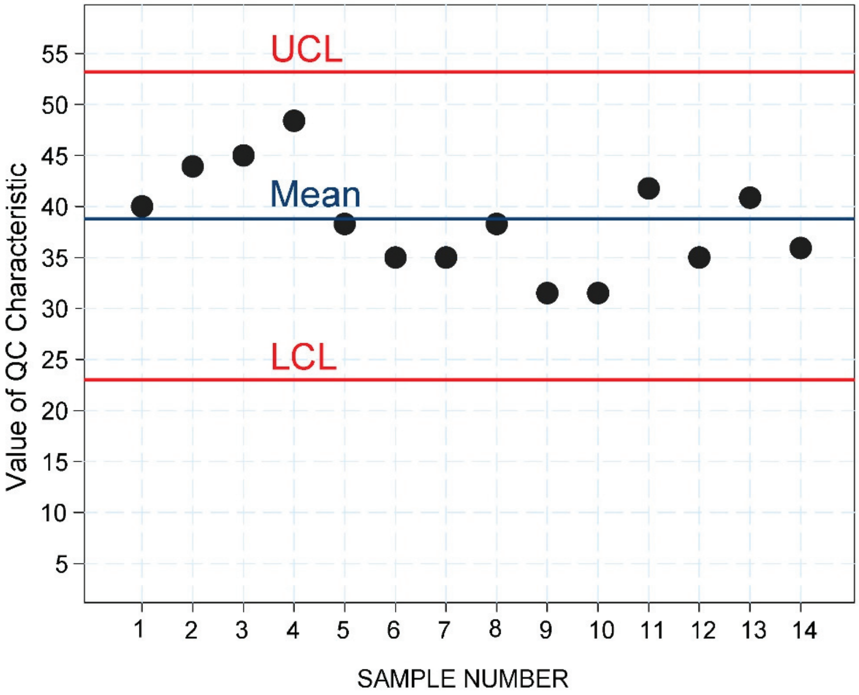


Figure 41: This figure shows the mean and UCL/LCL lines for the table of samples previously shown.

12.2 CONTROL CHARTS WITH SAMPLE SIZES

The previous section gave you the basics of control charts and introduced terms like mean, UCL and LCL. This section builds upon the basics and introduces the term sample sizes. Sometimes samples are taken in sets/groups of 3 to 10 samples per set or sometimes more. When you take a group of samples, there are two values that you can measure for each group. The first is the mean of that group of samples, the second is the range.

Subgroup						Mean	Max	Min	Range
1	28	29	33	29	37	31.2	37	28	9
2	28	32	26	26	33	29	33	26	7
3	27	40	32	35	25	31.8	40	25	15
4	29	39	38	35	26	33.4	39	26	13
5	33	35	31	34	28	32.2	35	28	7
6	25	29	36	34	25	29.8	36	25	11
7	38	32	28	37	32	33.4	38	28	10
8	33	31	39	27	25	31	39	25	14
9	32	39	40	26	26	32.6	40	26	14
10	36	27	29	38	32	32.4	38	27	11

The range is simply the difference between the largest and smallest value within the subgroup.

$$Range \rightarrow R = x_{max} - x_{min}$$

There are three types of charts that are created to monitor quality control. These charts are the R Chart, $\bar{X}$ (X-Bar) Chart and the s-Chart. The R-chart is a plot of the ranges of the subgroups. The $\bar{X}$ chart is a plot of the averages of the subgroups and the s-chart is the plot of the sample's standard deviations. Luckily the s-chart is much more complex and its equations are more involved, which indicates that the s-chart will most likely not be on the FE exam. Therefore, you should focus on the other two charts. In order to understand these charts you must also understand the following table, which shows the factors for computing the control limits and centerlines for these charts, based on the number of samples. The following guide introduces how the constants on the next table apply to the various charts.

12.2.1 Control Chart Factors

A_2: This control chart factor is used to find the UCL and LCL of the averages as a function of the grand mean and the average range.

$$UCL = \bar{\bar{X}} + A_2 * \bar{R}; LCL = \bar{\bar{X}} - A_2 * \bar{R}$$

$$Centerline = \bar{\bar{X}}$$

D_4 and D_3: These control chart factors are used to find the UCL and LCL of the average range.

$$UCL = \bar{R}D_4;\ LCL = \bar{R}D_3$$

$$Centerline = \bar{R}$$

Control Chart Factors

n	A2	A3	~~c4~~	B3	B4	~~d2~~	~~1/d2~~	~~d3~~	D3	D4
2	1.880	2.659	~~0.7979~~	0.000	3.267	~~1.128~~	~~0.8862~~	~~0.853~~	0.000	3.267
3	1.023	1.954	~~0.8862~~	0.000	2.568	~~1.693~~	~~0.5908~~	~~0.888~~	0.000	2.575
4	0.729	1.628	~~0.9213~~	0.000	2.266	~~2.059~~	~~0.4857~~	~~0.880~~	0.000	2.282
5	0.577	1.427	~~0.9400~~	0.000	2.089	~~2.326~~	~~0.4299~~	~~0.864~~	0.000	2.114
6	0.483	1.287	~~0.9515~~	0.030	1.970	~~2.534~~	~~0.3946~~	~~0.848~~	0.000	2.004
7	0.419	1.182	~~0.9594~~	0.118	1.882	~~2.704~~	~~0.3698~~	~~0.833~~	0.076	1.924
8	0.373	1.099	~~0.9650~~	0.185	1.815	~~2.847~~	~~0.3512~~	~~0.820~~	0.136	1.864
9	0.337	1.032	~~0.9693~~	0.239	1.761	~~2.970~~	~~0.3367~~	~~0.808~~	0.184	1.816
10	0.308	0.975	~~0.9727~~	0.284	1.716	~~3.078~~	~~0.3249~~	~~0.797~~	0.223	1.777
11	0.285	0.927	~~0.9754~~	0.321	1.679	~~3.173~~	~~0.3152~~	~~0.787~~	0.256	1.744
12	0.266	0.886	~~0.9776~~	0.354	1.646	~~3.258~~	~~0.3069~~	~~0.778~~	0.283	1.717
13	0.249	0.850	~~0.9794~~	0.382	1.618	~~3.336~~	~~0.2998~~	~~0.770~~	0.307	1.693
14	0.235	0.817	~~0.9810~~	0.406	1.594	~~3.407~~	~~0.2935~~	~~0.763~~	0.328	1.672

15	0.223	0.789	~~0.9823~~	0.428	1.572	~~3.472~~	~~0.2880~~	~~0.756~~	0.347	1.653
16	0.212	0.763	~~0.9835~~	0.448	1.552	~~3.532~~	~~0.2831~~	~~0.750~~	0.363	1.637
17	0.203	0.739	~~0.9845~~	0.466	1.534	~~3.588~~	~~0.2787~~	~~0.744~~	0.378	1.622
18	0.194	0.718	~~0.9854~~	0.482	1.518	~~3.640~~	~~0.2747~~	~~0.739~~	0.391	1.609
19	0.187	0.698	~~0.9862~~	0.497	1.503	~~3.689~~	~~0.2711~~	~~0.733~~	0.404	1.596
20	0.180	0.680	~~0.9869~~	0.510	1.490	~~3.735~~	~~0.2677~~	~~0.729~~	0.415	1.585
21	0.173	0.663	~~0.9876~~	0.523	1.477	~~3.778~~	~~0.2647~~	~~0.724~~	0.425	1.575
22	0.167	0.647	~~0.9882~~	0.534	1.466	~~3.819~~	~~0.2618~~	~~0.720~~	0.435	1.565
23	0.162	0.633	~~0.9887~~	0.545	1.455	~~3.858~~	~~0.2592~~	~~0.716~~	0.443	1.557
24	0.157	0.619	~~0.9892~~	0.555	1.445	~~3.895~~	~~0.2567~~	~~0.712~~	0.452	1.548
25	0.153	0.606	~~0.9896~~	0.565	1.435	~~3.931~~	~~0.2544~~	~~0.708~~	0.459	1.541

d_2 and $1/d_2$: These two terms are crossed out because these values are only used to determine A_2.

$$A_2 = \frac{3}{d_2\sqrt{n}};\ where\ n = sample\ size$$

d_3: This term is crossed out because it is only used to determine D_3 and D_4.

$$D_3 = 1 - \frac{3d_3}{d_2};\ D_4 = 1 + \frac{3d_3}{d_2};$$

B_3, B_4: These terms are used in s-charts to find UCL and LCL.

$$UCL_s = B_4\bar{s};\ LCL_s = B_3\bar{s}, where\ \bar{s}\ is\ the\ average\ standard\ deviation\ (centerline)$$

A_3: This term is used to find the UCL and LCL of the grand mean through the use of the average standard deviation.

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3\bar{s};\ LCL_{\bar{X}} = \bar{\bar{X}} - A_3\bar{s};$$

c_4: This term is crossed out because it is used primarily to find A3, through the following equation.

$$A_3 = \frac{3}{c_4\sqrt{n}};\ where\ n = sample\ size$$

12.2.1 R Chart

The R-chart plots the ranges for the various subgroups and the average (mean) of the ranges for all the subgroups. The average of the example is 11.1.

Subgroup						Mean	Max	Min	Range
1	28	29	33	29	37	31.2	37	28	9

2	28	32	26	26	33	29	33	26	7
3	27	40	32	35	25	31.8	40	25	15
4	29	39	38	35	26	33.4	39	26	13
5	33	35	31	34	28	32.2	35	28	7
6	25	29	36	34	25	29.8	36	25	11
7	38	32	28	37	32	33.4	38	28	10
8	33	31	39	27	25	31	39	25	14
9	32	39	40	26	26	32.6	40	26	14
10	36	27	29	38	32	32.4	38	27	11

$$\bar{R} = 11.1$$

Next, in order to find the UCL and LCL, you must use the following equations.

$$UCL = \bar{R}D_4;\ LCL = \bar{R}D_3$$

You should notice that the sample size is 5, thus the corresponding factors are as shown below:

$$D_4 = 2.115;\ D_3 = 0$$

$$UCL = 11.1 * 2.115 = 23.48;\ LCL = 0$$

$$Centerline = 11.1$$

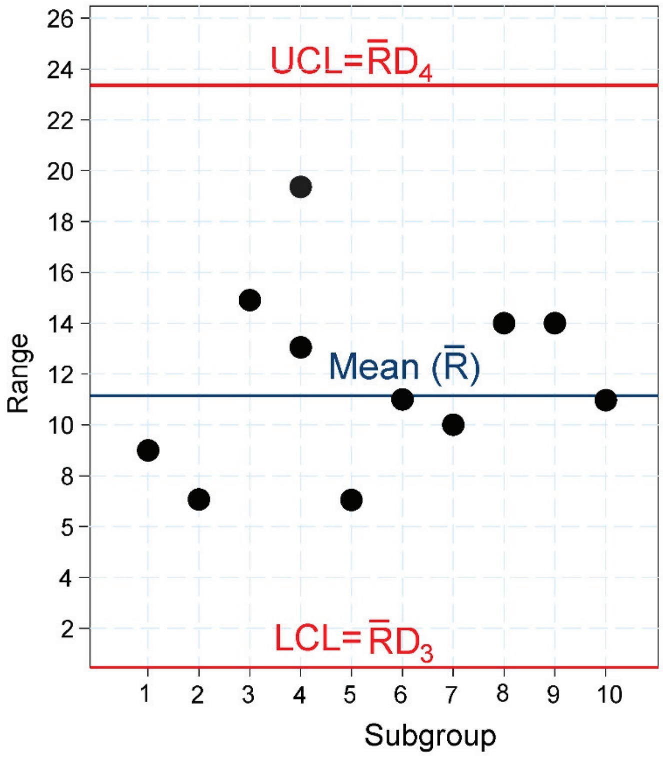


Figure 42: This figure shows the mean range value and the upper control limit and lower control limits based on the control factors and the previous equations.

12.2.2 X-Bar Chart

The X-Bar chart graphically shows the averages of each subgroup. The average of the averages is called the grand average and is shown as, $\bar{\bar{X}}$. This value serves as the centerline of the graph. The upper and lower control limits are found through the following equations.

Subgroup						Mean	Max	Min	Range
1	28	29	33	29	37	31.2	37	28	9
2	28	32	26	26	33	29	33	26	7
3	27	40	32	35	25	31.8	40	25	15
4	29	39	38	35	26	33.4	39	26	13
5	33	35	31	34	28	32.2	35	28	7
6	25	29	36	34	25	29.8	36	25	11
7	38	32	28	37	32	33.4	38	28	10
8	33	31	39	27	25	31	39	25	14
9	32	39	40	26	26	32.6	40	26	14
10	36	27	29	38	32	32.4	38	27	11

$$\bar{\bar{X}} = 31.7$$

Next, in order to find the UCL and LCL, you must use the following equations.

$$UCL = \bar{\bar{X}} + A_2 * \bar{R}; LCL = \bar{\bar{X}} - A_2 * \bar{R}$$

You should notice that the sample size is 5, thus the corresponding factor is as shown below:

$$A_2 = 0.577;$$

$$UCL = 31.7 + 0.577 * 11.1 = 38.1;\ LCL = 31.7 - 0.577 * 11.1 = 25.3$$

$$Centerline = 31.7$$

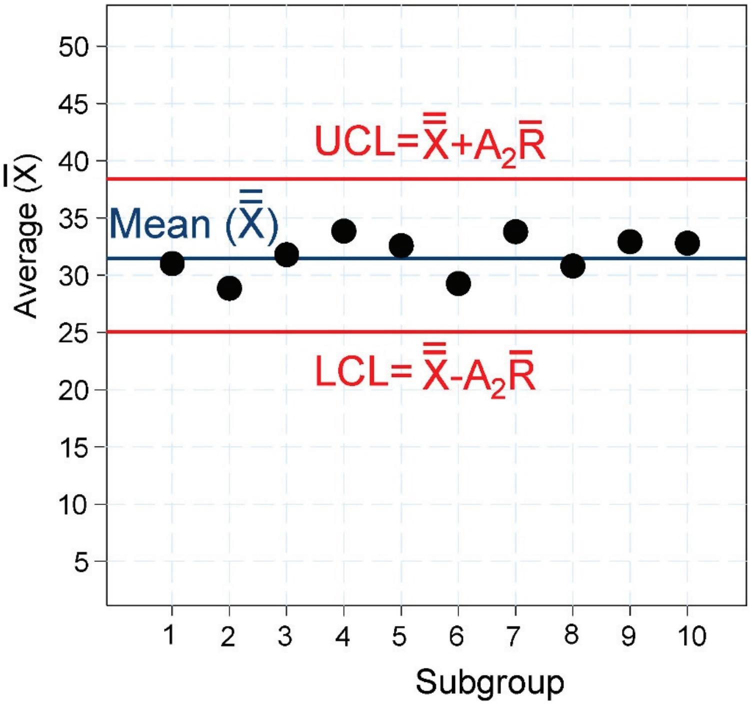


Figure 43: This figure plots the average values for each subgroup. The grand average, UCL and LCL values are also shown with their equations.

12.23 s-Chart

The s-chart plots the standard deviation values for each subgroup. The mean of the standard deviations ($\bar{s}$) is plotted as the centerline. The UCL and LCL lines are plotted in accordance with the below equations.

$$Centerline = \bar{s}$$

$$UCL_s = B_4\bar{s};\ LCL_s = B_3\bar{s};$$

Another version of the X-Bar chart can also be plotted with the following equations.

$$Centerline = \bar{\bar{X}}$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3\bar{s};\ LCL_{\bar{X}} = \bar{\bar{X}} - A_3\bar{s};$$

13.0 HYDRAULIC COMPONENTS

Hydraulics includes the equipment necessary to do work with liquid. This includes pumps, pipes, pressure vessels, control valves, actuators and connections, as shown in the simple hydraulic system below.

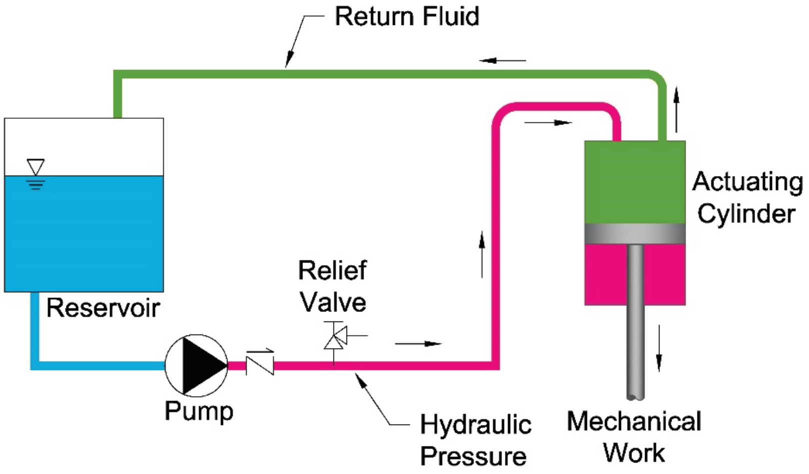


Figure 44: A simple hydraulic system consists of a reservoir that holds the hydraulic fluid, followed by a pump that pressurizes the fluid. The pressurized fluid in pink is then used to power an actuating cylinder to conduct mechanical work. In order to avoid over pressurization, there is a relief valve in the system. The green line shows the hydraulic fluid returning back to the reservoir when not needed.

Pressure vessels and piping were covered previously in this section and the control valves, actuators and connections are not covered in the *NCEES FE Reference Handbook*, so the only thing you need to know for Hydraulic Components is the pumps topic.

13.1 PUMPS

There are three main types of pumps, centrifugal, rotary and reciprocating pumps. Rotary and reciprocating pumps are positive displacement pumps. This guide will not cover positive displacement and reciprocating pumps in detail because they are not covered in the FE handbook.

Centrifugal pumps are the most common type of pumps used for condenser water applications. The following information is tailored to centrifugal pumps and should not be applied freely to positive displacement pumps.

Centrifugal pumps operate on the principle of "centrifugal force", which is the conversion of rotational kinetic energy imparted by rotating impellers onto the liquid to produce a flow rate (kinetic energy) at a certain pressure (pressure energy). Fluid enters the pump at the center or eye of the impeller. The rotating impellers then push the fluid to the outer edges, imparting a flow rate and pressure.

There are two main types of centrifugal pumps, in-line and end-suction. These two types are discussed below.

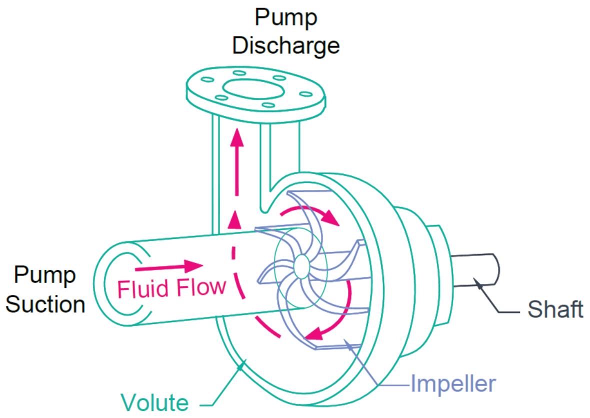


Figure 45: This figure shows a cut-away of an end-suction centrifugal pump. In a centrifugal pump, the impeller rotates, which pushes the fluid to the outer edges of the volute. The volute directs the fluid to the discharge. This centrifugal pump is an end suction type, because the suction and discharge are at 90 degrees to each other. (1) Fluid flows into the center of the impeller, (2) as the impeller rotates, (3) the centrifugal force pushes fluid to the edges (4) until the fluid travels out the pump discharge.

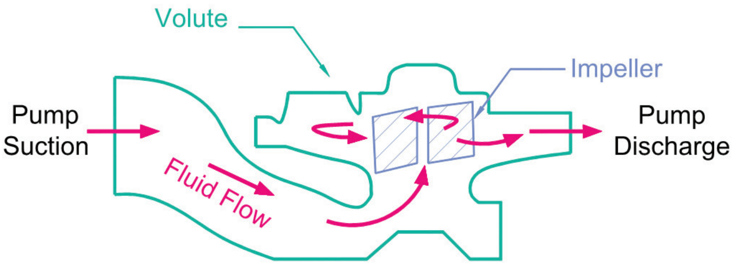


Figure 46: (1) Fluid flows into the center of the impeller, (2) as the impeller rotates,(3) the centrifugal force pushes the fluid to the edges (4) until the fluid travels out the pump discharge.

Within each centrifugal pump type (end suction & in-line) are horizontal and vertical type pumps, which are characterized by the orientation of the pump shaft as either horizontal or vertical. In addition, pumps can be further classified by the number of stages that the fluid proceeds through. Finally the last classification is how the pump is connected to the motor. Pumps can be long-coupled where the pump is connected to the motor by a flexible coupling or they can be close-coupled where the connection between the pump and motor is through a rigid coupling. The table below is a summary of the most common condenser water pump types.

Centrifugal Pump Type	**Description**
Horizontal, End Suction, Split Coupled:	A split coupled pump has two shafts (motor and pump shafts) connected by a coupler. Sometimes the coupler can be flexible.

Horizontal, End Suction, Close Coupled	There is a single shaft between the motor and pump.
Horizontal, In-Line, Split Coupled:	Similar to the end-suction, split coupled, except the suction and discharge are arranged in-line to one another.
Horizontal, In-Line, Close Coupled:	Similar to the end-suction, close coupled, except the suction and discharge are arranged in-line to one another.

The same four types of pumps also can be found with a vertical arrangement.

Another type of pump that is used for larger flow rates is the split case type. This centrifugal pump type has two chambers (split case) as opposed to the single chamber for end suction and in-line pumps.

13.2 PUMP AND SYSTEM CURVES

There are two curves that are needed for a pump selection: the pump curve and system curve. The pump curve is unique to the pump itself and the system curve is based on the piping system that the pump will be connected to. When the two curves are overlaid, their intersection will indicate the operation point of the pump.

13.2.1 Pump Curve

Pump curves are created by the manufacturers of the pumps through a series of tests and describe the operating performance for a specific impeller diameter and pump type. The curve plots the corresponding flow rates at varying pressure.

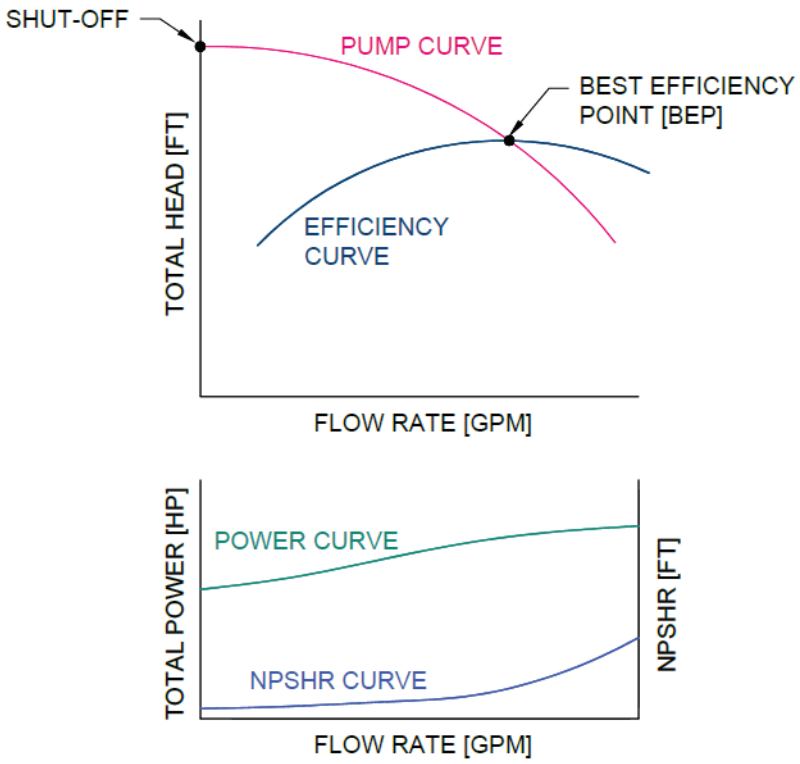


Figure 47: Pump Curves

In practice, engineers will use a pump selection tool, which makes it very easy to select pumps, but you should also know how to select pumps via the pump curves. A pump curve shows the pressure and flow rate relationship for a pump operating at various speeds or impeller diameters. If a pump speed is selected (1,200, 1,800 or 3,600) for a pump, then the pump curve graph will show multiple pump curves at various impeller diameters. If an impeller diameter is selected, then the pump curve graph will show multiple pump curves at various speeds. The following figure shows a specific pump speed, with multiple pump curves at various impeller diameters.

The graph also shows curves of pump horsepower in blue. These curves are created by calculating the horsepower based on the pressure, flow rate and pump/motor efficiency at a point. The efficiency curves shown on the figure are found based on a series of tests of the actual pump.

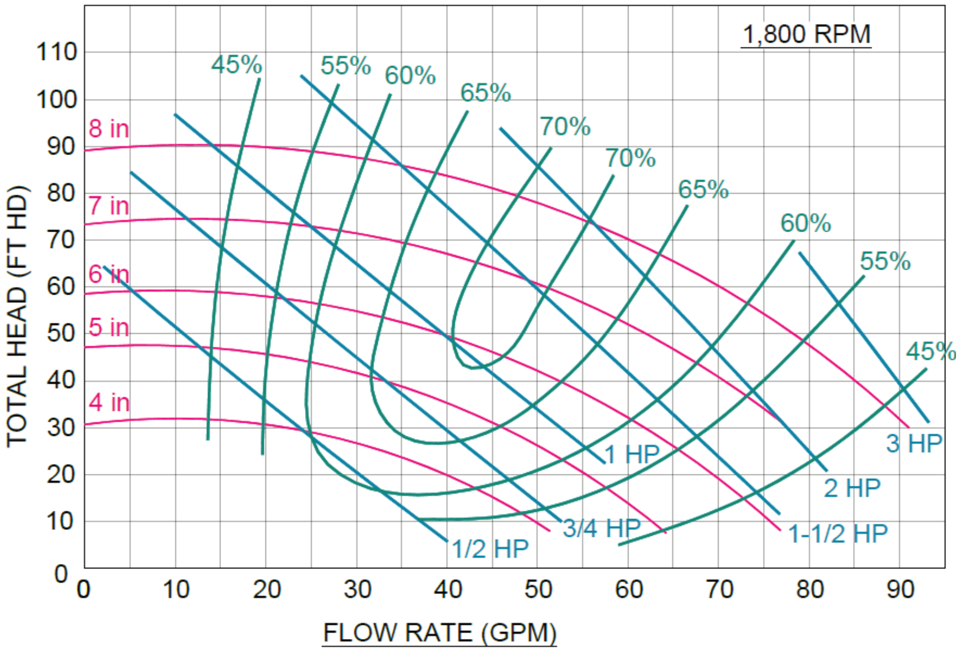


Figure 48: This sample pump curve shows the operating conditions for a pump at a specific pump speed. The red curves show the operating points for this pump at various impeller diameters. The red curves show the pressure that can be provided at various flow rates. The intersection of the red curve and the blue horsepower curve shows the horsepower required at a certain operating point. The intersection of the red curve and the green efficiency curves show the efficiency at a certain operating point

13.2.2 System Curve

The system includes the pipes, fittings, valves, and equipment that the fluid will travel through. The system curve graphs the pressure drops through the system at various flow rates. We will start by discussing a simple system, where there is no elevation gain, Δz, between the initial and final points or system pressurization.

During design, the engineer will calculate the pressure drop that the pump must overcome at the design flow rate. Once a single pressure and the flow rate is known, an estimated pressure at any other flow rate can be calculated with the following relationship.

$$\frac{P_1}{P_2}=\frac{GPM_1^2}{GPM_2^2}\quad [System\ Curve\ Relationship, no\ elevation\ gain\ \Delta z]$$

This relationship comes from the Darcy Weisbach equations, where the pressure is related closely to the square of the velocity. Since we are plotting a system, the diameter and fluid density do not change, so the pressure is also related to the square of the flow rate.

Using the relationship above, given pressure P_1 at flow rate GPM_1, we can estimate the pressure P_2 along the various flow rates, GPM_2 and plot them, as shown in the following graph. This same relationship also applies for fan systems.

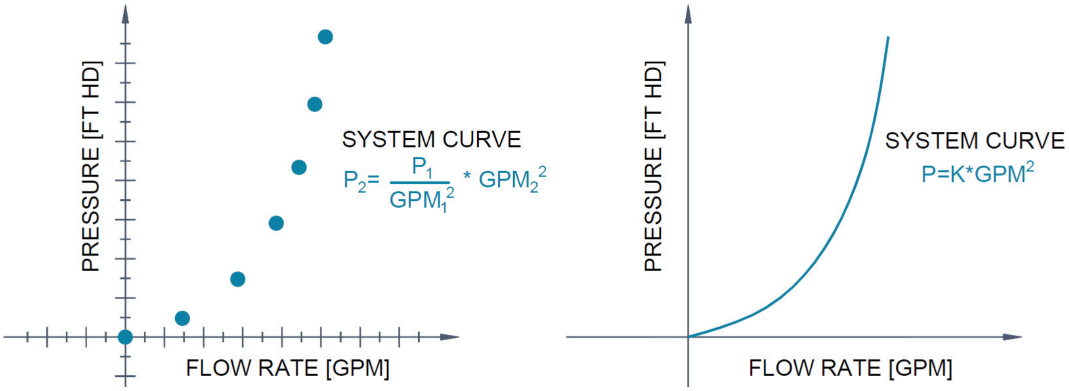


Figure 49: Plotting the system curve for a closed loop system with no additional static. The pressure is related to the square of the flow rate.

If, however, the pump is connected to an open loop system and the fluid must also overcome an elevation gain, or a pressurized closed loop system, then the entire curve shifts up by this amount of static pressure. Static pressure is the pressure not affected by the fluid velocity.

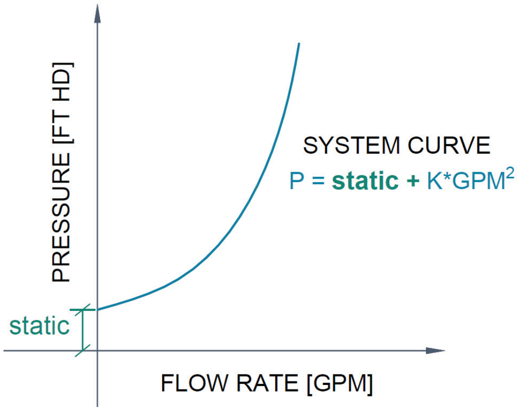


Figure 50: Generic system curve. The curve is shifted upward by the static pressure in the system.

13.2.3 Operation Point

Finally, the point where the pump curve and the system curve intersects is the operation point. This intuitively makes sense since the system and pump curves are set. Whatever flow rate the pump produces will result in a pressure drop based on the system. Therefore, a pump is selected where the intersection between the two curves will be as close as possible to the design conditions.

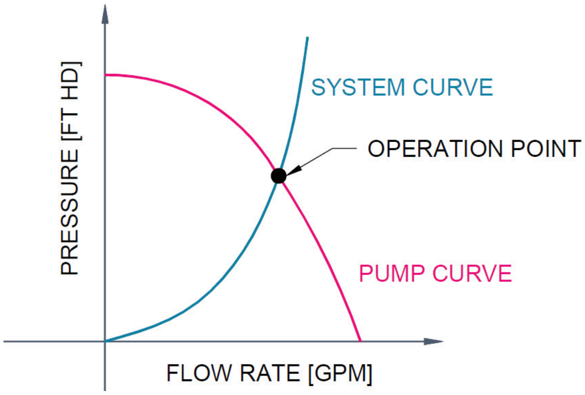


Figure 51: The operation point is found at the intersection of the system and pump curves.

13.3 PUMP AFFINITY LAWS

It is often necessary to determine how a pump will operate under differing operating conditions. The operating conditions of a pump that can most readily be changed are the impeller diameter and the rotational speed of the pump. In order to predict how a *centrifugal* pump will behave prior to changing the speed or the impeller diameter, the engineer can use the affinity laws shown below. These relationships affect the centrifugal pump curve and not the system curve.

The first set of affinity laws is that the flow rate (Q) is directly proportional to the size of the diameter of the pump impeller (D) and/or the rotational speed (N) of the pump.

$$\frac{Q_1}{Q_2} = \frac{D_1}{D_2}; if\ speed\ is\ held\ constant$$

$$\frac{Q_1}{Q_2} = \frac{N_1}{N_2}; if\ diameter\ is\ held\ constant$$

The second affinity law is that the total head (H) is directly proportional to the square of the size of the diameter of the pump impeller (D) and/or the square of the rotational speed (N) of the pump.

$$\frac{H_1}{H_2} = \frac{D_1^2}{D_2^2}; if\ speed\ is\ held\ constant$$

$$\frac{H_1}{H_2} = \frac{N_1^2}{N_2^2}; if\ diameter\ is\ held\ constant$$

The third affinity law is that the power (BHP) is directly proportional to the cube of the size of the diameter of the pump impeller (D) and/or the cube of the rotational speed (N) of the pump.

$$\frac{BHP_1}{BHP_2} = \frac{D_1^3}{D_2^3}; if\ speed\ is\ held\ constant$$

$$\frac{BHP_1}{BHP_2} = \frac{N_1^3}{N_2^3}; if\ diameter\ is\ held\ constant$$

13.3.1 Pump Similarity Laws

You may come across these formulas, if you encounter a question that compares two similar pumps. These formulas are called the similarity laws. These laws compare similar pumps within the same series of pumps. The previous formulas compared the original condition and new condition of the same pump. These formulas compare two similar pumps, with different diameters. In order to best understand what is meant by same series of pumps, visit a manufacturer’s website and you will see various pump series that have varying sizes within the same series. Within a series of pumps, a pump with “x” diameter impeller and “y” diameter volute can be compared to another pump in the same series of pumps, but with “2x” diameter impeller and “2y” diameter volute. The second pump is similar but has twice the diameter of the first pump.

$\frac{Q_1}{Q_2}$	=	$\left(\frac{D_1}{D_2}\right)^3 \left(\frac{N_1}{N_2}\right)$
$\frac{H_1}{H_2}$	=	$\left(\frac{D_1}{D_2}\right)^2 \left(\frac{N_1}{N_2}\right)^2$
$\frac{BHP_1}{BHP_2}$	=	$\left(\frac{D_1}{D_2}\right)^5 \left(\frac{N_1}{N_2}\right)^3 \left(\frac{\rho_1}{\rho_2}\right)$
$\frac{Q_1}{Q_2}$	=	$\left(\frac{D_1}{D_2}\right)^2 \left(\frac{H_1}{H_2}\right)^{.5}$
$\frac{N_1}{N_2}$	=	$\left(\frac{D_2}{D_1}\right) \left(\frac{H_1}{H_2}\right)^{.5}$
$\frac{BHP_1}{BHP_2}$	=	$\left(\frac{D_1}{D_2}\right)^2 \left(\frac{H_1}{H_2}\right)^{1.5} \left(\frac{\rho_1}{\rho_2}\right)$
$\frac{N_1}{N_2}$	=	$\left(\frac{D_2}{D_1}\right)^3 \left(\frac{Q_1}{Q_2}\right)$
$\frac{H_1}{H_2}$	=	$\left(\frac{D_2}{D_1}\right)^4 \left(\frac{Q_1}{Q_2}\right)^2$
$\frac{BHP_1}{BHP_2}$	=	$\left(\frac{D_2}{D_1}\right)^4 \left(\frac{Q_1}{Q_2}\right)^3 \left(\frac{\rho_1}{\rho_2}\right)$

$$where\ Q = flow\ rate, D = diameter, N = speed, H = head\ pressure, BHP = power, \rho = density$$

13.4 PUMP WITH A MODULATING VALVE

A pump with a modulating valve will change the system resistance curve but it will not change the pump curve. The decrease and increase of system pressure as the modulating valve opens and closes, respectively, causes the system curve to change as shown in the figure below.

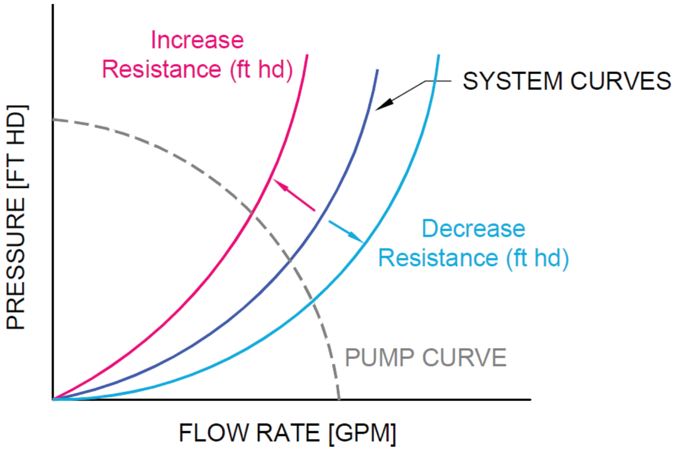


Figure 52: The system curve stretches right and left as the modulating valve opens and closes, causing a decrease and increase in overall system pressure drop.

13.5 PUMP WITH A VFD

The variable frequency drive (VFD) will vary the input power frequency, which will vary the rotational speed of the pump impeller. The pump normally operates at 60 Hz in America, 50 Hz in other countries. The 60 Hz electrical power speed will correspond to a rotational speed of 1,200, 1,800 and 3,600 RPM for the typical pumps. Sometimes the pump manufacturer will indicate a speed slightly below 1,200, 1,800 or 3,600. This is because the motor is an induction motor as opposed to a synchronous motor. This means that electrically the rotation will be 1,200, 1,800 or 3,600 RPM, but the shaft will lag slightly behind this rotation.

When a variable frequency drive changes the 60 HZ power to another frequency like 50 Hz, the rotational speed will also change by the same ratio.

$$Frequency\ Change \rightarrow 60\ Hz\ to\ 40\ Hz$$

$$Speed\ Change \rightarrow 1{,}750\ RPM\ x\ \left(\frac{40\ Hz}{60\ Hz}\right) = 1{,}167\ RPM$$

The shift in the pump curve due to speed adjustments from a VFD is illustrated in the figure below.

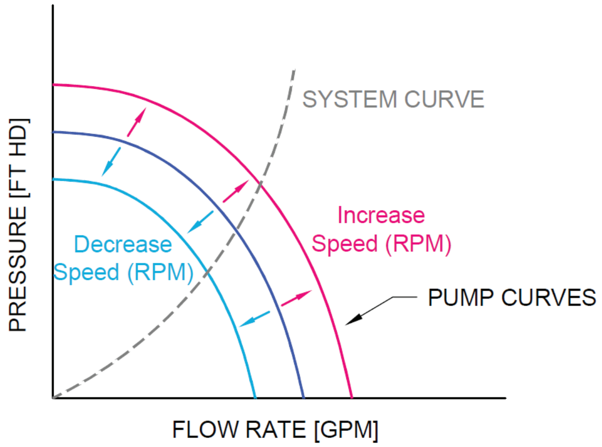


Figure 53: Pump curves of varying rotational speeds are parallel to each other.

When you choose the pump speed, you must first ensure that the pump manufacturer has that speed available. Lower pump speeds are preferred because the increased number of rotations will cause increased wear and tear. Bearings lifetimes are rated based on number of rotations, so if the rotations are reduced, then the bearings should have a longer life. This is true of all rotating objects.

13.6 MULTIPLE PUMPS

There are two ways multiple pumps can be arranged, (1) Parallel and (2) Series.

13.6.1 Pumps in Parallel

Pumps arranged in parallel will produce flow that is equal to the sum of the flow of each individual pump at every pressure point.

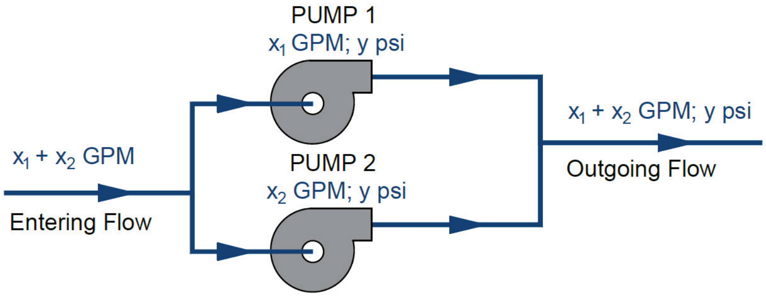


Figure 54: Pumps in parallel

On the pump curve, two pumps in parallel are graphed by selecting a pressure point, then adding the flow of each pump at that pressure point. See an example in the figure below. At 100 feet head, one pump produces 100 GPM, so 2 identical pumps will produce 200 GPM at 100 feet head. The operating point of each of the pumps in parallel has half the total flow and the same pressure rating.

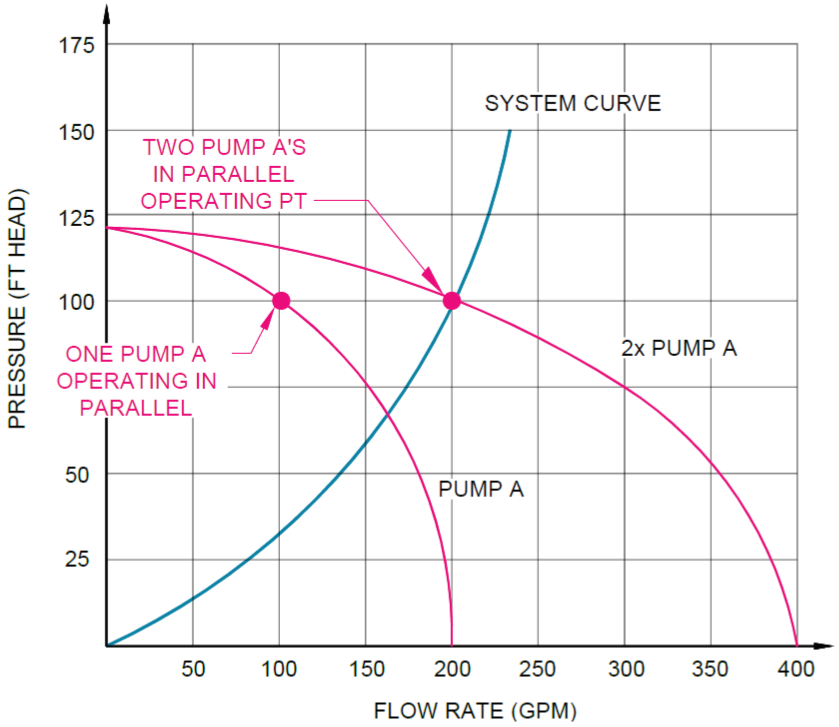


Figure 55: Two pumps operating in parallel. The flow rate of each pump is added and the pressure of each pump is the same.

Single versus Double Pump Operation Point:

Take another scenario where two pumps of equal size are installed in parallel, but one pump is turned off. Then what is the operation point of one pump on versus two pumps on? The new operation point will not simply be half the flow rate, but will depend on where the pump curve intersects the system curve. The single pump verses two pump operation point will vary as shown below.

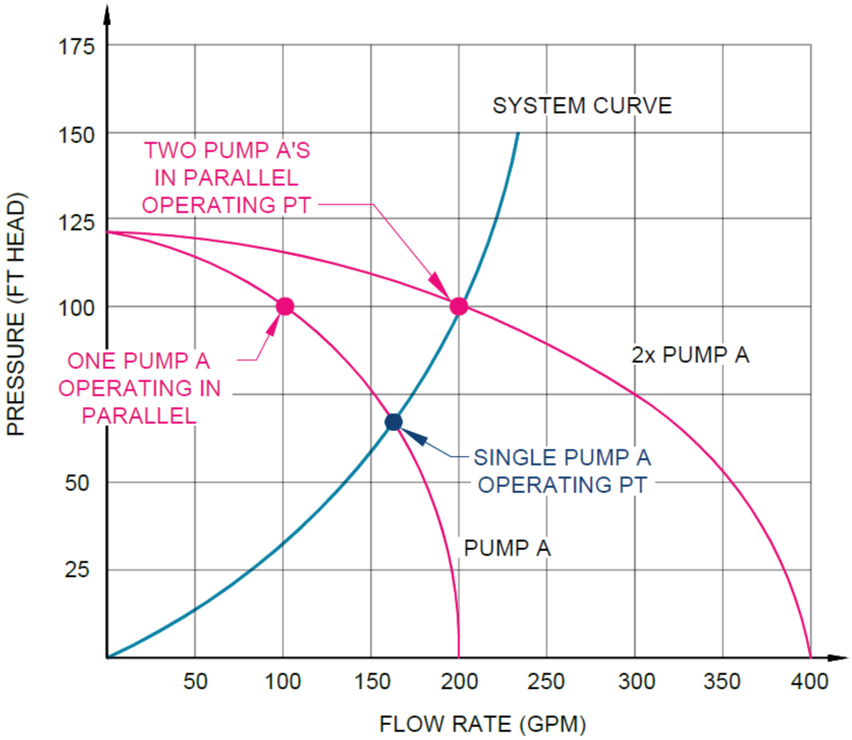


Figure 56: This figure shows the pump curve of a single pump on the left in red and two identical pumps operating in parallel on the right in red. The operation point for one pump on or two pumps on will vary along the system curve.

Realize that the system curve does not change when there is one or two pumps. The system curve is dependent on the piping system and how it behaves with various flow rates. Therefore, the system curve is not affected by pumps in parallel, but the intersection of the system curve with one pump curve or two pumps in parallel will change.

13.6.2 Pumps in Series

Pumps arranged in series will produce a pressure that is equal to the sum of the pressure of each individual pump at every flow rate point. In other words, the pressures are added and the flow rate remains the same. In the figure below, the flow leaving pump 1 must equal the flow entering pump 2, but as fluid enters each pump, pressure is added to the system.

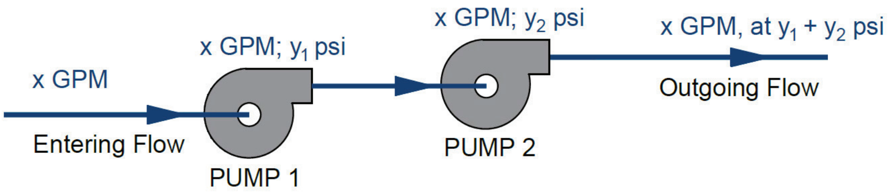


Figure 57: Pumps in series

The pump curve of pumps in series is found by going to a flow rate, then adding the pressure of each pump at that flow rate point. At 200 GPM, one pump produces 50 feet of head, so two identical pumps will produce 200 GPM at 100 feet head.

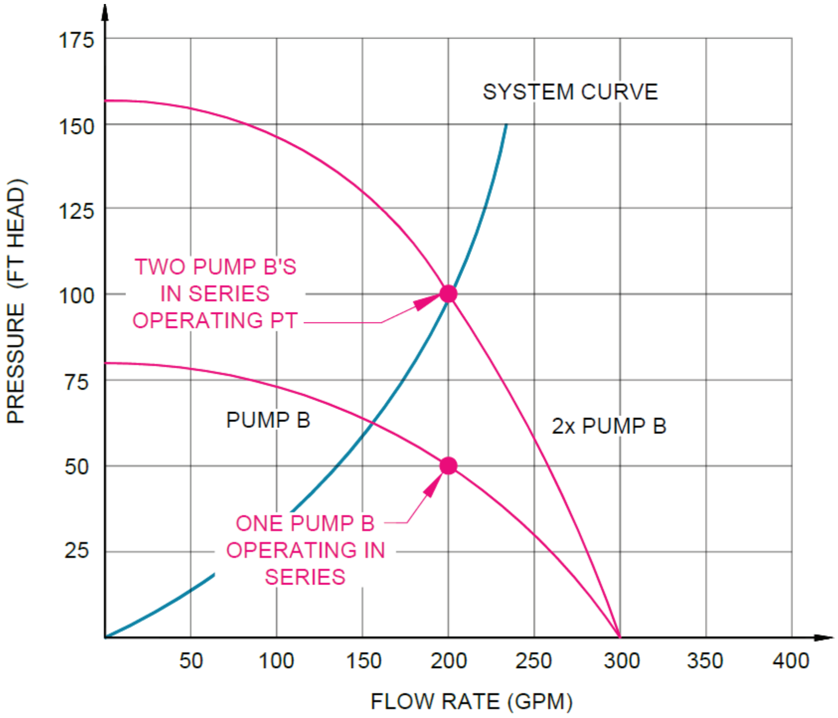


Figure 58: Two pumps operating in series. The pressure of each pump is added and the flow rate is the same.

Single versus Double Pump Operation Point:

Similar to the pumps in parallel, if one of the two pumps in series were turned off, then the operation point of a single pump be where the single pump curve intersects the system curve. The operation point of a single pump on verses two pumps on is shown in the figure below. Again, the system curve does not change when there is one or two pumps, but the intersection of one pump curve and two pumps in parallel will change.

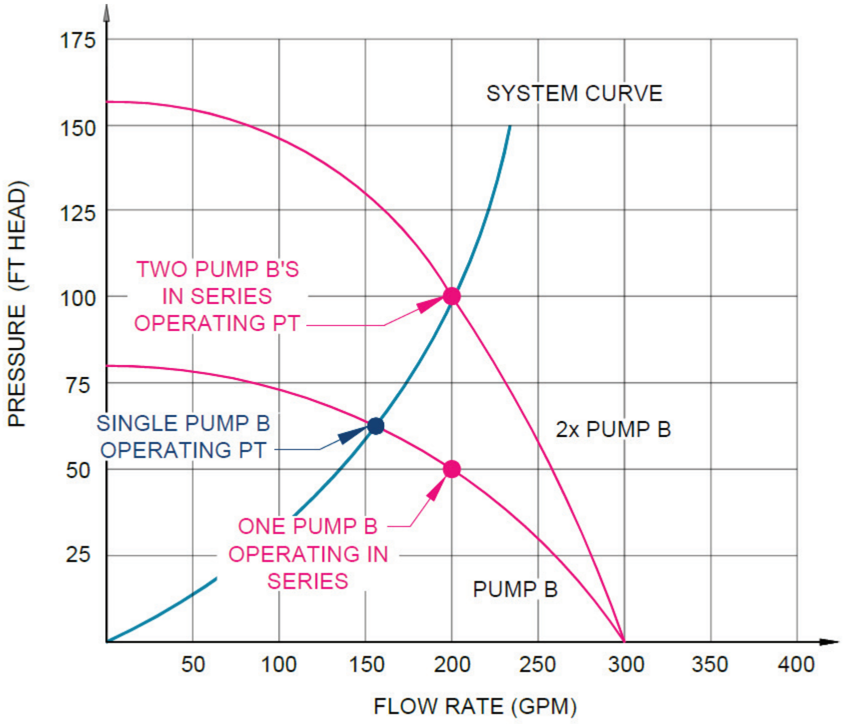


Figure 59: This figure shows the pump curve of a single pump on the bottom in red. The other pump curve occurs when two identical pumps are arranged in series.

13.7 NET POSITIVE SUCTION HEAD

The engineer must be able to properly determine net positive suction head in order to avoid cavitation. *Cavitation* occurs when the suction pressure (head) is less than the vapor pressure of the water. If the suction pressure is lower than the vapor pressure, then small vapor bubbles form. When these bubbles reach the pump, the pressure of the fluid increases and the bubbles implode causing damage to the impellers and other parts of the pump. This is what is known as cavitation.

Suction head is defined as the total pressure at the inlet of the pump and net positive suction head is the difference between the suction head at the inlet and the vapor pressure of the water at the inlet of the pump.

$$Net\ Positive\ Suction\ Head\ Available = Suction\ Head_{inlet\ of\ pump} - Vapor\ Pressure_{water}$$

Suction head is found by determining all the pressures acting upon the fluid whether positive or negative. The following figure best describes all the pressures that can be acting upon a pump.

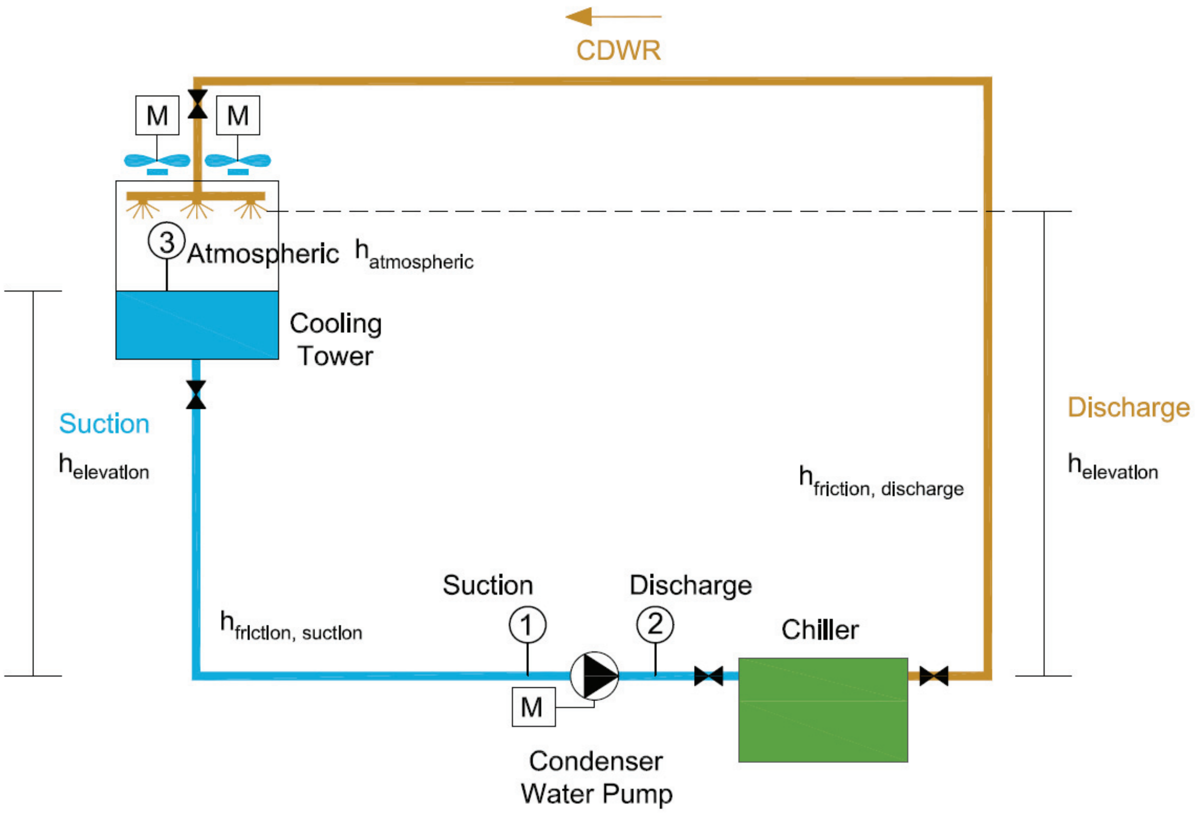

CDWR
M
M
③ Atmospheric $h_{atmospheric}$
Cooling Tower
Suction
$h_{elevation}$
Discharge
$h_{elevation}$
$h_{friction, discharge}$
$h_{friction, suction}$
Suction
①
Discharge
②
Chiller
M
Condenser Water Pump

Figure 60: This figure will be used to show how net positive suction head is determined.

Point 1 is the pressure measured at the suction of the pump. Please remember that the typical pressure gauge will only measure the static pressure.

Point 2 is the discharge pressure gauge. The difference in pressure between points “2” and “1” will be the pressure applied by the pump. There must be sufficient pressure at point “2” in order to overcome the pressure drop through the discharge piping, the equipment like the chiller and the elevation difference between the centerline of the pump and the discharge to atmosphere point.

Point 3 shows the atmospheric pressure. This is the absolute pressure acting on the fluid. If the system is located at sea level, then the pressure is equal to1 atmosphere (atm) or 14.7 PSIA or 33.9 feet of water.

$H_{\text{elevation, suction}}$: This pressure identifies the elevation difference between the top surface of the liquid and the pump centerline. This value can be positive or negative and is measured in “feet of head”.

$H_{\text{elevation, discharge}}$: This pressure identifies the elevation difference between the discharge location at the top of the cooling tower and the centerline of the pump. This value can be positive or negative and is measured in “feet of head”.

$H_{\text{friction, suction}}$: The friction pressure or head is the amount of pressure lost due to friction in the piping, fittings, equipment, valves, etc. leading from the fluid source to the pump on the suction side.

$H_{\text{friction, discharge}}$: The friction pressure or head is the amount of pressure lost due to friction in the piping, fittings, equipment, valves, etc. leading from the pump discharge to the discharge point at the top of the cooling tower.

There are two equations used to calculate the net positive suction head. The first equation is used when there is no pressure gauge at the suction of the pump. You take the atmospheric pressure, then add or subtract the elevation pressure on the suction piping. Next you subtract the friction in the suction piping leading to the suction of the pump. This current sum is the suction pressure at the pump. But lastly, you need to subtract out the vapor pressure of water. This will give you the net positive suction head available.

Equation/Method 1:

$$NPSHa = h_{atmospheric} \pm h_{elevation,suction} - h_{friction,suction} - h_{vapor\ pressure\ water}$$

The vapor pressure of the water is found by simply looking up fluid property tables and finding the vapor pressure at the operating temperature. Water is the most common fluid used in pumping systems and a table of corresponding vapor pressure and temperatures are shown below.

Water Temperature [°F]	Vapor Pressure [ft of head]
32	0.20
41	0.29
50	0.41
59	0.57
68	0.78
77	1.06
86	1.42
104	2.47
140	6.68
176	15.89
212	33.96

From the table above, it can be seen that as the temperature of the water increases, the pressure at which vaporization will occur also increases. The issue of cavitation becomes even more critical at higher temperatures.

The next equation is used when you have a pressure gauge reading at the suction of the pump. Remember that the pressure gauge will be in gauge pressure and it will only measure the static pressure. So in order to find the net positive suction head available, you need to convert to atmospheric pressure and then add in the velocity pressure to get the total pressure.

Equation/Method 2:

$$NPSHa = h_{suction\ pressure\ (gauge)} + h_{atmospheric} + h_{velocity\ pressure} - h_{vapor\ pressure\ water}$$

H velocity pressure: The velocity head pressure is the pressure due to the flowing liquid. Using the velocity pressure is an alternate method to calculating the suction pressure. This term is only used if there is a pressure gauge at the suction of the pump. The pressure gauge measures the static pressure at the pump. In order to determine the total suction pressure the static pressure and the velocity pressure are summed together. Typically the velocity head pressure is very small.

$$h_{velocity\ pressure} = \frac{V^2}{2g}[m\ or\ ft\ of\ head];$$

$$V = velocity\ in\ \frac{m}{s} or \frac{ft}{s};\ g = gravity = 9.81\frac{m}{s^2}\ or\ 32.2\frac{ft}{s^2}$$

The net positive suction head required (NPSHR) is an important criteria when selecting pumps. NPSHR is provided by the pump manufacturer and it is the minimum required pressure at the suction of the pump. The NPSHA must be higher than the NPSHR to prevent cavitation.

$$NPSHA > NPSHR$$

14.0 PNEUMATIC COMPONENTS

Pneumatics includes the equipment necessary to do work with air. This includes compressors, tubing, pressure vessels, control valves, actuators and connections, as shown in the simply pneumatic system.

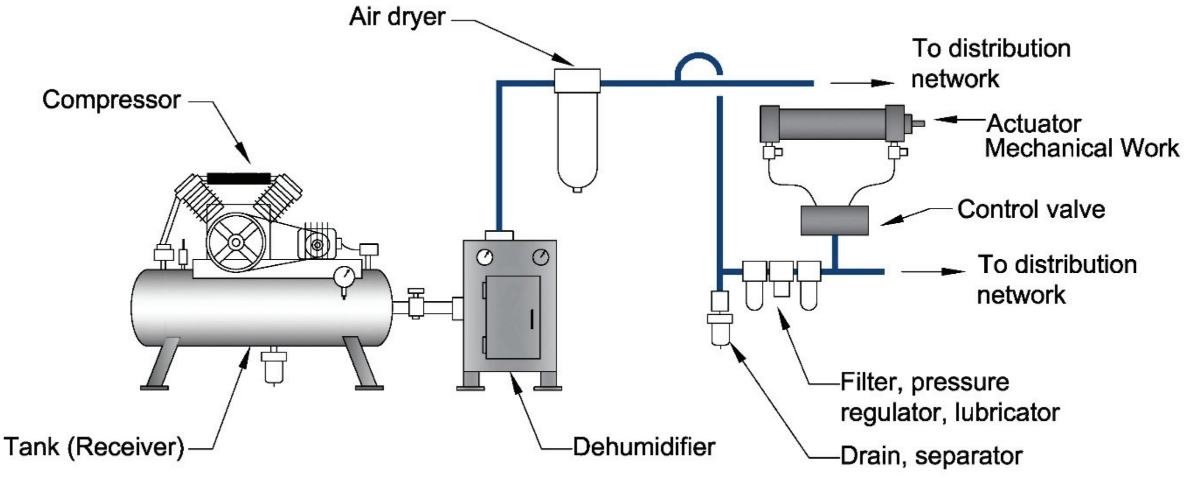

Air dryer
To distribution network
Compressor
Actuator Mechanical Work
Control valve
To distribution network
Filter, pressure regulator, lubricator
Tank (Receiver)
Dehumidifier
Drain, separator

Figure 61: A simple pneumatic system consists of a receiver that holds the compressed gas, followed by a compressor that pressurizes the gas. The pressurized gas then goes through a dehumidifier, air dryer filters and drains, before it finally reaches the actuator. The actuator is used to conduct mechanical work.

14.1 COMPRESSORS

The compressor raises the temperature and pressure of a gas. The compressor is where the work takes place. The compressor is also the driving force that moves the gas through the cycles and pneumatic systems.

The compression of gas typically occurs isentropically for exam purposes, meaning that there is no change in entropy. Since the compressor is not completely efficient there will be an increase in enthalpy as the heat generated by the compressor is transferred to the gas.

Entropy - a measure of the amount of disorder in a thermodynamic system.

Enthalpy - a measure of the total energy in a thermodynamic system (sensible and latent energy).

The engineer should be knowledgeable of the 5 different types of compressors and their advantages and disadvantages, in order to determine when they should be used. The five types of compressors are centrifugal, scroll, reciprocating, screw and rotary. A brief overview of the different types of compressors is shown below.

- Rotary: The rotary type compressor compresses refrigerant gas through positive displacement. Positive displacement simply means that the pressure of the gas is increased by reducing the volume.
- Scroll: Similar to the rotary type compressor, the scroll compressor uses positive displacement to increase the pressure of the gas.
- Screw: The screw compressor consists of two interlocking screws. The gas moves through the screw from the beginning thread to the end thread, increasing the pressure as it moves to the discharge side.
- Reciprocating: A reciprocating compressor compresses gas through positive displacement. A piston type movement compresses gas as it enters the cylinder.
- Centrifugal: Centrifugal compressors are not like positive displacement compressors, these compressors rely on a rotating impeller to use its centrifugal force to move the gas to the outside diameter of the rotating impeller, which increases the velocity of the gas. The increased velocity is then translated into increased pressure.

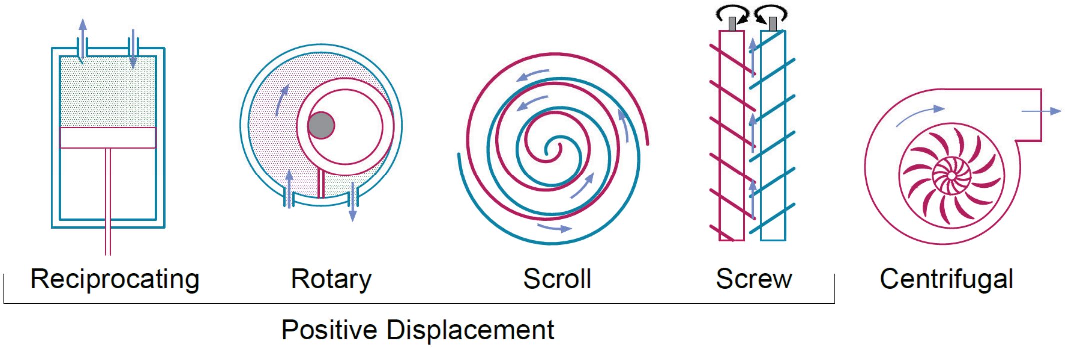


Figure 62: Compressor types

Another distinction between compressors is made between hermetic, semi-hermetic and open drive compressors. Hermetic is most often recognized when used in the phrase “hermetic seal”, which means airtight.

Hermetic: A hermetic compressor is airtight. The compressor and motor are located in a **welded** container, so no refrigerant can escape. Since the motor is located in the same enclosure as the compressor, the compressor needs to account for the motor heat.

Open Drive: An open drive compressor indicates that the compressor and refrigerant are located in an enclosure and out of the enclosure is a shaft connecting it to a motor. The motor is outside of the enclosure and the heat is lost to the space and not to compressor.

Semi-Hermetic: A semi-hermetic is similar to a hermetic compressor, except the motor and compressor are located in a **mechanically sealed** container, which can be opened without cutting into the enclosure unlike the hermetic compressor.

The main compressors that are used in pneumatics type exam problems are below. These types of compressors give you the best understanding of compressors for the exam and how they can be used on the exam.

A single stage piston compressor has one stage of compression. It is comprised of a cylinder, which is the chamber of which the piston moves. The piston is the bottom of the plunger shape. The rod or stem is connected to the piston which moves the piston up and down.

The compression process begins as the piston draws air into the cylinder through the one-way intake valve. The cylinder fills up with air and then the piston compresses the volume of air as it decreases the volume. The distance at which the piston compresses the air is called the stroke. The stroke is not the entire length of the cylinder, because there is some clearance involved. Thus the total volume of the compressed air is the area of the cylinder multiplied by the stroke.

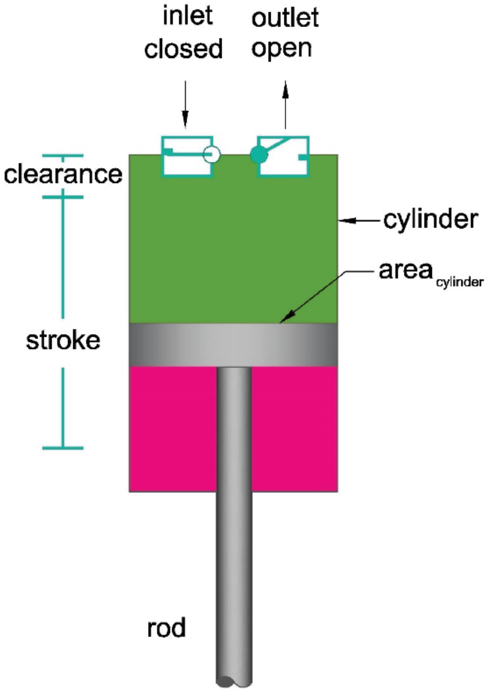

inlet
closed
outlet
open
clearance
cylinder
area cylinder
stroke
rod

Figure 63: A single stage piston, which shows the area of the cylinder and the stroke.

A two stage piston compressor is comprised of two cylinders that are situated in series. These multi stage compression systems are used because as the cylinder compresses the air, the air rises in temperature. When the temperature rises, the efficiency goes down. A compressor system will have multiple stages of compression but with cooling in between each stage to increase the efficiency of compression.

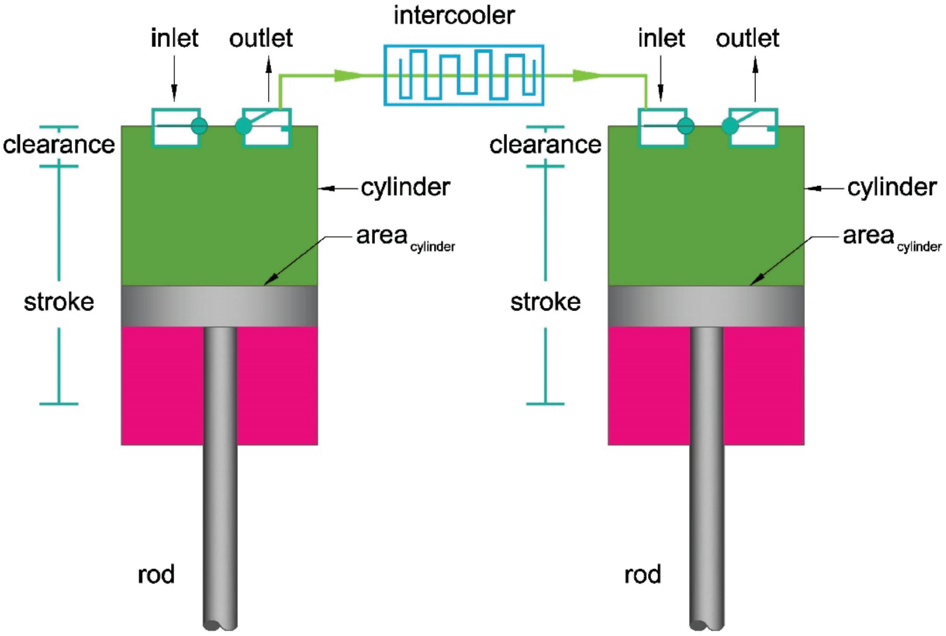

inlet
outlet
intercooler
inlet
outlet
clearance
cylinder
area cylinder
stroke
rod
clearance
cylinder
area cylinder
stroke
rod

Figure 64: A two stage piston, which shows the area of the cylinder and the stroke and an intermediate intercooler.

14.1.1 ACFM vs. SCFM

ACFM stands for actual cubic feet per minute and SCFM stands for standard cubic feet per minute. The term "standard cubic feet per minute" is a flow rate based on standard air temperature and pressure conditions of 14.7 PSIA, 68 degrees Fahrenheit and an air density of 0.075 pounds per cubic feet. The term "actual cubic feet per minute" is the air flow rate at the actual pressure and temperature conditions at the compressor. On the exam you may be asked to convert from one format to the other. Use the following equation to complete these conversions.

$$ACFM = SCFM * \left(\frac{P_{actual}}{14.7}\right) * \left(\frac{519}{T_{actual}}\right)$$

$$P_{actual} = actual\ pressure\ in\ absolute\ terms\ (psia); T_{actual} = actual\ temperature\ in\ Rankine\ (°R)$$

14.1.2 Compressor Work

A common skill, that is required of a professional engineer, is to determine the work done by the compressor. This work is shown as the difference between the compressor entering enthalpy (H1) and the leaving enthalpy (H2). The equation to determine the work of the compressor is shown below. This equation multiples the mass flow rate by the change in enthalpy between the discharge and suction conditions.

$$Q\ [W] = \left(H_{leaving} - H_{entering}\right)\left[\frac{J}{kg}\right] * (mass\ flow\ Rate)\left[\frac{kg}{s}\right] \quad [SI\ units]$$

$$Q\ [Btu/hr] = \left(H_{leaving} - H_{entering}\right)\left[\frac{Btu}{lb}\right] * (mass\ flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right] \quad [IP\ units]$$

$$H_{entering} = entering\ compressor\ enthalpy\ \left[\frac{J}{kg}\ or\ \frac{Btu}{lb}\right]$$

$$H_{leaving} = leaving\ compressor\ enthalpy\ \left[\frac{J}{kg}\ or\ \frac{Btu}{lb}\right]$$

This work can also be shown by the difference in temperature. Use the equation below to determine the work of the compressor.

$$Q\ [W] = c_p\left[\frac{J}{kg°C}\right] * \left(T_{leaving} - T_{entering}\right)[°C] * (mass\ Flow\ Rate)\left[\frac{kg}{s}\right] \quad [SI]$$

$$Q\ [Btu/hr] = c_p\left[\frac{Btu}{lbm°F}\right] * \left(T_{leaving} - T_{entering}\right)[°F] * (mass\ Flow\ Rate)\left[\frac{lb}{min}\right] * (60)\left[\frac{min}{hr}\right] \quad [IP]$$

$$T_{entering} = entering\ compressor\ temperature\ [°C\ or °F];$$

$$T_{leaving} = leaving\ compressor\ temperature\ [°C\ or °F]; c_p = heat\ capacity\ of\ gas$$

14.1.3 Compressor efficiency

Each compressor is provided with power, which is typically electricity. The compressor efficiency is found by dividing the actual work output calculated in the previous section by the input electricity power.

$$Efficiency_{compressor} = \frac{Work\ Out}{Power\ In}$$

The hardest part about solving for the compressor efficiency is making sure your units are correct.

If the input power and output work conditions are not known and you are given the construction of the compressor, then you may be able to find the efficiency of the compressor through this basic equation.

$$Efficiency_{compressor} = \frac{Actual\ flow\ rate}{Theoretical\ flow\ rate\ displaced\ by\ compressor}$$

The actual flow rate is the rate of air that is measured at the exit of the compressor. This value cannot be calculated and should be given to you in the exam. The theoretical flow rate displaced by the compressor is something that you should be able to calculate. The name should give you an idea of what this value entails. In a compressor cylinders compress a specific amount of air based on the number of cylinders, the diameter of cylinders and the length of the cylinders also known as stroke.

$$Volume_{compressor}[ft^3] = (\#\ of\ cylinders) * \left(\pi * \frac{D_{cylinder}^2}{4}\right) * Stroke$$

$$Stroke = length\ of\ cylinder\ (ft)$$

The above equation uses the number of cylinders multiplied by the area of a cylinder and the length of the cylinder to achieve a volume in cubic feet.

Next use the revolutions per minute (RPM) to determine the volumetric flow rate. A single revolution displaces one unit of volume, thus by multiplying the volume by the number of revolutions per minute, will result in a volumetric flow rate per minute.

$$Theoretical\ flow\ rate\ (CFM) = Volume_{compressor}(ft^3) * RPM$$

14.2 ACTUATORS

Actuators are used to convert the energy in fluid (gas or liquid) into mechanical power. The power produced by the actuator and all the equations will be dependent on the flow rate of the fluid, the pressure and the efficiency of the actuator. A hydraulic actuator is an actuator that uses liquid as power and a pneumatic actuator is an actuator that uses gas as power.

The typical questions asked with actuators are to calculate the output force, the required pressure to output a force, the speed of the cylinder, the flow needed for a cylinder speed and the cylinder area required.

14.2.1 Cylinder force

The force that a cylinder can produce is a result of the pressure of the fluid within the cylinder and the area that the cylinder acts upon.

$$Force\ (lbf) = Pressre\ (psi) * Area\ (in^2)$$

In the equation above, the pressure describes the pressure of the hydraulic or pneumatic fluid. The area is the area that is conducting the mechanical power.

14.2.2 Fluid pressure

If the force required and the cylinder area is given, then you can use the reverse of the previous equation to find the required pressure to achieve the mechanical work.

$$Pressure\ (psi) = \frac{Force\ (lbf)}{Area(in^2)}$$

14.2.3 Cylinder speed

The next type of question is determining the cylinder speed, which is based upon the volume of the cylinder and the flow rate of the pressurized fluid. As the fluid fills the cylinder, it moves the cylinder. The rate at which the cylinder is moved can be found with the below equation.

$$Cylinder\ speed\ \left(\frac{ft}{\mathrm{min}}\right) = \frac{Volumetric\ flow\ rate\ (\frac{ft^3}{\mathrm{min}})}{Area_{cylinder}(ft^2)}$$

14.2.4 Fluid flow

The next type of question is the opposite of the cylinder speed type question. If you are given the required cylinder speed and the area of the cylinder, then you must determine the required fluid flow. The below equation can be used for this type of problem.

$$Volumetric\ flow\ rate\ \left(\frac{ft^3}{\mathrm{min}}\right) = Area_{cylinder}(ft^2) * Cylinder\ speed\ \left(\frac{ft}{\mathrm{min}}\right)$$

14.2.5 Bulk modulus

The term Bulk Modulus is a property of a fluid that describes the compressibility of the fluid. Bulk modulus, β, is defined in the equation below. This term is used to determine how much a hydraulic fluid will compress when a pressure is applied. This is especially useful when completing problems when a cylinder force is applied to a hydraulic fluid.

$$\beta = \left(\frac{\Delta P}{\Delta V/V}\right); psi$$

In this equation, V is the original volume of the fluid and delta V is the change in volume, when a pressure change is applied to the fluid. Bulk modulus has units of pressure. For example, if a hydraulic fluid has a bulk modulus of 200,000 psi and a pressure of 1,000 psi applied to the fluid ($V = 10\ in^3$), then the change in volume can be found with the below equation.

$$\beta = \left(\frac{\Delta P}{\Delta V/V}\right); psi$$

$$\Delta V = \left(\frac{\Delta P * V}{\beta}\right) = \left(\frac{1{,}000\ psi * 10\ in^3}{200{,}000\ psi}\right)$$

$$\Delta V = 0.05\ in^3$$

As you can see from the previous example, if a fluid has a smaller bulk modulus then the fluid is more compressible. If the fluid has a larger bulk modulus then the fluid is less compressible. Compressibility and the bulk modulus are inversely related.

15.0 ELECTROMECHANICAL COMPONENTS

Electromechanical components are mechanical components that use electricity to operate. These components are typically motors and generators. These components were discussed in *Section 5.0 Electricity & Magnetism*. Please refer to that section for everything that you need for the FE exam on electromechanical components.

16.0 GEOMETRIC DIMENSIONING AND TOLERANCING

Geometric dimensioning and tolerancing or GD&T is used primarily in Machine Design technical drawings. For this topic, you need to be familiar with the common symbols and their meanings. The NCEES FE Reference Handbook gives you a list of the symbols and terminology that you should know but it does not explain how to use or interpret those symbols and terminology. The following section will go into more detail on all of the necessary symbols and terminology that you should know.

16.1 FEATURE CONTROL FRAME

Geometric dimensioning and tolerancing provides more control than simply dimensioning and tolerancing. You can add dimensions and tolerances, but by adding in the geometrical component, you can ensure that the design intent is properly manufactured. Geometric dimensioning and tolerancing will be presented with a feature control frame. An example of a feature control frame is presented below. It consists of the geometric characteristic symbol, the tolerance, material condition and the reference datum.

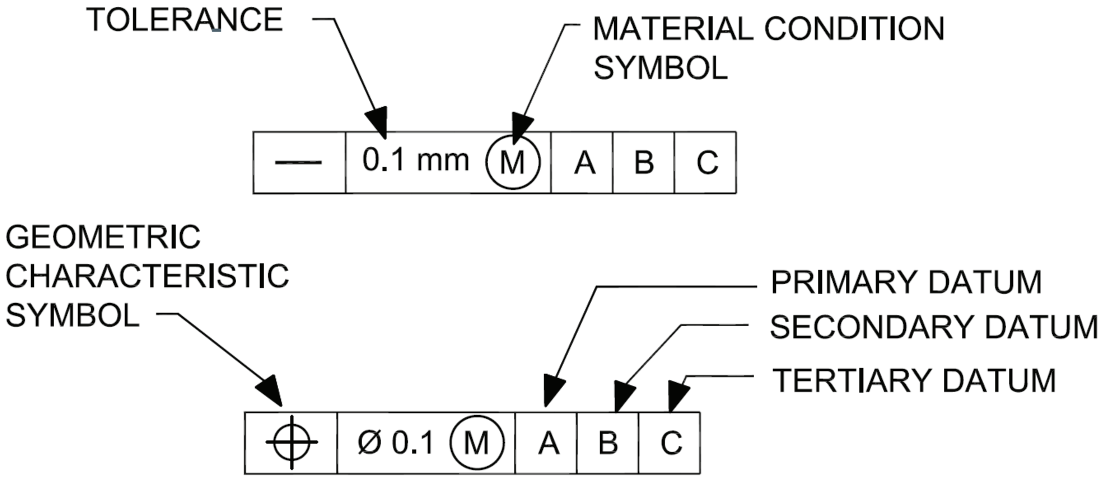


Figure 65: The feature control frame consists of four main sections, the geometric characteristic, tolerance, material condition and datum. The feature control frame will also be coupled with the actual dimension as necessary.

Each part of the feature control frame will be discussed in the following sections in the following order, datums, tolerance zone, material condition modifiers and finally geometric characteristic symbol.

16.2 DATUMS

Datums are a reference plane, axis or point for the dimensions and tolerances. There are a few ways that datums (plane, axis, point) may appear on the exam. First, the datum plane can be placed on a surface directly with a triangle and leader as shown in example (a). The datum plane can also be placed on a surface with a leader and the datum symbol shown on the horizontal portion of the leader as in example (b). Lastly, the datum symbol can be placed on a dimension extension line as in example (c). Please note that in example (c) the datum triangle symbol is not on the dimension arrow.

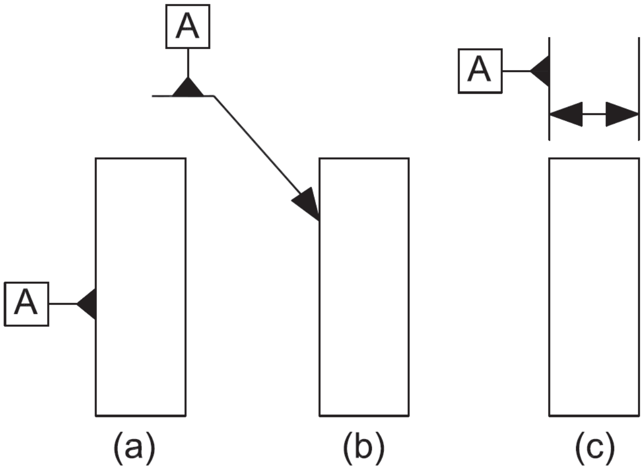


Figure 66: Datums can be applied to surfaces in three different ways. The datum triangle symbol can be filled in or not filled in. The filled in triangle is shown in this figure.

The actual surface is not the datum, but the imaginary plane created at that location becomes the datum. A datum can also be assigned to an axis. The axis is the center line or center plane between two dimension extensions or between two dimensions. The datum must be placed on the dimension line or extension of the dimension line in order for the axis to be the datum.

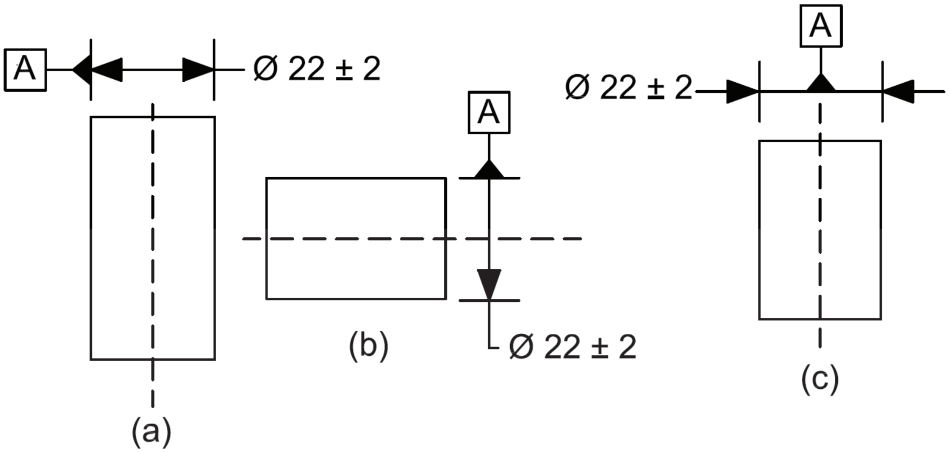


Figure 67: This figure shows three different ways a datum can be assigned to the center plane or axis.

16.3 TOLERANCE ZONE

Before discussing geometric characteristic symbols, you need to understand tolerance zones. The feature control frame consists of a tolerance amount. This value creates the tolerance zone. If the tolerance amount has a diameter symbol, then the tolerance zone is the diameter of a cylindrical or spherical zone, where the tolerance amount is equal to the diameter. If there is no diameter symbol, then the tolerance zone is between two straight lines or planes. The distance between those lines or planes is equal to the tolerance amount.

16.4 MATERIAL CONDITION MODIFIERS

Material condition modifiers are used to provide more information and constraints to the tolerance zone. You should be familiar with the following material condition modifiers, (1) MMC, (2) LMC and (3) RFS. These modifiers tell you what is the constraining tolerance and will create the boundary condition for the feature.

Maximum Material Condition (MMC): Maximum material condition is shown by a "M" in a circle. It describes the condition where the maximum amount of material exists. If it is a hole, then it is the smallest hole size. If it is a pin or shaft, then it is the largest size.

Least Material Condition (LMC): The least material condition is shown by a "L" in a circle. It describes the condition where the least amount of material exists. If it is a hole, then it is the largest hole size. If it is a pin or shaft, then it is the smallest size.

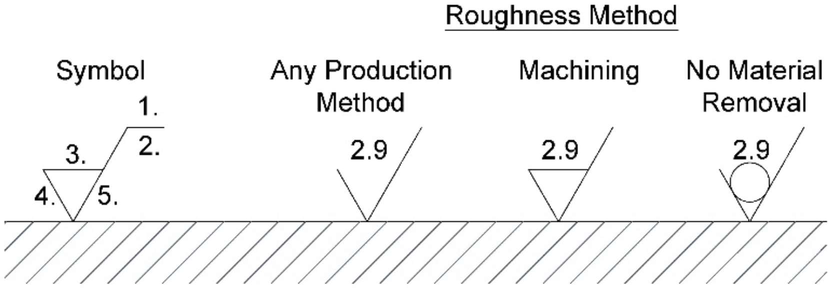
Regardless of Feature Size (RFS): Regardless of feature size indicates a geometric tolerance that will be independent of the dimension of the part. Regardless of feature size is the default.

16.5 GEOMETRIC CHARACTERISTIC SYMBOLS

The geometric characteristic symbols provide the final part of the feature control frame. The geometry characteristic symbols tell you how to apply the tolerance zone and any material condition modifiers. Please read through the descriptions in the NCEES Mechanical FE Reference Handbook. The handbook provides examples for each type of geometric characteristic symbol.

16.6 SURFACE CONDITIONS

Surface finishes can be important for the fits and tolerances of components, because the roughness of a component's surface may affect the tolerance of the component. If a tight tolerance is required, then the roughness should be minimized. Also if a figure is rough, then it will affect the straightness of the surface. The following method is used to specify the roughness for a component. The roughness value, #3, is shown in units, μm.


Roughness Method
Symbol
Any Production Method
Machining
No Material Removal
1.
2.
3.
4.
5.
2.9
2.9
2.9

Symbol Legend

1. Production Method, Treatment or Coating
2. Sampling Length
3. Roughness
4. Allowance
5. Lay Direction

Figure 68: The roughness of a surface can be specified with the above symbology.

17.0 Practice Exam Problems

17.1 Practice Problem 1 – Quality and Reliability

A machined product is measured and the following shows six samples, each with 17 observations per sample.

Sample	Mean	Range
1	3.06	0.42
2	3.15	0.50
3	3.11	0.41
4	3.13	0.46
5	3.06	0.46
6	3.09	0.45

n	A2	A3	~~c4~~	B3	B4	~~d2~~	~~1/d2~~	~~d3~~	D3	D4
17	0.203	0.739	~~0.9845~~	0.466	1.534	~~3.588~~	~~0.2787~~	~~0.744~~	0.378	1.622
18	0.194	0.718	~~0.9854~~	0.482	1.518	~~3.640~~	~~0.2747~~	~~0.739~~	0.391	1.609
19	0.187	0.698	~~0.9862~~	0.497	1.503	~~3.689~~	~~0.2711~~	~~0.733~~	0.404	1.596
20	0.180	0.680	~~0.9869~~	0.510	1.490	~~3.735~~	~~0.2677~~	~~0.729~~	0.415	1.585
21	0.173	0.663	~~0.9876~~	0.523	1.477	~~3.778~~	~~0.2647~~	~~0.724~~	0.425	1.575
22	0.167	0.647	~~0.9882~~	0.534	1.466	~~3.819~~	~~0.2618~~	~~0.720~~	0.435	1.565

What are the LCL and UCL for the mean chart?

(A) $UCL = 0.45; LCL = 0.40$

(B) $UCL = 3.10; LCL = 3.01$

(C) $UCL = 3.19; LCL = 3.01$

(D) $UCL = 3.15; LCL = 3.06$

17.2 PRACTICE PROBLEM 2 – QUALITY AND RELIABILITY

A machined product is measured and the following shows six samples, each with 15 observations per sample.

Sample	Mean	Range
1	78.1	14.1
2	72.1	8.5
3	63.7	18.1
4	89.4	18.9
5	80.8	9.2
6	78.8	14.7

n	A2	A3	~~c4~~	B3	B4	~~d2~~	~~1/d2~~	~~d3~~	D3	D4
15	0.223	0.789	~~0.9823~~	0.428	1.572	~~3.472~~	~~0.2880~~	~~0.756~~	0.347	1.653
16	0.212	0.763	~~0.9835~~	0.448	1.552	~~3.532~~	~~0.2831~~	~~0.750~~	0.363	1.637
17	0.203	0.739	~~0.9845~~	0.466	1.534	~~3.588~~	~~0.2787~~	~~0.744~~	0.378	1.622
18	0.194	0.718	~~0.9854~~	0.482	1.518	~~3.640~~	~~0.2747~~	~~0.739~~	0.391	1.609
19	0.187	0.698	~~0.9862~~	0.497	1.503	~~3.689~~	~~0.2711~~	~~0.733~~	0.404	1.596
20	0.180	0.680	~~0.9869~~	0.510	1.490	~~3.735~~	~~0.2677~~	~~0.729~~	0.415	1.585

What are the LCL and UCL for the range chart?

(A) $UCL = 23.0; LCL = 4.8$

(B) $UCL = 13.9; LCL = 13.9$

(C) $UCL = 18.9; LCL = 8.5$

(D) $UCL = 13.9; LCL = 8.5$

17.3 PRACTICE PROBLEM 3 - GEARS

A worm gear is mated with a double threaded worm. The worm rotates at 600rpm and has a pitch diameter of 4 inches. If the worm gear has 36 teeth, what will be the rotational speed of the worm gear?

(A) 33 rpm

(B) 46 rpm

(C) 220 rpm

(D) 230 rpm

17.4 PRACTICE PROBLEM 4 – BEARINGS

A manufacturer's dynamic capacity rating for a ball bearing is 70,000 lbs, based on 90% reliability. If the bearing is under a load of 12,000 lbs and operating in high temperatures conditions, what is the bearing rating life in terms of operating hours? Assume the bearing is operating at 3600 rpm and a life adjustment factor of 0.8 for the high operating temperatures.

(a) 660 hours

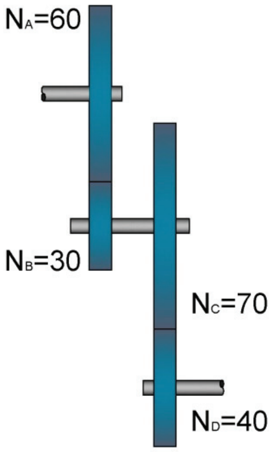
(b) 740 hours

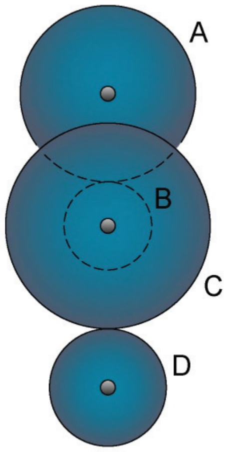
(c) 920 hours

(d) 1100 hours

17.5 PRACTICE PROBLEM 5 – GEARS

A train gear configuration is shown in the figure below. The shaft at gear A rotates at 3600 rpm, counterclockwise. The number of teeth for each gear is given in the figure. What is the rotational speed at gear D?





(a) 1030 rpm, counterclockwise

(b) 3200 rpm, clockwise

(c) 4100 rpm, counterclockwise

(d) 13,000 rpm, counterclockwise

17.6 PRACTICE PROBLEM 6 - SPRINGS

A helical compression spring has a total number of 20 coils and has square/ground ends. The shear modulus is 15 x 10^6 psi. The design load is 100 pounds and the spring has a mean diameter of 4” and a wire diameter of 0.25 inches. What is the design deflection?

(a) 2 inches

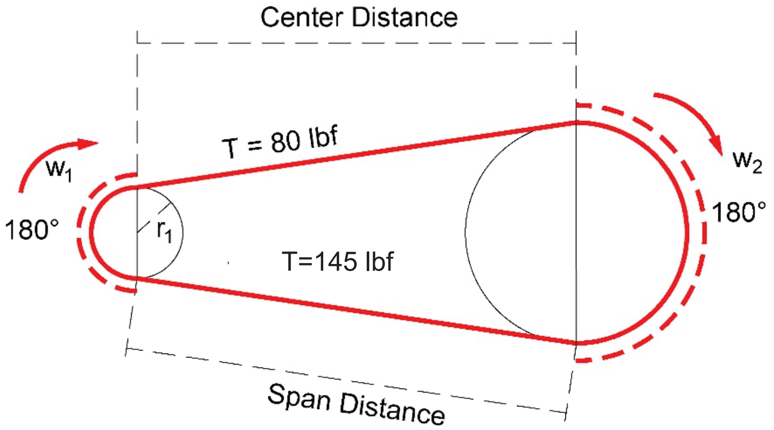
(b) 6 inches

(c) 10 inches

(d) 16 inches

17.7 PRACTICE PROBLEM 7 – BELTS & PULLEYS

A pulley as shown below has the below tension on the tight and slack sides. The angle of wrap will be increased with another pulley from 180 degrees to 200 degrees. What is the increase in torque capacity at the large pulley? Assume the slack tension remains the same. The coefficient of friction is 0.2.



(a) 16%

(b) 33%

(c) 49%

(d) 71%

17.8 PRACTICE PROBLEM 8 - MANUFACTURABILITY

A nominal 2" shaft is designed to have a maximum value of 2.0025" and a minimum value of 1.995. The hole for the shaft is designed to have a minimum value of 2.00" and a maximum value of 2.0015". What is the largest clearance?

(a) .0035"

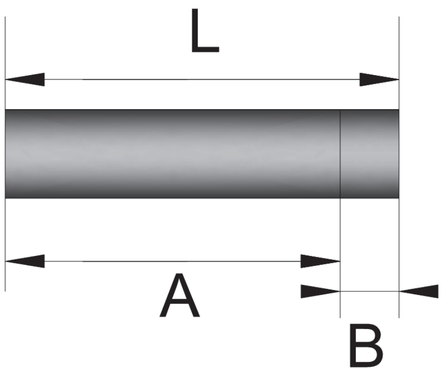
(b) .0040"

(c) .0055"

(d) .0065"

17.9 PRACTICE PROBLEM 9 - MANUFACTURABILITY

The following shaft is constructed from two pieces, "A" and "B". A has dimensions 1.500 ± .001. B has dimensions of 0.500 ± .005 in. What is the tolerance of the overall length?



(a) ±.001 in

(b) ±.025 in

(c) ±.005 in

(d) ±.006 in

17.10 PRACTICE PROBLEM 10 – PRESSURE VESSEL

What is the stress in a cylindrical pressure vessel when the fluid inside of the vessel is at a pressure of 10,000 psi? The vessel is 10 feet long and has a diameter of 4 feet. The vessel is 3” thick.

The answer is most nearly,

(a) 100,000 psi

(b) 150,000 psi

(c) 195,000 psi

(d) 215,000 psi

17.11 PRACTICE PROBLEM 11 – PRESSURE DROP FITTINGS

20 GPM of water at 50 F is flown through a 2” plastic pipe (2.067” ID). The Moody friction factor is .025. What is the pressure drop (PSI) through 3 elbows and 2 valves?

K_elbow = 0.8

K_valve = 1.7

(a) 0.14 psi

(b) 0.28 psi

(c) 0.37 psi

(d) 0.51 psi

17.12 PRACTICE PROBLEM 12 – NET POSITIVE SUCTION HEAD

A cooling tower is located such that the fluid level in the basin is 10 ft above the centerline for the suction of the condenser water pump. The water is at an average temperature of 86 F. The friction loss from the cooling tower basin to the suction of the pump is approximately 15 ft of head. What is the net positive suction head available at the suction side of the pump with a flow rate of 400 GPM?

$$h_{vapor\ pressure} = 1.4\ ft\ of\ head$$

(a) 12.6 ft of head

(b) 17.6 ft of head

(c) 27.6 ft of head

(d) 30.1 ft of head

17.13 PRACTICE PROBLEM 13 – HYDRAULIC COMPONENTS

A centrifugal pump, pumps 50 GPM at 25 feet of an unknown fluid of density 45 lbm/ft^3. What is the mechanical work produced by the pump? Assume 100% efficient motor and pump.

The answer is most nearly,

(a) 1/10 HP

(b) 1/5 HP

(c) 1/4 HP

(d) 1/2 HP

17.14 PRACTICE PROBLEM 14 – HYDRAULIC COMPONENTS

A pump is sized for 200 GPM at 150 ft of head. If the speed is decreased by 10%, what is the new design pressure of the pump? Assume the diameter remains the same.

(a) 100 ft of head

(b) 122 ft of head

(c) 135 ft of head

(d) 150 ft of head

17.15 PRACTICE PROBLEM 15 – JOINING METHODS

A bolt is initially used to secure two components with a force of 500 lbf. The bolt has a stiffness coefficient of 1.8 lbf/in and the clamping stiffness coefficient is 13 lbf/in. If an external load of 1,000 lbf is applied to the components, then what is the total amount of force acting upon the bolt? The bolt is SAE grade no. 1, 1” with a proof strength of 55 kips and a tensile strength of 75 kips.

(a) 622 lbf

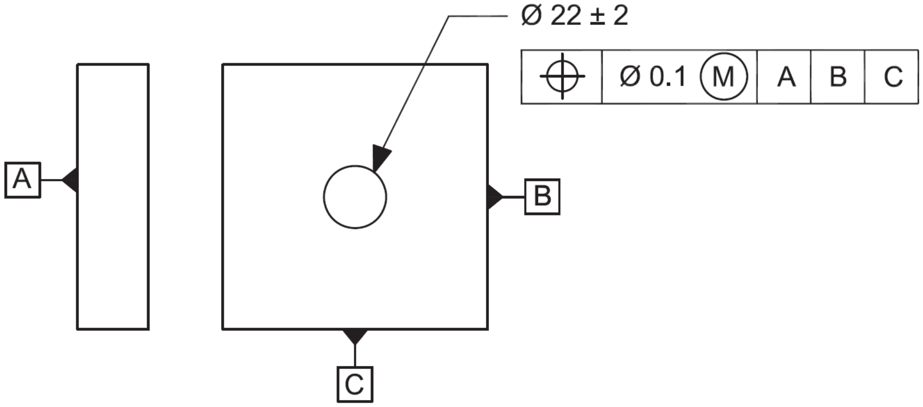
(b) 999 lbf

(c) 1,381 lbf

(d) 1,500 lbf

17.16 PRACTICE PROBLEM 16 – GEOMETRIC DIMENSIONING & TOLERANCING

What is the MMC and LMC of the hole in the figure below?



(a) LMC: 22.1 & MMC: 21.9

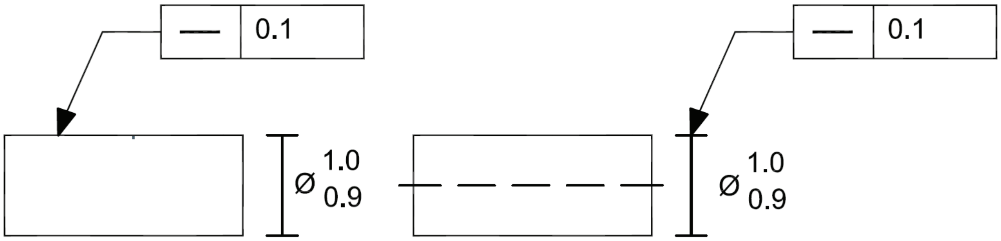
(b) LMC: 24 & MMC: 20

(c) LMC: 24.1 & MMC: 19.9

(d) LMC: 23.9 & MMC: 20.1

17.17 PRACTICE PROBLEM 17 – GEOMETRIC DIMENSIONING & TOLERANCING

Which of the following statements is true about the figures of the pin below?



(a) Left figure controls the flatness of the rectangle.

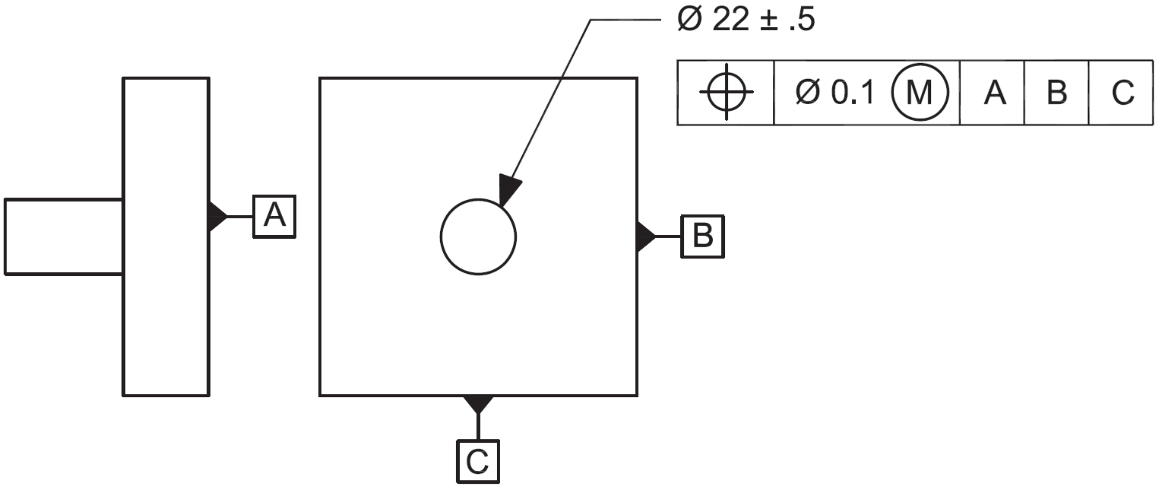
(b) Right figure controls the straightness of the center axis.

(c) A datum is required in both figures.

(d) Left and right figure controls the flatness.

17.18 PRACTICE PROBLEM 18 – GEOMETRIC DIMENSIONING &TOLERANCING

What is the MMC, LMC and Virtual Condition of the pin in the figure below?



(a) LMC: 21.5, MMC: 22.5, VC: 22.6

(b) LMC: 21, MMC: 22, VC: 22.5

(c) LMC: 21.5, MMC: 22.5, VC: 21.5

(d) LMC: 21.4, MMC: 22.6, VC: 22.6

18.0 Solutions

18.1 Solution 1 – Quality and Reliability

A machined product is measured and the following shows six samples, each with 17 observations per sample.

Sample	Mean	Range
1	3.06	0.42
2	3.15	0.50
3	3.11	0.41
4	3.13	0.46
5	3.06	0.46
6	3.09	0.45

What are the LCL and UCL for the mean chart?

(A) $UCL = 0.45; LCL = 0.40$

(B) $UCL = 3.10; LCL = 3.01$

(C) $UCL = 3.19; LCL = 3.01$

(D) $UCL = 3.15; LCL = 3.06$

The first step is to find $\bar{\bar{X}}$.

$$\bar{\bar{X}} = 3.10$$

Next find, $\bar{R}$

$$\bar{R} = 0.45$$

Now use the control chart constants table and find the variable A2 for sample size 17.

n	A2	A3	~~c4~~	B3	B4	~~d2~~	~~1/d2~~	~~d3~~	D3	D4
17	0.203	0.739	~~0.9845~~	0.466	1.534	~~3.588~~	~~0.2787~~	~~0.744~~	0.378	1.622
18	0.194	0.718	~~0.9854~~	0.482	1.518	~~3.640~~	~~0.2747~~	~~0.739~~	0.391	1.609
19	0.187	0.698	~~0.9862~~	0.497	1.503	~~3.689~~	~~0.2711~~	~~0.733~~	0.404	1.596
20	0.180	0.680	~~0.9869~~	0.510	1.490	~~3.735~~	~~0.2677~~	~~0.729~~	0.415	1.585
21	0.173	0.663	~~0.9876~~	0.523	1.477	~~3.778~~	~~0.2647~~	~~0.724~~	0.425	1.575
22	0.167	0.647	~~0.9882~~	0.534	1.466	~~3.819~~	~~0.2618~~	~~0.720~~	0.435	1.565

$$UCL = \bar{\bar{X}} + A_2 * \bar{R}; LCL = \bar{\bar{X}} - A_2 * \bar{R}$$

$$UCL = 3.10 + 0.203 * 0.45; LCL = 3.10 - 0.203 * 0.45$$

$$UCL = 3.19; LCL = 3.01$$

The correct answer is most nearly, **(C)** $\boldsymbol{UCL = 3.19; LCL = 3.01}$

18.2 Solution 2 – Quality and Reliability

A machined product is measured and the following shows six samples, each with 15 observations per sample.

Sample	Mean	Range
1	78.1	14.1
2	72.1	8.5
3	63.7	18.1
4	89.4	18.9
5	80.8	9.2
6	78.8	14.7

What are the LCL and UCL for the range chart?

(A) $UCL = 23.0; LCL = 4.8$

(B) $UCL = 13.9; LCL = 13.9$

(C) $UCL = 18.9; LCL = 8.5$

(D) $UCL = 13.9; LCL = 8.5$

First find, $\bar{R}$

$$\bar{R} = 13.9$$

Next, use the following equations to find UCL and LCL.

$$LCL = D_3 * \bar{R} = 13.9 * 0.347 = 4.8$$

$$UCL = D_4 * \bar{R} = 13.9 * 1.653 = 23.0$$

Use the charts to find the control chart variables.

n	A2	A3	~~c4~~	B3	B4	~~d2~~	~~1/d2~~	~~d3~~	D3	D4
15	0.223	0.789	~~0.9823~~	0.428	1.572	~~3.472~~	~~0.2880~~	~~0.756~~	0.347	1.653
16	0.212	0.763	~~0.9835~~	0.448	1.552	~~3.532~~	~~0.2831~~	~~0.750~~	0.363	1.637
17	0.203	0.739	~~0.9845~~	0.466	1.534	~~3.588~~	~~0.2787~~	~~0.744~~	0.378	1.622
18	0.194	0.718	~~0.9854~~	0.482	1.518	~~3.640~~	~~0.2747~~	~~0.739~~	0.391	1.609

19	0.187	0.698	~~0.9862~~	0.497	1.503	~~3.689~~	~~0.2711~~	~~0.733~~	0.404	1.596
20	0.180	0.680	~~0.9869~~	0.510	1.490	~~3.735~~	~~0.2677~~	~~0.729~~	0.415	1.585

The correct answer is most nearly, **(A)** $\boldsymbol{UCL = 23.0; LCL = 4.8.}$

18.3 SOLUTION 3 - GEARS

A worm gear is mated with a double threaded worm. The worm rotates at 600rpm and has a pitch diameter of 4 inches. If the worm gear has 36 teeth, what will be the rotational speed of the worm gear?

(A) 33 rpm

(B) 46 rpm

(C) 220 rpm

(D) 230 rpm

The speed ratios of the worm and worm gear use the following relationship.

$$\frac{n_{wg}}{n_w} = \frac{N_w}{N_{wg}}$$

For the worm gear, N_{wg} is equal to the number of teeth. For the worm, N_w, is equal to the number of threads.

Therefore, the rotational speed is equal to:

$$n_{wg} = n_w \left(\frac{N_w}{N_{wg}}\right)$$

$$n_{wg} = 600rpm * \left(\frac{2}{36}\right) = 33.3\ rpm$$

The answer is **(A) 33 rpm**

18.4 SOLUTION 4 – BEARINGS

A manufacturer's dynamic capacity rating for a ball bearing is 70,000 lbs, based on 90% reliability. If the bearing is under a load of 12,000 lbs and operating in high temperatures conditions, what is the bearing rating life in terms of operating hours? Assume the bearing is operating at 3600 rpm and a life adjustment factor of 0.8 for the high operating temperatures.

The bearing life rating formula from ISO 281 is given as:

$$L10\ (rpm) = a_1 * a_2 * a_3 * \left(\frac{C_r}{P_r}\right)^e$$

Where, a_1 is the reliability adjustment factor, a_2 is the material adjustment factor, and a_3 is the temperature adjustment factor. At 90% reliability, $a_1 = 1$. The material is assumed to be typical, therefore $a_2=1$. The temperature adjustment factor is given as, $a_3=0.8$.

For ball bearings, e=3.

Solve for L10, millions of revolutions.

$$L10\ (10^6\ rev) = 1 * 1 * 0.8 * \left(\frac{70{,}000\ lbs}{12{,}000\ lbs}\right)^{3.0} = 159$$

To find the hours, use the equation:

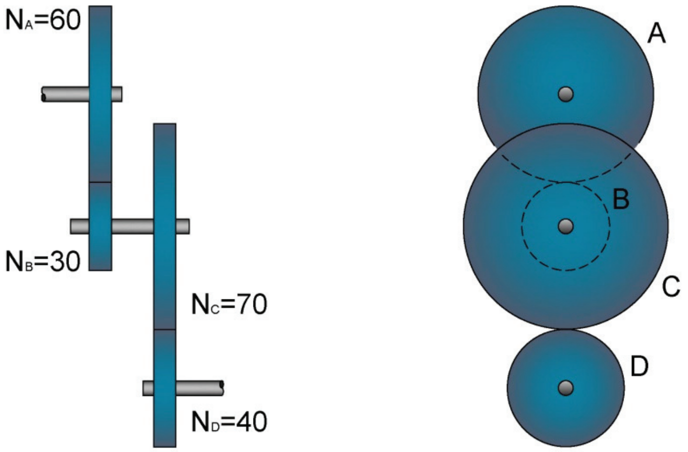
$$L_{10}h = \frac{10^6}{60 * n} * L_{10}$$

$$L_{10}h = \frac{10^6}{60 * 3600} * 159 = 735\ hours$$

The correct answer is most nearly, **(b) 740 hours**

18.5 SOLUTION 5 – GEARS

A train gear configuration is shown in the figure below. The shaft at gear A rotates at 3600 rpm, counterclockwise. The number of teeth for each gear is given in the figure. What is the rotational speed at gear D?



The speed of gears A and B is inversely proportional to number of teeth in each gear.

$$\frac{speed\ gear\ A}{speed\ gear\ B} = \frac{N_B}{N_A}$$

The shaft between gears B and C causes gear C to rotate at the same speed at gear B.

Then, find the speed of gear D.

$$\frac{speed\ gear\ B}{speed\ gear\ D} = \frac{speed\ gear\ C}{speed\ gear\ D} = \frac{N_D}{N_C}$$

The equation can be simplified to the following:

$$\frac{input\ speed}{output\ speed} = \frac{N_B}{N_A}\frac{N_D}{N_C}$$

Solve for the output speed:

$$output\ speed = input\ speed * \frac{N_A}{N_B}\frac{N_C}{N_D}$$

$$speed\ D = 3600 * \frac{60}{30}\frac{70}{40} = 12{,}600\ rpm$$

Gear A rotates counterclockwise, so Gear B will rotate clockwise. Gears B and C are on the same shaft, therefore, Gear C also rotates clockwise. Gear D rotates counterclockwise.

The correct answer is **D) 12,600 rpm counterclockwise**

18.6 SOLUTION 6 – SPRINGS

A helical compression spring has a total number of 20 coils and has square/ground ends. The shear modulus is 15 x 10^6 psi. The design load is 100 pounds and the spring has a mean diameter of 4” and a wire diameter of 0.25 inches. What is the design deflection?

$$\delta = \frac{8FD_m^3 n_a}{d_w^4 G}$$

$$Square\ \&\ Grounded \rightarrow n_a = N_t - 2 = 20 - 2 = 18$$

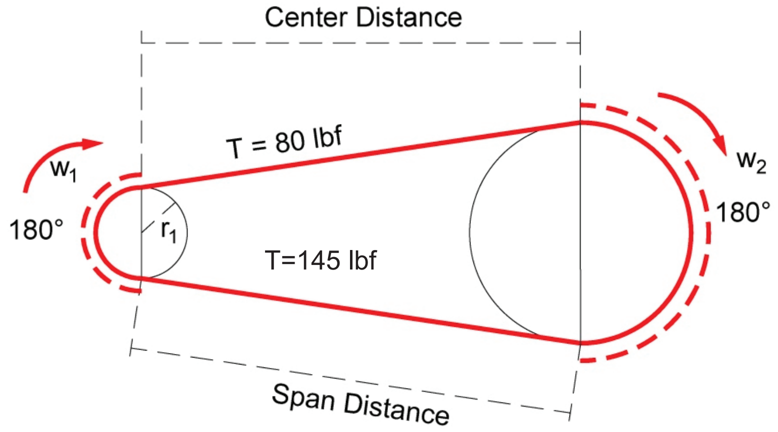
$$\delta = \frac{8 * 100\ lbs * (4\ in)^3 * 18}{(0.25\ in)^4 * 15 * 10^6\ psi}$$

$$\delta = 15.7\ inches$$

The correct answer is most nearly, **(d) 16 inches.**

18.7 SOLUTION 7 – BELTS & PULLEYS

A pulley as shown below has the below tension on the tight and slack sides. The angle of wrap will be increased with another pulley from 180 degrees to 200 degrees. What is the increase in torque capacity at the large pulley? Assume the slack tension remains the same. The coefficient of friction is 0.2.



The ratio of the tight and slack forces is equal to the natural logarithm of the angle of wrap and the coefficient of friction.

$$\theta_{org} = 180° * \left(\frac{\pi}{180}\right) = 3.14; \theta_{new} = 200° * \left(\frac{\pi}{180}\right) = 3.49;$$

$$\frac{F_{tight,org}}{F_{slack,org}} = e^{0.2*3.14}; \frac{F_{tight,new}}{F_{slack,org}} = e^{0.2*3.49}$$

$$\frac{F_{tight,org}}{F_{slack,org}} = e^{0.2*3.14}; \frac{F_{tight,new}}{F_{slack,org}} = e^{0.2*3.49}$$

$$\frac{F_{tight,org}}{F_{slack,org}} = 1.87; \frac{F_{tight,new}}{F_{slack,org}} = 2.01$$

Next use the ratios and the fact that the new slack tension is the same as the original slack tension.

$$Pulley\ Torque\ 2 \rightarrow T_{2,org} = r_2 * (1.87F_{slack} - F_{slack})$$

$$Pulley\ Torque\ 2 \rightarrow T_{2,new} = r_2 * (2.01F_{slack} - F_{slack})$$

Finally, cancel out the slack tension value and radius value and find the percentage difference from the original to the new conditions.

$$\frac{T_{2,new}}{T_{2,org}} = \frac{r_2 * (2.01F_{slack} - F_{slack})}{r_2 * (1.87F_{slack} - F_{slack})} = \frac{r_2 * (2.01 - 1) * (F_{slack})}{r_2 * (1.87 - 1) * (F_{slack})} = 1.16$$

The correct answer is most nearly **(a) 16% increase**.

18.8 SOLUTION 8 - MANUFACTURABILITY

A nominal 2” shaft is designed to have a maximum value of 2.0025” and a minimum value of 1.995. The hole for the shaft is designed to have a minimum value of 2.00” and a maximum value of 2.0015”. What is the largest clearance?

The maximum clearance occurs at the maximum hole value and the minimum shaft value.

$$Max\ Hole = 2.0015";\ Min\ Shaft\ =\ 1.995"$$

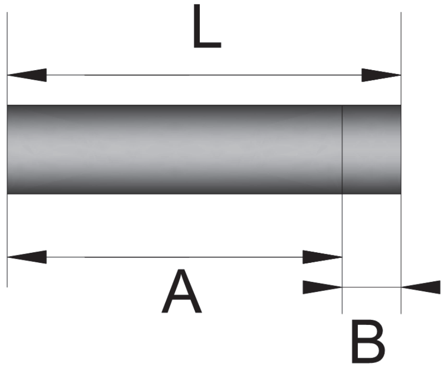
The difference between these two values is the clearance.

$$Max\ Clearance = 2.0015 \text{ - } 1.995. = 0065"$$

The correct answer is most nearly, **(d).0065”**.

18.9 SOLUTION 9 - MANUFACTURABILITY

The following shaft is constructed from two pieces, “A” and “B”. A has dimensions 1.500 ± .001. B has dimensions of 0.500 ± .005 in. What is the tolerance of the overall length?



The tolerances for “A” and “B” are simply added together.

The correct answer is most nearly, **(d) ±.006 in**.

18.10 SOLUTION 10 – PRESSURE VESSEL

What is the stress in a cylindrical pressure vessel when the fluid inside of the vessel is at a pressure of 10,000 psi? The vessel is 10 feet long and has an outer diameter of 4 feet. The vessel is 3” thick.

$$for\ cylindrical\ thin\ walled\ pressure\ vessels \rightarrow\ \sigma = \frac{PR}{t}$$

$$\sigma = \frac{10{,}000\ psi * \left(\frac{4\ ft}{2}\right) * 12\ in/ft}{3\ in}$$

$$\sigma = 80{,}000\ psi$$

$$cylindrical\ thick\ walled\ pressure\ vessels \rightarrow\ \sigma_t = P_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 10{,}000 \frac{24^2 + 21^2}{24^2 - 21^2} = 75{,}333\ psi$$

$$\sigma = 75{,}333\ psi$$

The answer is most nearly, (A), 100,000 psi.

18.11 SOLUTION 11 – PRESSURE DROP FITTINGS

20 GPM of water at 50 F is flown through a 2” plastic pipe (2.067” ID). The Moody friction factor is .025. What is the pressure drop (PSI) through 3 elbows and 2 valves?

K_{elbow} = 0.8

K_{valve} = 1.7

$$\text{Multiply GPM by } \frac{1}{448.83} \text{ to get} \frac{\text{FT}^3}{\text{sec}}.$$

$$20 \text{ GPM} * \frac{1}{448.83} = 0.04456 \frac{\text{FT}^3}{\text{sec}}.$$

$$\text{Area} = \text{PI} * \frac{\left(\frac{2.067}{12}\right)^2}{4} = 0.0233 \text{ ft}^2$$

$$\text{Velocity} = .\frac{04456 \text{ ft}^3/\text{sec}}{0.0233 \text{ ft}^2} = 1.91 \frac{ft}{sec}$$

Find the velocity pressure

$$VP = \frac{V^2}{2G}$$

$$Pressure\ loss = (3 * K_{elbow} + 2 * K_{valve}) * \frac{V^2}{2G}$$

$$Pressure\ loss = (3 * 0.8 + 2 * 1.7) * \frac{1.91 \frac{ft}{sec}^2}{2 * 32.2 \frac{ft}{sec^2}};$$

$$Pressure\ loss = 5.8 * .0566\ ft\ of\ head$$

$$Pressure\ loss = 0.32$$

Next convert to PSI

$$h = 0.32\ ft\ of\ head * \frac{0.433\ psi}{1\ ft\ of\ head} = 0.14\ PSI$$

The answer is most nearly, A, 0.14 psi.

18.12 SOLUTION 12 – NET POSITIVE SUCTION HEAD

A cooling tower is located such that the fluid level in the basin is 10 ft above the centerline for the suction of the condenser water pump. The water is at an average temperature of 86 F. The friction loss from the cooling tower basin to the suction of the pump is approximately 15 ft of head. What is the net positive suction head available at the suction side of the pump with a flow rate of 400 GPM?

Assume sea level.

For this question use the Net Positive Suction Head Available equation:

$$NPSHA = h_{pressure} + h_{elevation} - h_{friction} - h_{vapor\ pressure}$$

The pressure at the basin is simply atmospheric pressure.

$$h_{pressure} = 14.7\ psi\ or\ 33.96\ ft\ of\ head$$

The elevation pressure is the difference in height between the top of the fluid and the centerline of the pump.

$$h_{elevation} = 10\ ft\ of\ head$$

The friction loss is given.

$$h_{friction} = 15\ ft\ of\ head$$

Refer to your NCEES FE Handbook for the vapor pressure.

$$h_{vapor\ pressure} = 1.4\ ft\ of\ head$$

Find NPSHA

$$NPSHA = 33.96 + 10 - 15 - 1.4 = 27.6\ ft\ of\ head$$

The answer is most nearly, (C), 27.6 ft of head.

18.13 SOLUTION 13 – HYDRAULIC COMPONENTS

A centrifugal pump, pumps 50 GPM at 25 feet of head of an unknown fluid of density 45 lbm/ft^3. What is the mechanical work produced by the pump? Assume 100% efficient motor and pump.

Refer to the *NCEES FE Reference Handbook for the pump power equation. You need the* g_c *for English units only.*

$$Mechanical\ work\ (W) = \frac{p(\frac{g}{g_c})HQ}{n_{pump} * n_{motor}}$$

First convert 50 GPM to cubic feet per second.

$$50\frac{gal}{min} * \left(\frac{ft^3}{7.48\ gal}\right) * \left(\frac{min}{60\ sec}\right) = 0.11\frac{ft^3}{s}$$

Now plug in the new volumetric flow rate into the equation from the *NCEES FE Reference Handbook*.

$$Mechanical\ work\ (W) = \frac{45\frac{lbm}{ft^3} * \frac{32.2\frac{ft}{s^2}}{32.2\frac{lbm-ft}{lbf-s^2}} * 25\ ft * 0.11\frac{ft^3}{s}}{n_{pump} * n_{motor}}$$

$$W = 123.75\frac{ft-lbf}{s}$$

Finally convert using the *NCEES FE Reference Handbook* to horsepower.

$$W = 123.75\frac{ft-lbf}{s} * \left(\frac{hp}{550\ \frac{ft-lbf}{s}}\right) = 0.22\ HP$$

The answer is most nearly, (B), 1/4 HP.

18.14 SOLUTION 14 – HYDRAULIC COMPONENTS

A pump is sized for 200 GPM at 150 ft of head. If the speed is decreased by 10%, what is the new design pressure of the pump? Assume the diameter remains the same.

Use the affinity laws.

$$\frac{Q_1}{Q_2} = \left(\frac{N_1}{N_2}\right)^2 ;if\ diameter\ is\ held\ constant$$

$$\frac{150}{Q_2} = \left(\frac{x}{0.9x}\right)^2$$

$$150 * 0.9^2 * x = Q_2 * x$$

$$Q_2 = 121.5\ ft\ head$$

18.15 SOLUTION 15 – JOINING METHODS

A bolt is initially used to secure two components with a force of 500 lbf. The bolt has a stiffness coefficient of 1.8 lbf/in and the clamping stiffness coefficient is 13 lbf/in. If an external load of 1,000 lbf is applied to the components, then what is the total amount of force acting upon the bolt? The bolt is SAE grade no. 1, 1" with a proof strength of 55 kips and a tensile strength of 75 kips.

For this problem, use the following equation from the *NCEES FE Reference handbook*. The other information about the bolt is unnecessary.

$$F_b = F_i + \frac{k_b}{k_b + k_c} F_e$$

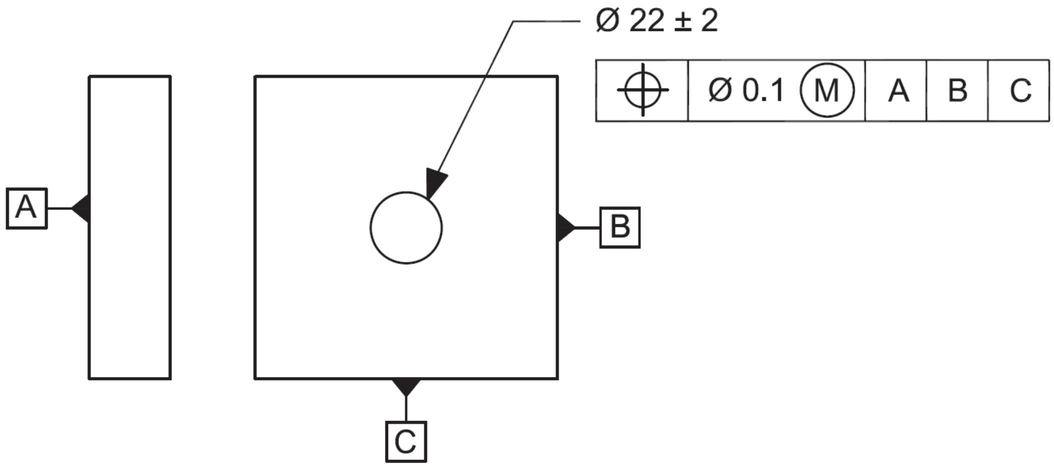
$$F_b = 500\ lbf + \frac{1.8 \frac{lbf}{in}}{1.8 \frac{lbf}{in} + \frac{13 lbf}{in}} 1{,}000\ lbf$$

$$F_b = 621.6\ lbf$$

The correct answer is most nearly, **(a) 622 lbf**.

18.16 SOLUTION 16 – GEOMETRIC DIMENSIONING & TOLERANCING

What is the MMC and LMC of the hole in the figure below?



For a hole, the maximum material condition occurs at the smallest hole size. The smallest hole size is 20. The least material condition occurs at the largest hole size, which is 24. The geometric tolerance applies to the position of the hole as shown by the geometric position symbol.

The correct answer is most nearly, **(b) LMC: 24 & MMC: 20.**

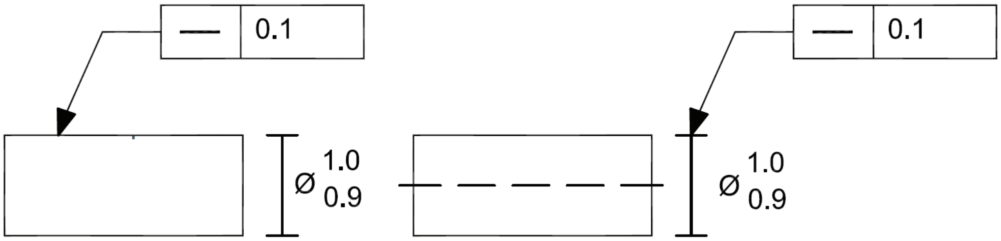
Another possible problem is to solve for the virtual condition. The virtual condition is found in the reference handbook.

$$External\ Feature\ (Shaft\ or\ Pin) \rightarrow Virtual\ Condition:\ MMC + Geometric\ Tolerance$$

$$Internal\ Feature\ (Hole\ or\ slot) \rightarrow Virtual\ Condition:\ MMC - Geometric\ Tolerance$$

18.17 SOLUTION 17 – GEOMETRIC DIMENSIONING & TOLERANCING

Which of the following statements is true about the figures of the pin below?



(a) Left figure controls the flatness of the rectangle. [A diameter symbol is shown, this means the figure is of a cylinder.]

(b) Right figure controls the straightness of the center axis. [Yes, this is correct, because the feature control frame is applied to the dimension extension lines.]

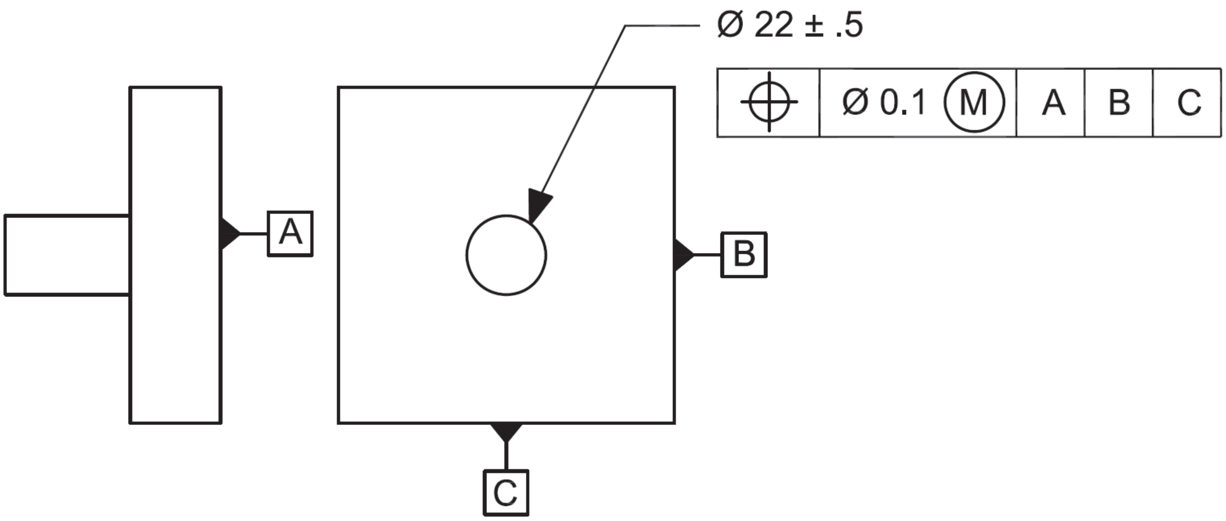
(c) A datum is required in both figures. [Datums are not required for straightness.]

(d) Left and right figure controls the flatness. [The geometric symbol is for straightness, not flatness.]

The correct answer is most nearly, **(b) Right figure controls the straightness of the center axis.**

18.18 SOLUTION 18 – GEOMETRIC DIMENSIONING & TOLERANCING

What is the MMC, LMC and Virtual Condition of the pin in the figure below?



For a pin, the maximum material condition occurs at the largest pin size. The largest pin size is 22.5. The least material condition occurs at the smallest pin size, which is 21.5. The geometric tolerance applies to the position of the hole as shown by the geometric position symbol. This allows you to find the virtual condition.

$$External\ Feature\ (Pin) \rightarrow Virtual\ Condition\colon\ MMC + Geometric\ Tolerance$$

$$External\ Feature\ (Pin) \rightarrow Virtual\ Condition\colon\ 22.5 + 0.1 = 22.6$$

The correct answer is most nearly, **(a) LMC: 21.5, MMC: 22.5, VC: 22.6.**

(a) LMC: 21.5, MMC: 22.5, VC: 22.6

(b) LMC: 21, MMC: 22, VC: 22.5

(c) LMC: 21.5, MMC: 22.5, VC: 21.5

(d) LMC: 21.4, MMC: 22.6, VC: 22.6

15 – Checklist Items

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	Check List Item		Yes/No?
1	**Mathematics**		
A	**Analytic geometry**		
	Can you find the equation of a line, given two points?		
	Can you find the equations of lines that are parallel or perpendicular to another line?		
	Can you find the distance between two points?		
	Are you comfortable with the LOG and LN mathematical properties?		
	Are you comfortable with using LOG and LN functions on your calculator?		
	Are you comfortable with using all of the trig functions and their right angle triangle relationships? SOH - CAH - TOA		
	Can you use the law of sines and cosines?		
	Can you quickly apply trigonometric identities?		
B	**Calculus**		
	Can you evaluate the derivative for common functions? Find the slope at a certain point on any function.		
	Can you solve for the derivative of various functions?		
	Can you evaluate the integral for common functions? Find the area under a curve.		
	Can you solve for the integral of various functions?		
C	**Linear algebra**		
	Can you solve for two unknowns with two equations?		
	Can you use matrices to solve multiple equations with multiple unknowns?		
	Can you use the matrices functions on your calculator?		
	Can you solve for matrix determinants, dot products and cross products?		
D	**Vector analysis**		
	Can you switch between rectangular format and polar format?		
	Can you use your calculator with rectangular values and polar values?		
	Can you conduct vector operations, like addition, multiplication, etc.?		
E	**Differential equations**		
	Can you solve a first order and second order differential equation?		
	Can you evaluate a first order and second order differential equation?		
F	**Numerical methods**		
	Can you convert between decimal and binary numbering methods?		

	Can you follow the curve fitting process?		
	Can you follow the least squares process?		
2	**Probability & Statistics**		
A	**Probability distributions**		
	Can you calculate mean, mode, and median?		
	Can you calculate standard deviation?		
	Can you calculate the probability of an outcome using the binomial distribution?		
	Can you calculate the probability of an outcome using the normal distribution?		
	Can you calculate the probability of an outcome using the t-distribution?		
	Can you calculate the probability of an outcome using the x^2-distribution?		
B	**Regression & curve fitting**		
	Can you follow the linear, square and cubed fit processes?		
	Can you interpret the coefficient of determination?		
	Can you calculate the coefficient of determination for a simple linear fit?		
3	**Ethics & Professional Practice**		
A	**Codes of ethics**		
	Did you read the NCEES Model Law and Model Rules		
	https://ncees.org/about/publications/		
B	**Agreements and contracts**		
	Have you viewed the Engineers Joint Contract Documents Committee knowledgebase? This website is the go to source for engineering contracts.		
	https://www.ejcdc.org/knowledgebase/		
C	**Ethical and legal considerations**		
	Did you view the National Society of Professional Engineers ethics section?		
	https://www.nspe.org/resources/ethics		
D	**Professional liability**		
	Have you viewed the Engineers Joint Contract Documents Committee knowledgebase? This website has many liability cases and their results. It gives you an idea on the extent of professional liability and how to protect yourself.		
	https://www.ejcdc.org/knowledgebase/		
E	**Public health, safety and welfare**		
	Did you read the NCEES Model Law and Model Rules		

	https://ncees.org/about/publications/		
	Did you view the National Society of Professional Engineers ethics section?		
	https://www.nspe.org/resources/ethics		
4	**Engineering Economics**		
A	**Time value of money**		
	What is the present value? Can you convert from future value to present value and vice versa?		
B	**Cost, including incremental, average, sunk and estimating**		
	Can you use cost indices?		
	Can you scale equipment costs?		
	Can you identify a sunk cost? Can you identify an ongoing cost? Can you identify a variable versus fixed cost?		
C	**Economic analyses**		
	What is the future value? Can you convert annualized values and present values to a future value?		
	What is the annualized value? Can you convert future values and present values to an annualized value?		
	What is the rate of return? Can you calculate the rate of return?		
D	**Depreciation**		
	Can you calculate depreciation through the various methods like sum of years digits, straight line and MACRS?		
5	**Electricity & Magnetism**		
A	**Charge, current, voltage, power and energy**		
	What is current? Do you understand the basic equations for electric DC circuits?		
	What is voltage? Do you understand the basic equations for electric DC circuits?		
	Can you calculate the charge for a number of electrons or protons? Can you find the charge for an electron in the handbook?		
	Can you calculate current based on flow of charge?		
	Can you calculate power with current, voltage and resistance?		
	Can you calculate energy based on power usage?		
B	**Current and voltage laws**		
	What is resistance? Do you understand the basic equations for electric DC circuits? How do you calculate equivalent resistance for resistors in series and in parallel?		

	Can you use KVL and KCL for a simple circuit?		
C	**Equivalent circuits**		
	Can you combine capacitors in parallel/series?		
	Can you combine inductors in parallel/series?		
	Can you combine resistors in parallel/series?		
D	**AC circuits**		
	What is power factor? When is it used?		
	Can you calculate the frequency of a waveform?		
	Can you calculate the RMS or MAX of a waveform?		
	Can you combine the impedances of capacitors, inductors and/or resistors?		
E	**Motors and generators**		
	What are the equations for power? How do you calculate mechanical horsepower for a fan and pump? How do you determine the motor horsepower and electricity usage?		
6	**Statics**		
A	**Resultants of force systems**		
	Can you use your calculator with rectangular values and polar values?		
	Can you calculate the force due to gravity?		
B	**Concurrent force systems**		
	Can you combine force vectors? Can you combine force vectors given to you in either polar or rectangular format?		
C	**Equilibrium of rigid bodies**		
	Can you draw a free body diagram of a system, like a beam, pulley or any other mechanical component and the force vectors?		
	Can you calculate the force due to a spring?		
	Can you calculate the forces in a cable/pulley system at various angles (x, y-components)?		
	Can you calculate the forces in a beam system, including the reactionary forces? Can you interpret the different types of supports and the various reactionary forces (moment, x, y, z-forces)?		
	Can you calculate the balancing moment for beam systems?		
D	**Frames and trusses**		
	Can you use the method of joints to find the forces at various members within a truss? Can you draw the free body diagrams?		
	Can you use the method of sections to find the forces at various members within a truss? Can you draw the free body diagrams?		
E	**Centroids**		

	Can you calculate the centroid for an amorphous 2-D component? Or at least understand the process and how it is completed mathematically?		
	Can you use the FE Handbook to calculate the centroid for various shapes?		
F	**Moments of inertia**		
	Can you calculate the moments of inertia for various shapes?		
	Can you calculate the moments of inertia for two shapes combined together with the parallel axis theorem?		
	Can you calculate the radius of gyration?		
G	**Static friction**		
	Can you calculate the force due to friction for a block on a slope, when the block is still? Can you calculate the friction with and without an external force?		
	Can you calculate the force due to friction for belts and pulleys, when there is no movement? Can you calculate the friction with gravity and/or an external force?		
7	**Dynamics, Kinematics and Vibrations**		
A	**Kinematics of particles**		
	Are you comfortable with the various kinematics equations in the FE Handbook? Can you calculate the linear displacement, linear velocity and linear acceleration?		
	Can you use the kinematics equations with projectiles?		
	Are you comfortable with the various angular kinematics equations in the FE Handbook? Can you calculate the angular displacement, angular velocity and angular acceleration? Are you comfortable working in radians and degrees with angular motion?		
B	**Kinetic friction**		
	Can you calculate the force due to friction for a block on a slope, when the block is moving? Can you calculate the friction with and without an external force?		
	Can you calculate the force due to friction for belts and pulleys, when there is movement? Can you calculate the friction with gravity and/or an external force?		
C	**Newton's second law for particles**		
	Can you calculate linear momentum for multiple objects?		
	Can you calculate angular momentum for multiple objects?		
	Can you calculate the mass moment of inertia for multiple objects?		
D	**Work-energy of particles**		

	Are you comfortable using the FE Handbook conservation of energy equations? Potential energy, kinetic energy, spring energy, etc.?		
	Are you comfortable using the conservation of energy equations with work?		
	Are you comfortable using conservation of energy equations in various directions (x & y & z)?		
E	**Impulse-momentum of particles**		
	Can you calculate impulse, with force and time? Can you calculate momentum of one ore multiple objects?		
F	**Kinematics of rigid bodies**		
	Are you comfortable with the various kinematics equations in the FE Handbook? Can you calculate the linear displacement, linear velocity and linear acceleration?		
	Can you use the kinematics equations with projectiles?		
	Are you comfortable with the various angular kinematics equations in the FE Handbook? Can you calculate the angular displacement, angular velocity and angular acceleration? Are you comfortable working in radians and degrees with angular motion?		
G	**Kinematics of mechanisms**		
	Are you comfortable with the common slider-crank mechanism? Can you calculate the velocity, linear displacement, acceleration of points along the mechanism?		
	Can you convert rotational movement to linear movement with a mechanism?		
H	**Newton's second law for rigid bodies**		
	Can you calculate linear momentum for multiple objects?		
	Can you calculate angular momentum for multiple objects?		
	Can you calculate the mass moment of inertia for multiple objects?		
I	**Work-energy of rigid bodies**		
	Are you comfortable using the FE Handbook conservation of energy equations? Potential energy, kinetic energy, spring energy, etc.?		
	Are you comfortable using the conservation of energy equations with work?		
	Are you comfortable using conservation of energy equations in various directions (x & y & z)?		
J	**Impulse-momentum of rigid bodies**		
	Can you calculate momentum for inelastic and elastic collisions? Can you use the coefficient of restitution to calculate the final velocities?		
	Can you calculate impulse, with force and time? Can you calculate momentum of one ore multiple objects?		
K	**Free and forced vibrations**		

	Can you use the spring energy to create a conservation of energy equation?		
	Can you find the natural frequency of a system?		
	Can you calculate the stiffness of a system?		
8	**Mechanics of Materials**		
A	**Shear and moment diagrams**		
	Can you create shear and moment diagrams for various beams, even though the shear and moment diagrams are provided in the FE Handbook?		
	Can you interpret the shear and moment diagrams provided in the FE Handbook?		
	Can you solve for reactionary forces and determine the force/moment at various points on a beam?		
B	**Stress types (axial, bending, torsion, shear)**		
	Can you calculate axial forces for various situations? Can you determine whether a force is axial and what equation to use for various situations?		
	Can you calculate shear forces for various situations? Can you determine whether a force is due to shear and what equation to use for various situations?		
	Can you calculate bending forces for various situations? Can you determine whether a force is due to bending and what equation to use for various situations?		
	Can you calculate torsion forces for various situations? Can you determine whether a force is due to torsion and what equation to use for various situations?		
C	**Stress transformations**		
	Can you use the transformation equations to transform stresses in one set of axes to another set of axes?		
D	**Mohr's circle**		
	Can you use Mohr's circle to show the relationship between tension, compression and torsion stresses? Can you use the Mohr's circle to also determine failure?		
E	**Stress and strain caused by axial loads**		
	Can you calculate shear stresses due to a force?		
	Can you calculate shear strain as a result of shear stress?		
	Can you use the Poisson's ratio to determine the relationship between the shear and axial strain/stress?		
	Can you use thermal expansion to determine stress and strain?		
F	**Stress and strain caused by bending loads**		
	Can you calculate the section modulus? Can you calculate the stress in a beam due to bending?		

	Can you find the maximum moment of a beam with a given force diagram?		
	Can you calculate beam deflection?		
G	**Stress and strain caused by torsion**		
	Can you calculate the stress and strain as a result of torsion?		
	Can you calculate the polar moment of inertia for various shaft configurations?		
	Can you use the parallel axis theorem to combine shafts to find the combined polar moment of inertia?		
H	**Stress and strain caused by shear**		
	Can you calculate shear stresses due to a force?		
	Can you calculate shear strain as a result of shear stress?		
	Can you use the Poisson's ratio to determine the relationship between the shear and axial strain/stress?		
	Can you use thermal expansion to determine stress and strain?		
I	**Combined loading**		
	Can you calculate the effect of multiple stresses acting upon an object?		
J	**Deformations**		
	Can you calculate the strain energy after a force and some displacement (deformation)? Can you determine the difference between elastic and inelastic deformations? Can you calculate deformation as a result of axial and shear loads?		
K	**Columns**		
	Can you calculate if buckling will occur? Can you calculate the critical buckling load?		
9	**Material Properties and Processing**		
A	**Chemical, electrical, mechanical, physical and thermal properties**		
	Are you comfortable converting between conductivity and resistivity? Can you find these properties in the FE Handbook for various materials?		
	Are you comfortable with the various hardness scales and finding those properties in the FE Handbook for various materials?		
	Are you comfortable with the following material properties like density, reflectance, transmittance, melting point, thermal expansion and heat capacity?		
B	**Stress-strain diagrams**		
	Can you interpret stress-strain diagrams? Are you familiar with elastic limit, yield strength, ultimate strength?		
	Are you familiar with the term Young's Modulus? Yield strength? Proportional limit?		

	Are you familiar with the terms Poisson's ratio and Shear modulus? Can you calculate axial and shear strain?		
	Are you comfortable with the term Ultimate tensile strength?		
C	**Engineered materials**		
	Are you familiar with the term, alloys?		
D	**Ferrous metals**		
	None		
E	**Nonferrous metals**		
	None		
F	**Manufacturing processes**		
	Are you familiar with machining, molding and heat treatment in the machine design area of engineering?		
G	**Phase diagrams**		
	Can you interpret phase diagrams? Can you follow various thermal processes like heating and quenching on the diagram to determine the final properties of the material? Can you use the lever rule?		
	Are you comfortable using in particular the iron-carbide diagram?		
H	**Phase transformation, equilibrium, and heat treating**		
	Are you comfortable with the term, annealing?		
	Can you use the time temperature transformation graph to determine the material's final properties?		
	Are you comfortable with the processes of hardening, tempering and quenching?		
I	**Materials selection**		
	None		
J	**Surface conditions**		
	Are you familiar with interpreting diagrams that indicate surface conditions?		
K	**Corrosion mechanisms and control**		
	Are you comfortable with the galvanic series graph and using that graph to determine which materials will corrode and which ones will not?		
	Can you follow the oxidation and reduction processes?		
L	**Thermal failure**		
	Are you familiar with the term, creep?		
M	**Ductile or brittle behavior**		
	None		
N	**Fatigue**		
	Can you use a Goodman diagram to predict failure? Can you create a Goodman diagram?		
O	**Crack propagation**		

	Can you use the FE Handbook crack propagation formulas for various crack types?		
10	**Fluid Mechanics**		
A	**Fluid properties**		
	Can you convert between Fahrenheit and Rankine?		
	Can you convert between Celsius and Kelvin?		
	Can you convert between gauge pressure and absolute pressure?		
	What are the units of viscosity? Do you understand the difference between absolute and kinematic viscosities? Do you have quick access to values of viscosity of water?		
	What are the various equations for Reynolds number?		
	What are the units of density and specific volume? Can you use your handbook to access to the values of density and specific volume of water at different temperatures and pressures?		
	What are the units of specific gravity? Can you use your handbook to access the values of specific gravity of typical fluids?		
	What are the units of heat capacity? Can you calculate the energy transfer due to a fluid's temperature change?		
B	**Fluid statics**		
	Can you calculate the pressure at any point within a piping system or within a vessel? Can you calculate the pressure due to the fluid above a point?		
	Can you solve manometer type problems?		
	Can you find forces on submerged areas?		
	Can you calculate buoyancy forces?		
C	**Energy, impulse, and momentum**		
	Can you create an energy balance equation for fluids (kinetic, pressure, elevation)?		
	Can you calculate impulse and momentum on a fluid?		
	Can you calculate the force (x & y) upon an elbow with the impulse-momentum equations?		
D	**Internal flow**		
	Can you calculate Reynolds number? Are you familiar with finding the properties of fluids in the FE Handbook? Can you determine whether or not a fluid flow is laminar or turbulent?		
	Can you find the hydraulic diameter?		
	Can you use Bernoulli's equation to calculate the pressure at any point in a piping system? Can you calculate velocity pressure, pressure due to elevation and static pressure?		
E	**External flow**		

	Can you use the Manning equation? Can you calculate open channel flow?		
F	**Incompressible flow**		
	Can you calculate friction loss with the Darcy Weisbach equation? Can you determine the friction factor with the Moody Diagram? Can you calculate the relative roughness?		
	Are you familiar with finding piping properties like inner diameter and roughness in the FE Handbook?		
	Can you use the Hazen Williams equation to calculate friction loss?		
	Can you use Charles' Law to calculate hydraulic power for pressurized fluids?		
G	**Compressible flow**		
	Can you calculate the Mach Number?		
	Can you calculate the bulk modulus?		
	Can you use the conservation of energy equations to solve for pressure, velocity for fluids flowing through diffusers, nozzles and converging-diverging nozzles?		
H	**Power and efficiency**		
	Can you calculate power from fluid flow and pressure?		
	Can you convert mechanical fluid power to required pump/fan power with efficiency?		
I	**Performance curves**		
	Can you interpret pump and fan curves?		
	Can you find the operating point of a pump/fan given its performance curve and the system curve?		
	Are you familiar with the relationship with the relationship between pressure and flow rate in a system curve?		
J	**Scaling laws for fans, pumps and compressors**		
	Can you calculate the change in flow rate, pressure or power based on a change in speed or impeller diameter for fans, pumps and compressors?		
11	**Thermodynamics**		
A	**Properties of ideal gases and pure substances**		
	What is enthalpy? What are the units of enthalpy? What is the enthalpy of liquid? Enthalpy of vapor? Enthalpy of evaporation?		
	What is the relationship between temperature and pressure? How does the boiling temperature of a liquid relate to pressure?		
	What is entropy? What are the units of entropy?		
	What is specific volume? What are its units? How does it relate to density?		

	What is quality? What is the quality at the saturated liquid curve? What is the quality at the saturated vapor curve?		
B	**Energy transfers**		
	Can you conduct energy balances on various energy transfers like those in boilers, condensers, evaporators, heat exchangers, etc.?		
C	**Laws of thermodynamics**		
	Are you comfortable with the 1st law of thermodynamics?		
	Are you comfortable with the 2nd law of thermodynamics?		
	Are you comfortable with the 3rd law of thermodynamics?		
D	**Processes**		
	What does throttling mean? What processes involve throttling? What typically remains constant in a throttling process?		
	What does isentropic mean? What processes typically are isentropic?		
	What does isobaric mean? What processes typically are isobaric?		
E	**Performance of components**		
	Are you comfortable calculating energy balances and mass balances on boilers, condensers, evaporators, turbines, heat exchangers, expansion valves and combustion chambers?		
F	**Power cycles, thermal efficiency and enhancements**		
	Are you comfortable with the Brayton cycle? Compressor, combustion chamber, turbine, exhaust? Can you calculate the efficiency of the system? Can you calculate the effect of heat recovery systems? Can you compare ideal to actual movements in the cycle due to inefficiencies?		
	Are you comfortable with the Rankine cycle? Pump, boiler, turbine, condenser? Can you calculate the efficiency of the system? Can you calculate the effect of heat recovery systems? Can you compare ideal to actual movements in the cycle due to inefficiencies?		
	Can you navigate the steam tables to find the properties of steam throughout the Rankine cycle?		
G	**Refrigeration and heat pump cycles and coefficients of performance**		
	Are you comfortable with the vapor compression cycle? Evaporator, compressor, condenser and expansion valve? Can you calculate the efficiency of the system? Can you compare ideal to actual movements in the cycle due to inefficiencies?		
	Can you navigate a refrigeration pressure enthalpy diagram?		
H	**Non-reacting mixture of gases**		
	Can you calculate final conditions of a gas mixture? Assume no phase change.		
I	**Psychrometrics**		

	Do you understand the difference between Dry Bulb and Wet Bulb temperatures? How are the two different temperatures measured?		
	Do you understand the difference between Relative and Specific Humidity? Relative humidity is in comparison to what reference point? What is the difference between Specific Humidity and Humidity ratio? What does it mean when relative humidity is equal to 100 %(relation to dry bulb/wet bulb temperatures)? What does relative humidity equal for dry air?		
	What are the units of specific volume? How is specific volume related to density? What is the standard specific volume and density used in typical problems?		
	What values are needed to calculate sensible heat? What does sensible heat indicate? What is the formula for calculating sensible heat (quick and easy version)? Sensible heat gain/loss is indicated by what movement on the psychrometric chart. What HVAC equipment affects Sensible Heat only?		
	What values are needed to calculate latent heat? What does latent heat indicate? What is the formula for calculating latent heat (quick and easy version)? Latent heat gain/loss is indicated by what movement on the psychrometric chart. . What HVAC equipment affects Latent Heat only?		
	What is the formula for calculating enthalpy (quick and easy version)? How is enthalpy related to sensible and latent heat? What is the		
	What is the difference between dew point and wet bulb temperature? When does the dew point of an air mixture equal the wet bulb temperature?		
	What is the saturation curve? What is relative humidity equal to at the saturation curve?		
	What is the latent heat of vaporization of water [US units]? What is the specific heat capacity of air [US units]?		
	What is the sensible heat ratio? What does the sensible heat ratio compare?		
J	**Heating, ventilating and air-conditioning (HVAC) processes**		
	Given two points on a psychrometric chart and the CFM, can you find the sensible heat gain/loss? Given a single point, CFM and the sensible heat gain/loss, can you find the final point?		
	Given two points on a psychrometric chart and the CFM, can you find the latent heat gain/loss? Given a single point, CFM and the latent heat gain/loss, can you find the final point?		
	Given two points on a psychrometric chart and the CFM, can you find the enthalpy gain/loss? Given a single point, CFM, sensible heat ratio and the change in enthalpy, can you find the final point?		
	Can you determine the final points of two different air mixtures?		
K	**Combustion and combustion products**		

	Can you use the combustion chemical reactions to calculate theoretical air to fuel ratios? Are you familiar with the chemical reactions in the FE Handbook?		
	Can you balance chemical reactions?		
	Can you calculate percent excess air?		
	Are you familiar with the HHV and LHV values in the FE Handbook? Can you calculate the heat released from various fuels?		
12	**Heat Transfer**		
A	**Conduction**		
	Do you understand conduction and the equations governing conduction? Can you distinguish between the three modes of heat transfer?		
	What is the U-Factor? What are the units of the U-factor? Can you convert between U-Factor, R-Value and k-Value?		
	What is the R-value? What are the units of the R-Value? Can you convert between U-Factor, R-Value and k-Value?		
	What is the k-value? What are the units of the k-value? Can you convert between U-Factor, R-Value and k-Value?		
	What is the overall heat transfer coefficient? Can you determine the overall heat transfer coefficient of a wall or roof assembly?		
B	**Convection**		
	Do you understand convection and the equations governing convection? Can you distinguish between the three modes of heat transfer?		
	Can you solve for the convective heat transfer coefficient for various situations?		
C	**Radiation**		
	Do you understand radiation and the equations governing radiation? Can you distinguish between the three modes of heat transfer?		
	What is emissivity? Can you use emissivity and solve for heat transfer due to radiation?		
	What is the Boltzmann's constant? Can you use it to solve for heat transfer due to radiation?		
D	**Thermal resistance**		
	Can you navigate the FE Handbook to find the thermal resistance properties of various materials?		
E	**Transient processes**		
	Can you calculate the Biot number?		
F	**Heat exchangers**		

	Can you solve for LMTD for parallel and counterflow heat exchangers?		
	Can you conduct an energy balance on a heat exchanger to solve for various properties of the hot and cold fluids?		
G	**Boiling and condensation**		
	What is a boiler? How do you determine the capacity of a boiler? How do you determine the efficiency of a boiler?		
	Can you conduct an energy balance on a steam-to-water or a steam-to-air coil?		
	What is the relationship between temperature and pressure? How does the boiling temperature of a liquid relate to pressure?		
	What is enthalpy? What are the units of enthalpy? What is the enthalpy of liquid? Enthalpy of vapor? Enthalpy of evaporation?		
	What is quality? What is the quality at the saturated liquid curve? What is the quality at the saturated vapor curve?		
	Are you comfortable navigating the steam tables?		
	What is quality? What is the quality at the saturated liquid curve? What is the quality at the saturated vapor curve?		
	What is super heat? How does it relate to the saturated vapor point?		
	What is sub-cooling? How does it relate to the saturated liquid point?		
13	**Measurements, Instrumentation and Controls**		
A	**Sensors**		
	Are you able to use the strain gauge equations in the FE Handbook?		
B	**Block diagrams**		
	Can you combine functions in series and in parallel?		
	Can you combine functions with a feedback loop?		
C	**System response**		
	Can you determine the system response via the Routh Test?		
	Can you calculate steady state error for various system types in response to various inputs (step, ramp, parabolic)?		
D	**Measurement uncertainty**		
	Can you determine measurement uncertainty via propagation of error?		
14	**Mechanical Design and Analysis**		

A	**Stress analysis of machine elements**		
	None		
B	**Failure theories and analysis**		
	Can you calculate factor of safety? Can you find the ultimate strength of various materials with your FE Handbook?		
	Can you use Mohr Theory for brittle materials and ductile materials?		
	Can you compare maximum normal stress to ultimate strength to determine failure or safety factor?		
	Are you comfortable using Von Mises theory for complicated stresses?		
C	**Deformation and stiffness**		
	Previously discussed		
D	**Springs**		
	Are you familiar with the various types of springs?		
	Can you calculate the number of active coils for various types of springs?		
	Can you determine the k value for various types of springs with various properties like pitch, active coils, diameter, etc.?		
	Can you calculate the deflection of spring in response to a force and vice versa?		
	Are you comfortable performing a fatigue analysis on a spring and other components that are subject to forces/stresses that occur at certain frequencies?		
	Can you use the Goodman fatigue criteria?		
E	**Pressure vessels**		
	Are you comfortable distinguishing between hoop, longitudinal and radial stresses?		
	Can you determine whether a pressure vessel is thin or thick?		
	Can you calculate the various stresses for thin or thick walled pressure vessels?		
F	**Bearings**		
	Are you familiar with the relationship between the life of a bearing and the load on a bearing?		
G	**Power screws**		
	Are you comfortable calculating the lowering and lifting torque for various power screws? Can you calculate the locking condition too?		
	Can you calculate the torque and power for the power screws based on the lifting and lowering torques?		
H	**Power transmission**		
	Can you use the gear ratio to calculate the rotational speeds and torques for various gear arrangements?		

	Can you calculate the transmission of force, torque and power for the various gear arrangements?		
	Can you analyze the shafts connected to the gears? Can you determine the stresses upon the shafts for the various situations? Can you use the Von Mises theory to determine if failure will occur?		
I	**Joining methods**		
	Are you familiar with welding, brazing, bolts, screws as joining methods?		
	Can you calculate clamping forces for bolts and screws?		
	Can you calculate the forces and moments for simple bolt arrangements?		
J	**Manufacturability**		
	Are you comfortable using tolerances to determine fits? Can you use the hole and shaft basis?		
	Are you familiar with geometric dimensioning and tolerancing?		
K	**Quality and reliability**		
	Can you use the control charts/tables in the FE Handbook to determine UCL and LCL?		
L	**Hydraulic components**		
	Can you determine friction loss in a hydraulic system, see fluids section too?		
	Can you calculate hydraulic pump work? Hydraulic pump power? Hydraulic pump efficiency?		
	Can you convert between fluid pressure and force?		
L	**Pneumatic components**		
	Can you convert between ACFM and SCFM?		
	Can you calculate compressor work? Compressor power? Compressor efficiency?		
	Can you convert between air pressure and force?		
L	**Electromechanical components**		
	Previously discussed		
M	**Geometric Dimensioning & Tolerancing**		
	Can you interpret geometric tolerancing? Can you interpret the various geometric symbols, tolerance zones, datums and material condition? Can you interpret a feature control frame?		

16 – Full Exam

MECHANICAL FE TEXTBOOK

www.engproguides.com

1.0 FULL EXAM PROBLEMS

Total time for the 110 questions is 5 hours and 20 minutes.

-- START SESSION 1 –

1.1 PROBLEM 1 – MATHEMATICS

Convert the following value to polar form.

$$5 + \sqrt{-49}$$

(a) $8.6\angle 55°$

(b) $5 + 7\angle 50°$

(c) $8.6\angle -55°$

(d) $5\angle -55°$

1.2 PROBLEM 2 – MATHEMATICS

Find the product of the following two vectors, a & b.

$$a = 5 + 6i;\ \ b = \sqrt{2} - 7i;$$

(a) $56\angle 28°$

(b) $49\angle 14°$

(c) $56\angle -28°$

(d) $49\angle -14°$

1.3 PROBLEM 3 – MATHEMATICS

Find the product of the following two vectors, a & b.

$$a = 5 + 6i; \;\; b = \sqrt{2} - 7i;$$

(a) $7 - 42i$

(b) $56\angle 28°$

(c) $7 + 42i$

(d) $49 - 26.5i$

1.4 PROBLEM 4 - MATHEMATICS

Which of the following is not a solution to the below equation?

$$x^5 + 32 = 0$$

(a) -0.62 + 1.9i

(b) -2

(c) 0.62 + 1.9i

(d) 1.9 + 0.62i

1.5 PROBLEM 5 – MATHEMATICS

The following a-b-c triangle has sides and an angle shown below. What is the length of the remaining side?

$$a = 20; B = 75°; c = 30;$$

(a) 23

(b) 27

(c) 31

(d) 990

1.6 PROBLEM 6 - MATHEMATICS

Find the inflection point of the following equation.

$$y = e^x - 5x^2$$

(a) (2.3, -16.5)

(b) (-2.3, -16.5)

(c) (16, 2)

(d) (2.3, 0)

1.7 PROBLEM 7 – MATHEMATICS

Find the partial derivative of the following function.

$$f(x,y) = 4\sin(3x - 7y)$$

$$\frac{\partial^2 f}{\partial x^2} = ?$$

(a) $4\cos(3x - 7y)$

(b) $-12\cos(3x - 7y)$

(c) $12\sin(3x - 7y)$

(d) $-36\sin(3x - 7y)$

1.8 PROBLEM 8 - MATHEMATICS

Find the equation of the plane that is parallel to the below plane and travels through the point (-2, 2, 2).

$$10 = 10i - 2j + 2k$$

(a) 5i + 2j + 2k = 10

(b) 10i -2j - 2k = 20

(c) 5i + j + k = 5

(d) -5i + j – k = 20

1.9 PROBLEM 9 – MATHEMATICS

Find the equation for the function, y.

$$y'' + 20y' = 0$$

$$y(0) = 5; y'(0) = 2$$

(a) $4.9 + 0.1e^{10x}$

(b) $5.1 - 0.1e^{-20x}$

(c) $5.1 + 0.1e^{20x}$

(d) $5.0 + 0.1e^{-2x}$

1.10 PROBLEM 10 – PROBABILITY & STATISTICS

A manufacturing process is shown to produce satisfactory products at an 80% satisfactory rate. What is the probability that only 7 out of 10 products will be satisfactory?

(a) 12%

(b) 20%

(c) 32%

(d) 70%

1.11 PROBLEM 11 - PROBABILITY & STATISTICS

A population has a mean of 25 and a standard deviation of 5. The population is assumed to follow a normal distribution. What is the probability that a sample will be greater than 34?

(a) 3.6%

(b) 8.1%

(c) 13.2%

(d) 96%

1.12 PROBLEM 12 - PROBABILITY & STATISTICS

What is the t-value for the following scenario? A sample group of 10 widgets is selected. The sample group has a mean of 110 and a standard deviation of 8. The population mean is estimated as 100.

(a) 0.9

(b) 3.1

(c) 4.0

(d) 14.3

1.13 PROBLEM 13 - PROBABILITY & STATISTICS

A fair, 6-sided die is rolled 9 times. What is the probability that a 4 is rolled only once?

(a) 0.15

(b) 0.35

(c) 0.66

(d) 0.81

1.14 PROBLEM 14 - PROBABILITY & STATISTICS

A fair, 6-sided die is rolled 8 times. What is the probability that an even number is rolled less than 4 times?

(a) 0.03

(b) 0.10

(c) 0.22

(d) 0.36

1.15 PROBLEM 15 – ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly NOT true regarding engineers issuing public statements?

(a) Engineers shall issue public statements only in an objective and truthful manner.

(b) Engineers may always express publicly technical opinions.

(c) Engineers may always issue criticisms on technical matters that are paid for by interested parties.

(d) Engineers shall issue public statements along with any pertinent information and the date indicating when it was current.

1.16 PROBLEM 16 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly true regarding public safety?

(a) Engineers may take assignments when not qualified by education or experience, as long as they disclose this information to their client.

(b) Engineers may affix their stamp to plans that have not been prepared under their direction or control.

(c) Engineers must not take assignments outside of their qualifications.

(d) Engineers may coordinate and assume responsibility for other engineers' work as long as the engineers are qualified.

1.17 PROBLEM 17 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly false regarding professional reputation?

(a) Engineers in private practice can review the work of another engineer for the same client, without informing the other engineer.

(b) Engineers working for the government are entitled to review and evaluate the work of other engineers when so required by their employment duties.

(c) Engineers in sales are entitled to make engineering comparisons of represented products with products of other suppliers.

(d) Engineers may conduct an engineering review of another engineer's work, if the engineers are working for different clients.

1.18 PROBLEM 18 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following would legally protect a design logo?

(a) Trademark

(b) Copyright

(c) Industry secret

(d) Patent

1.19 PROBLEM 19 – ENGINEERING ECONOMICS

A new transformer costs $5,000 and has an annual maintenance cost of $250. If the salvage value is $1000 at the end of its 20 year life, then what is the annual value of this transformer? Assume an interest rate of 5%.

(a) $620

(b) $250

(c) -$250

(d) -$620

1.20 PROBLEM 20 - ENGINEERING ECONOMICS

Background: A 200 HP pump operates for 1500 hours in the year. The motor is 90% efficient and the power factor is 0.88. Energy cost is $0.23 per kilowatt-hour and monthly demand cost is $8.90/KVA.

Problem: How much does it cost to operate the pump in one year?

(a) $73,000

(b) $75,000

(c) $77,000

(d) $85,000

1.21 PROBLEM 21 - ENGINEERING ECONOMICS

A new machine is installed in order to increase productivity. This new machine costs $160,000 and has an ongoing operating and maintenance cost of $500 per month. The new machine will save $2,000 per month and will have a salvage value of $15,000 after 10 years. If the interest rate is 4%, then what is the annual cost of the new machine?

(a) $500

(b) $6,500

(c) $11,500

(d) $19,000

1.22 PROBLEM 22- ENGINEERING ECONOMICS

A new energy efficient motor will replace an existing 480 V, 3 PH, 60 HZ, 10 HP motor. The old motor had an efficiency of 90% and the new motor has an efficiency of 95%. All other properties are assumed to be the same. The horsepower value is the output power. Assume electricity costs are $0.25 per kWh and the motor runs 24 hours a day and 7 days a week. The motor runs constantly at a load of 9 HP. What is the yearly cost savings achieved by switching to the new motor?

(a) $860 per year

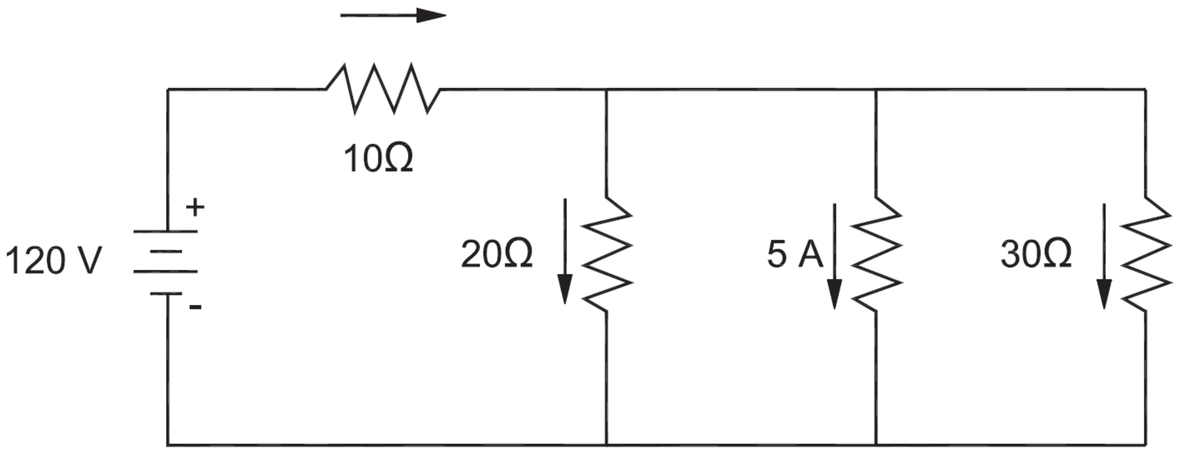
(b) $1,450 per year

(c) $3,210 per year

(d) $63,000 per year

1.23 PROBLEM 23 – ELECTRICITY & MAGNETISM

What is the current through the 10 ohm resistor?



(a) 4.1 A

(b) 5.5 A

(c) 8.2 A

(d) 10.9 A

1.24 PROBLEM 24 - ELECTRICITY & MAGNETISM

What is the combined impedance due to a resistor and capacitor in series? Assume a 120 V, 60 Hz system. The resistor has a resistance of 20 ohms and the capacitor has a capacitance of 200 μf.

(a) 7.3 Ω

(b) 20 - 13.3j Ω

(c) 20 + 13.3j Ω

(d) 33.3 Ω

1.25 PROBLEM 25 - ELECTRICITY & MAGNETISM

An inductor has an inductance of 100 mH at a frequency of 60 Hz. What is the impedance of the inductor?

(a) 4 ohms

(b) 38 ohms

(c) 47 ohms

(d) 107 ohms

1.26 PROBLEM 26 - ELECTRICITY & MAGNETISM

A 3 phase, 6 pole, synchronous generator must produce 50 Hz power. What speed should the generator be operated?

(a) 100 RPM

(b) 750 RPM

(c) 1,000 RPM

(d) 2,000 RPM

1.27 PROBLEM 27 - ELECTRICITY & MAGNETISM

Two wires of the same material have the following lengths and diameters.

$$Wire\ A \rightarrow Length = L; Diameter = d;$$

$$Wire\ B \rightarrow Length = 2L; Diameter = 0.5d;$$

Which of the following statements is most nearly true?

(a) The resistivity of wire A is twice the resistance of wire B.

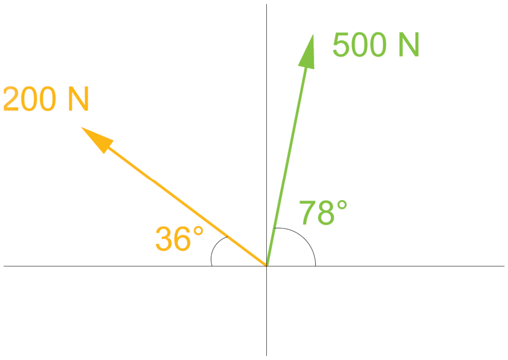
(b) The resistance of wire B is twice the resistance of wire A.

(c) The resistance of wire B is four times the resistance of wire A.

(d) The resistance of wire B is eight times the resistance of wire A.

1.28 PROBLEM 28 – STATICS

Given the two force vectors shown in the vector below, find the x and y components of the combined force vector.



(a) F_x = -58 N; F_y = 607 N;

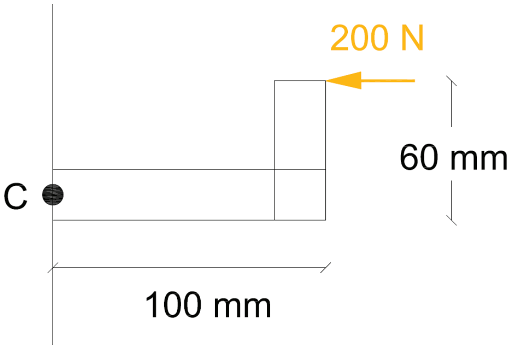
(b) F_x = -158 N; F_y = 75 N;

(c) F_x = -256 N; F_y = 444 N;

(d) F_x = -781 N; F_y = 856 N;

1.29 PROBLEM 29 – STATICS

What is the moment about point C, due to the force shown? Neglect the weight of the member.



(a) 12 N-m

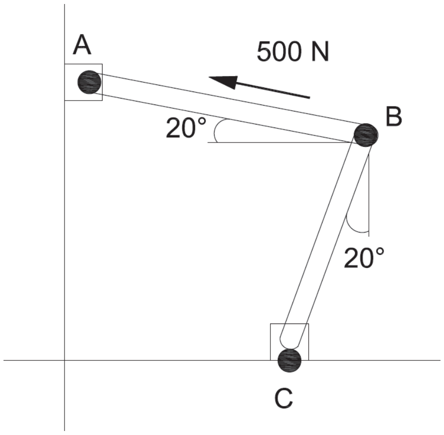
(b) 18 N-m

(c) 24 N-m

(d) 32 N-m

1.30 PROBLEM 30 – STATICS

Member BC = 23 mm. What is the moment about point C? Assume the two members are static.



(a) 1.9 N-m

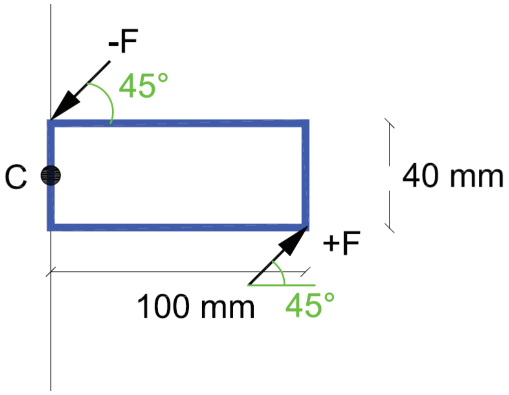
(b) 11.5 N-m

(c) 21.3 N-m

(d) 45.1 N-m

1.31 PROBLEM 31 – STATICS

One couple acts on the beam below. Assume F = 100 N, what is the couple moment about the center of the beam?



(a) 9.9 N-m

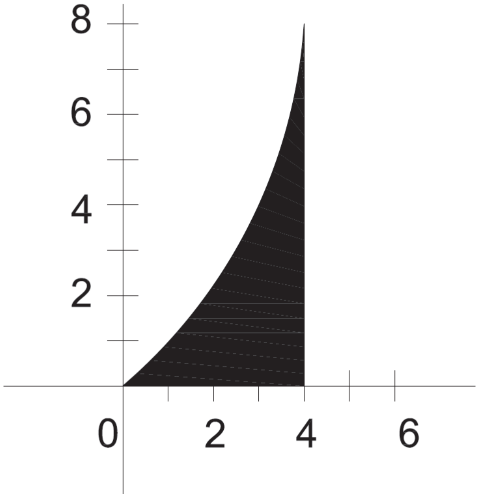
(b) 17.3 N-m

(c) 25.5 N-m

(d) 41.9 N=m

1.32 PROBLEM 32 – STATICS

Find the centroid of the following figure. $F(x) = (1/2)x^2$



(a) [2, 5.6]

(b) [2, 2.4]

(c) [3, 2.4]

(d) [3, 5.6]

1.33 PROBLEM 33 – STATICS

It is found that the minimum horizontal force required to move a 0.5 kg from rest is 2.5 N. What is the coefficient of static friction? Assume a flat, horizontal surface.

(a) 0.51

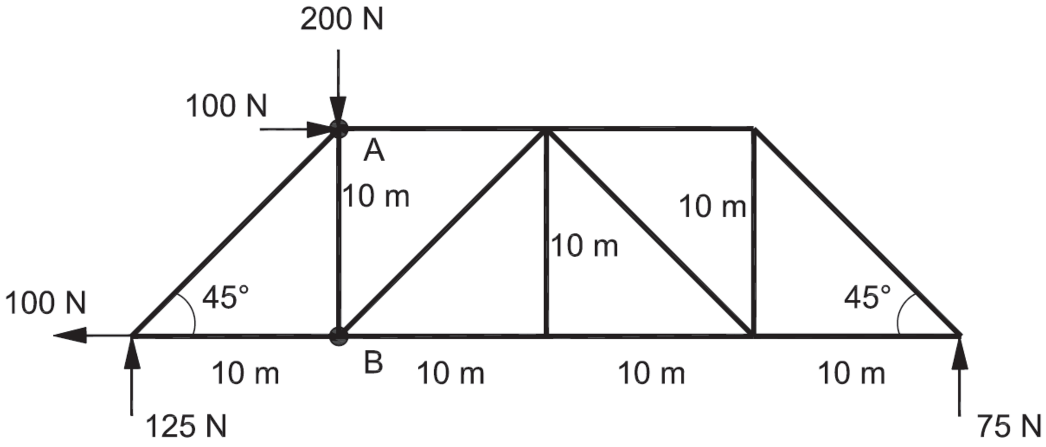
(b) 0.96

(c) 2.26

(d) 5.00

1.34 PROBLEM 34 – STATICS

What is the force in member A-B?


200 N
100 N
A
10 m
10 m
10 m
100 N
45°
45°
B
10 m
10 m
10 m
10 m
125 N
75 N

(a) Unable to determine

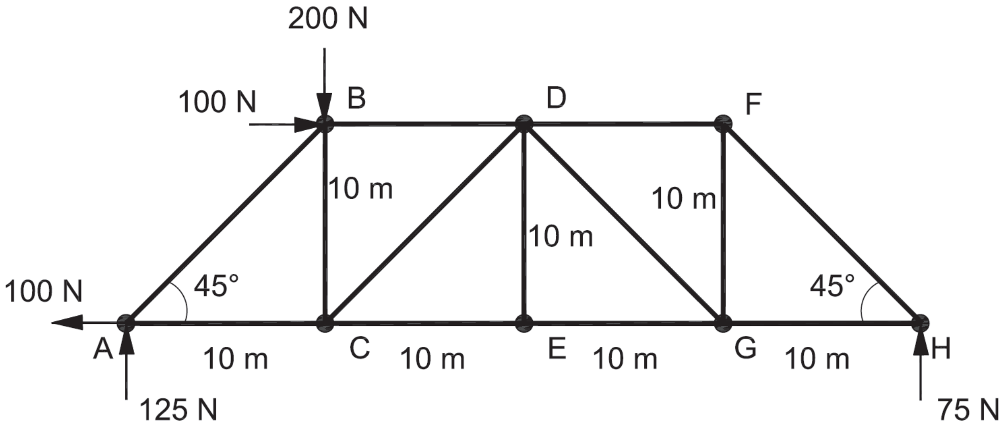
(b) 25 N

(c) 50 N

(d) 75 N

1.35 PROBLEM 35 – STATICS

What is the force in member A-C?


200 N
100 N
B
D
F
10 m
10 m
10 m
100 N
45°
45°
A
C
E
G
H
10 m
10 m
10 m
10 m
125 N
75 N

(a) Unable to determine

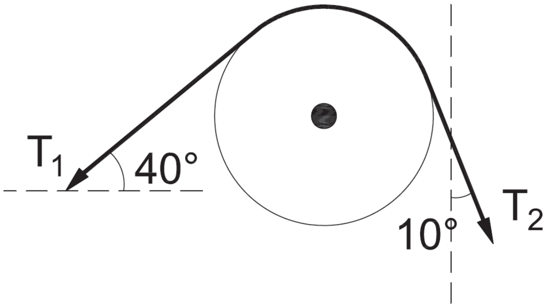
(b) 50 N

(c) 175 N

(d) 225 N

1.36 PROBLEM 36 – STATICS

The coefficient of static friction for the pulley is 0.2. The tension T1 is 100 N. What is the required tension T2 to overcome the static friction? Assume a weightless pulley.



(a) 115 N

(b) 123 N

(c) 135 N

(d) 152 N

1.37 PROBLEM 37 – DYNAMICS, KINEMATICS AND VIBRATIONS

The location of a particle as a function of time is described by the equation below. What is the location of the particle when the velocity is equal to zero? Time is units of seconds and distance is in units of meters.

$$x(t) = \frac{1}{3}t^3 - 4t^2 + 16t$$

(a) 21.3 m

(b) 43.1 m

(c) 88.2 m

(d) 105 m

1.38 PROBLEM 38 – DYNAMICS, KINEMATICS AND VIBRATIONS

The acceleration of a particle as a function of time is described by the equation below. What is the governing equation for velocity? V(0) = 5 m/s. Time is units of seconds and distance is in units of meters.

$$a(t) = 4t^2$$

(a) $v(t) = 8t + 5$

(b) $v(t) = 8t$

(c) $v(t) = (4/3)t^3$

(d) $v(t) = (4/3)t^3 + 5$

1.39 PROBLEM 39 – DYNAMICS, KINEMATICS AND VIBRATIONS

A motor is spinning at 1800 RPM. It is decelerated at a constant deceleration. The motor is at rest after 10 seconds. How many revolutions were completed in the 10 seconds?

(a) 150 revolutions

(b) 225 revolutions

(c) 300 revolutions

(d) 475 revolutions

1.40 PROBLEM 40 – DYNAMICS, KINEMATICS AND VIBRATIONS

A net force acts upon a 7.2 kg mass for 2 seconds. The net force varies in accordance with the following function. What is the velocity at t = 2 seconds. Assume the mass is initially at a speed of 1 m/s.

$$F(t) = 5 + t^2 \; [N]$$

(a) 1.76 m/s

(b) 2.76 m/s

(c) 3.76 m/s

(d) 5.66 m/s

1.41 PROBLEM 41 – DYNAMICS, KINEMATICS AND VIBRATIONS

A force of 10 N is applied to a mass of 5 kg for 3 seconds. A force of 10 N is applied to a mass of 50 kg for 3 seconds. Which of the masses has the largest momentum?

(a) 5 kg mass

(b) 50 kg mass

(c) Momentums are equal

(d) Unable to determine

1.42 PROBLEM 42 – DYNAMICS, KINEMATICS AND VIBRATIONS

A vibrating system is described as having a spring constant of 1,000 N/m and a mass of 50 kg. The viscous damping coefficient is 400 N-s/m. What is the damped frequency?

(a) 2 rad/s

(b) 4.5 rad/s

(c) 7.8 rad/s

(d) 10 rad/s

1.43 PROBLEM 43 – DYNAMICS, KINEMATICS AND VIBRATIONS

A system is described as having a spring constant of 1,000 N/m and a mass of 50 kg. The system is also viscously damped with a damping coefficient of 0.2. A sinusoidal force is applied to the system. What is the change in amplitude when the frequency is increased from the natural frequency to 2x the natural frequency?

(a) Increased by a factor of 8.0.

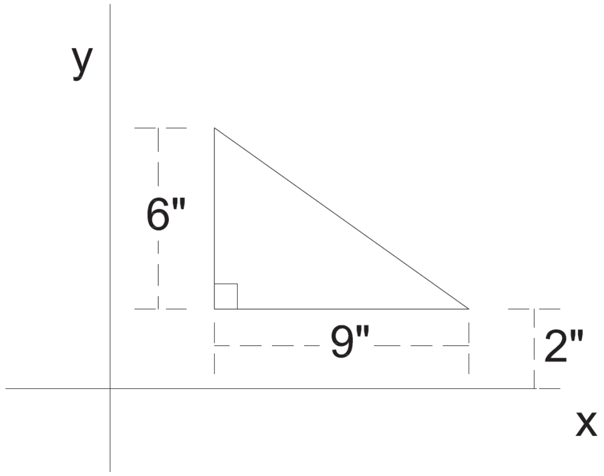
(b) Increased by a factor of 2.1.

(c) Reduced by a factor of 5.0.

(d) Reduced by a factor of 7.8.

1.44 PROBLEM 44 – DYNAMICS, KINEMATICS AND VIBRATIONS

Find the area moment of inertia of the figure below about the x-axis.



(a) 88 in^4

(b) 162 in^4

(c) 270 in^4

(d) 486 in^4

1.45 PROBLEM 45 – DYNAMICS, KINEMATICS AND VIBRATIONS

A 6" diameter steel shaft has a length of 24". What is the mass moment of inertia about the centerline along the length of the shaft?

(a) 900 lb-in^2

(b) 3,440 lb-in^2

(c) 4,100 lb-in^2

(d) 6,270 lb-in^2

1.46 PROBLEM 46 – DYNAMICS, KINEMATICS AND VIBRATIONS

A projectile is at an initial velocity of 10 m/s at an angle of 30 degrees above the horizontal. If the projectile is initially at an elevation of 45 m, then at what time will the projectile be at an elevation of 0 m?

(a) 3.6 s

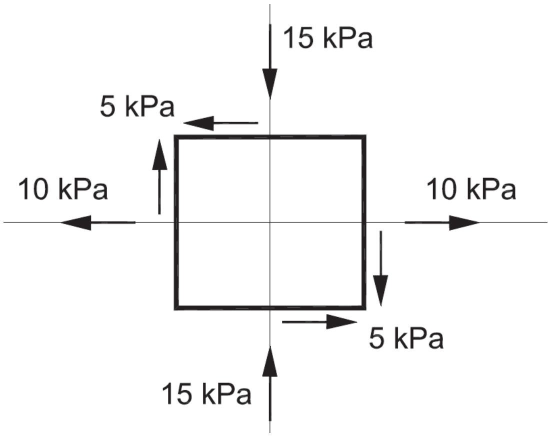
(b) 2.6 s

(c) 10.4 s

(d) 24.4 s

1.47 PROBLEM 47 – MECHANICS OF MATERIALS

What are the principal stresses?



(a) -2.5 & 2.5 kPa

(b) 6.5 & -11.5 kPa

(c) 10.5 & -15.5 kPa

(d) 2.5 & -11.5 kPa

1.48 PROBLEM 48 – MECHANICS OF MATERIALS

A steel rod has a diameter of 150mm and a length of 10m. The rod is secured and locked in place at both ends. The temperature varies by 50 °C. What is the maximum stress developed in the rod?

(a) 117 MPa

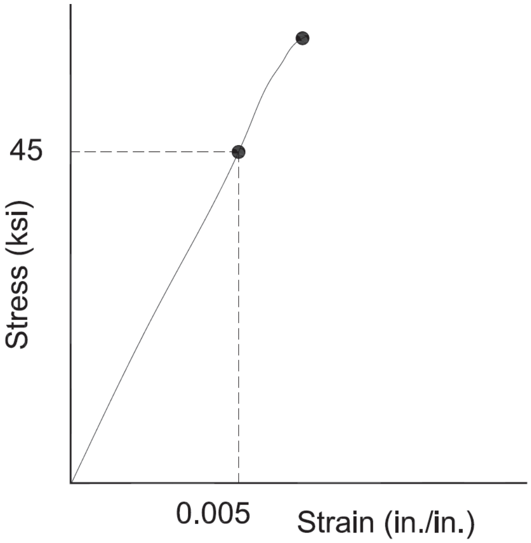
(b) 230 MPa

(c) 429 MPa

(d) 1,020 MPa

1.49 PROBLEM 49 – MECHANICS OF MATERIALS

A 10” long, 1” D shaft is put through a stress-strain test. The results are shown below. When a load of 10,000 lbf is applied to shaft in tension, the diameter is reduced by 0.1%. What is the change in length of the shaft?



(a) .001”

(b) .004”

(c) .009”

(d) .014”

1.50 PROBLEM 50 – MECHANICS OF MATERIALS

Which of the following rods will have the largest longitudinal change in length?

(a) 25 mm diameter, 1 m long steel rod, undergoing a 50 C change in temperature.

(b) 50 mm diameter, 1 m long aluminum rod, undergoing a 40 C change in temperature.

(c) 25 mm diameter, 1 m long copper rod, undergoing a 50 C change in temperature.

(d) 50 mm diameter, 1 m long bronze rod, undergoing a 40 C change in temperature.

1.51 PROBLEM 51 – MECHANICS OF MATERIALS

A steel pipe has an inner diameter of 50 mm and an outer diameter of 63 mm. The pipe is subject to a torque of 5,000 N-m. The pipe is 10 m long. One end is fixed. The other end is free. What is the shear stress?

(a) 150 MPa

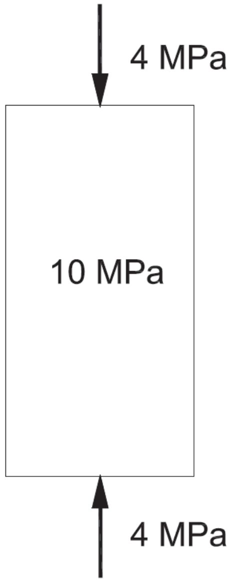
(b) 277 MPa

(c) 301 MPa

(d) 499 MPa

1.52 PROBLEM 52 – MECHANICS OF MATERIALS

A cylindrical pressure vessel has an internal pressure of 10 MPa. It is also subject to a compressive stress of 4 MPa. The pressure vessel has an inner diameter of 250 mm and a thickness of 5 mm. What is the stress in the up-down direction?



(a) 14 MPa

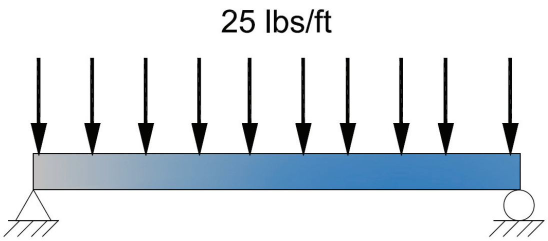
(b) 34 MPa

(c) 121 MPa

(d) 125 MPa

1.53 PROBLEM 53 – MECHANICS OF MATERIALS

What is the maximum stress in the following beam? The beam has a length of 25 feet. The beam is 6” wide by 6” thick. It is supported as shown in the figure below and is subject to the below force.



(a) 650 psi

(b) 1,200 psi

(c) 3,395 psi

(d) 10,240 psi

1.54 PROBLEM 54 – MECHANICS OF MATERIALS

Determine the critical buckling force of a steel column (fixed on both ends) that has the following properties.

$$L = 10m; I = 150\ cm^4$$

(a) 51 kN

(b) 95 kN

(c) 120 kN

(d) 442 kN

1.55 PROBLEM 55 – MECHANICS OF MATERIALS

Which of the following statements is true, regarding thin walled, cylindrical pressure vessels?

(a) Hoop stress is double longitudinal stress, so failure typically occurs on longitudinal joints of the cylinder.

(b) Hoop stress is one-half longitudinal stress, so failure typically occurs on longitudinal joints of the cylinder.

(c) Hoop stress is double longitudinal stress, so failure typically occurs on circumferential joints of the cylinder.

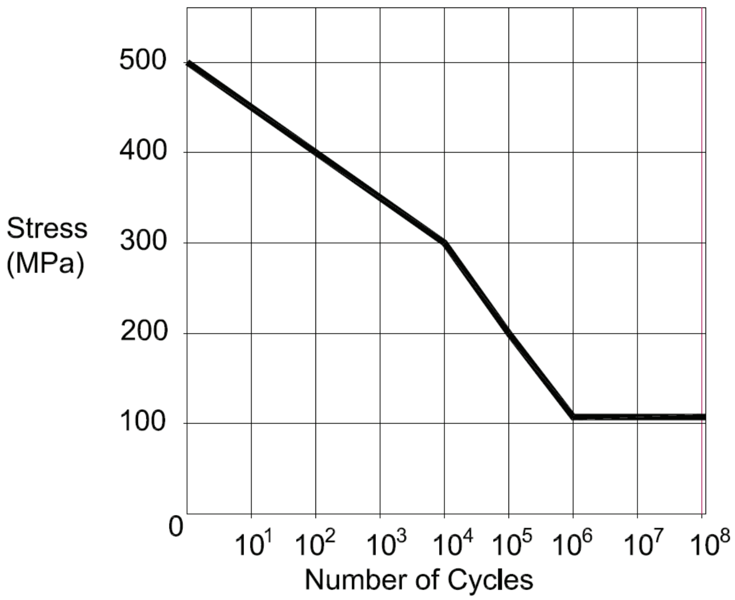
(d) Hoop stress is one-half longitudinal stress, so failure typically occurs on circumferential joints of the cylinder.

-- END SESSION 1 --

-- START SESSION 2 --

1.56 PROBLEM 56 – MATERIAL PROPERTIES & PROCESSING

Which of the following is most nearly the fatigue limit? Given the below fatigue failure graph.



(a) 500 MPa

(b) 300 MPa

(c) 200 MPa

(d) 100 MPa

1.57 PROBLEM 57 – MATERIAL PROPERTIES & PROCESSING

A material will undergo a fatigue test with a maximum stress of 50 MPa and a minimum stress of -80 MPa. What will be the stress amplitude?

(a) 130 MPa

(b) 80 MPa

(c) 65 MPa

(d) 50 MPa

1.58 PROBLEM 58 – MATERIAL PROPERTIES & PROCESSING

A 4340 steel is subject to an exterior crack of length 50 mm (Y = 1.1). What is the critical value of stress?

(a) 20 MPa

(b) 59 MPa

(c) 82 MPa

(d) 105 MPa

1.59 PROBLEM 59 – MATERIAL PROPERTIES & PROCESSING

Use the iron carbide diagram in the FE Reference Handbook to determine the weight of ferrite in a 250 kg austenite (1% carbon) that has been cooled down to 700 °C.

(a) 85 kg

(b) 107 kg

(c) 146 kg

(d) 213 kg

1.60 PROBLEM 60 – MATERIAL PROPERTIES & PROCESSING

Which of the following materials is the anode of the following chemical reactions?

$$Pb_{(s)} + SO_4^{2-} \rightarrow PbSO_{4\,(s)} + 2e^-$$

$$PbO_{2\,(s)} + 4H^+ + SO_4^{2-} + 2e^- \rightarrow PbSO_{4\,(s)} + 2H_2O$$

(a) Pb (s)

(b) SO_4

(c) $PbSO_4$

(d) PbO_2 (s)

1.61 PROBLEM 61 – MATERIAL PROPERTIES & PROCESSING

What is the electric potential of the following chemical reaction?

$$Ag + Fe^{3+} \rightarrow Fe^{2+} + Ag^+$$

(a) -0.09 V

(b) 0.09 V

(c) -1.5 V

(d) 1.5 V

1.62 PROBLEM 62 – MATERIAL PROPERTIES & PROCESSING

Which of the following statements is NOT true about creep failure?

(a) Creep is more of a concern at higher temperatures than lower temperatures.

(b) Creep failure occurs at a point slightly above yield strength.

(c) Creep is more of a concern under constant stress as opposed to cyclical stress.

(d) Creep is a concern in boilers and combustion engines.

1.63 PROBLEM 63 – MATERIAL PROPERTIES & PROCESSING

Which of the following metal combinations would corrode the fastest?

(a) Al [-1.662 V] & Cu [+0.337 V]

(b) Zn [-0.763 V] & Cu [+0.337 V]

(c) Iron [-0.440 V] & Copper [+0.337 V]

(d) Magnesium [-2.363] & Iron [-0.440 V]

1.64 PROBLEM 64 – FLUID MECHANICS

A fluid has is traveling at a velocity of 8 ft/sec through a 6" pipe. The fluid has a density of 60 lb/ft^3 and has a kinematic viscosity of 1.5 x 10^{-5} ft^2/sec. What is the Reynolds number?

(a) 270,000

(b) 540,000

(c) 1,115,000

(d) 3,200,000

1.65 PROBLEM 65 – FLUID MECHANICS

A duct with air flowing at 0.075 lb/ft3 has a pitot tube connected to a manometer. The manometer liquid has a density of 900 kg/m^3. The pressure differential is measured as a height of 4 centimeters. The pressure differential is equal to the velocity pressure of the air in the duct. What is the velocity of air in the duct?

(a) 27 m/s

(b) 41 m/s

(c) 57 m/s

(d) 81 m/s

1.66 PROBLEM 66 – FLUID MECHANICS

A Venturi meter has the following diameters, D_1 = 200 mm and D_2 = 100 mm. A manometer is connected between the two locations. The manometer has a delta reading of 100 mm. The density of the manometer liquid is 5,000 kg/m^3. What is the volumetric flow rate of water (density = 1,000 kg/m$^{3)}$ through the meter? C_v = 1.

(a) $1.056\,\frac{m^3}{s}$

(b) $0.194\,\frac{m^3}{s}$

(c) $0.059\,\frac{m^3}{s}$

(d) $0.0254\,\frac{m^3}{s}$

1.67 PROBLEM 67 – FLUID MECHANICS

A centrifugal, chilled water pump is rated at 300 GPM, 50 feet head, 78% efficiency, 1100 RPM. If the pump is run at 1700 RPM, then what will be the new flow rate and pressure ratings?

(a) 300 GPM & 120 FT HD

(b) 460 GPM & 120 FT HD

(c) 300 GPM & 50 FT HD

(d) 460 GPM & 80 FT HD

1.68 PROBLEM 68 – FLUID MECHANICS

A fan provides 2,000 m^3/h of air at a change in pressure of 500 Pa. The fan has an efficiency of 65%. What is the fan power? The air density is 1.2 kg/m^3.

(a) 55 W

(b) 122 W

(c) 209 W

(d) 430 W

1.69 PROBLEM 69 – FLUID MECHANICS

Which of the following statements is most nearly true about net positive suction head?

(a) Net positive suction head available must be less than net positive suction head required to avoid cavitation at a pump.

(b) Net positive suction head is equal to the suction pressure available.

(c) Net positive suction head is the pressure at the pump suction head that is greater than the vapor pressure of the fluid at the operating temperature.

(d) Both (a) & (c)

1.70 PROBLEM 70 – FLUID MECHANICS

An air compressor has an inlet pressure of 125 kPa. The air compressor has a pressure ratio of 5. The inlet temperature is 290 K. The isentropic efficiency of the compressor is 85%. What is the outlet temperature of the compressor? Air, k = 1.4.

(a) 290 K

(b) 340 K

(c) 460 K

(d) 490 K

1.71 PROBLEM 71 – FLUID MECHANICS

A fan is rated at 120 L/s at 450 Pa. The fan is used to transport air at 1.22 kg/m^3. What is the mass of air transported in an 8 hour period?

(a) 520 kg

(b) 1,970 kg

(c) 2,950 kg

(d) 4,200 kg

1.72 PROBLEM 72 – FLUID MECHANICS

Which of the following fluids is the most turbulent? Assume each fluid has the same hydraulic diameter and the same velocity.

(a) Air at 80 °F and 1 atm.

(b) Air at 100 °F and 1 atm.

(c) Water at 80 °F and 1 atm.

(d) Water at 100 °F and 1 atm.

1.73 PROBLEM 73 – FLUID MECHANICS

Which of the following factors has the least of an effect on the pressure drop due to friction in laminar flow in a pipe?

(a) Pipe roughness

(b) Fluid velocity

(c) Fluid dynamic viscosity

(d) Pipe diameter

1.74 PROBLEM 74 – FLUID MECHANICS

At extremely turbulent flow (Reynolds greater than 4 x 10^7), the pressure drop due to friction is independent of which term?

(a) Pipe roughness

(b) Pipe diameter

(c) Velocity

(d) Reynolds number

1.75 PROBLEM 75 – THERMODYNAMICS

A boiler & superheater is used to provide 100 kg/s of steam to a turbine. The condenser temperature is 50 C. The outlet of the superheater is at 500 C and 1 MPa. The isentropic efficiency of the turbine is 80%. The temperature entering the condenser is 50 C. What is the power generated by the turbine? Hint: This problem follows the Rankine Cycle.

(a) 80 MW

(b) 300 MW

(c) 350 MW

(d) 500 MW

1.76 PROBLEM 76 – THERMODYNAMICS

Water is heated from a saturated liquid at 145 C to a vapor quality of 50%. What is the work done by the water? Assume a constant pressure process and a mass of 1 kg of water.

(a) 92,500 J

(b) 138,750 J

(c) 225,000 J

(d) 341,000 J

1.77 PROBLEM 77 – THERMODYNAMICS

Two pounds of methane is burned with oxygen. Assume a perfectly balanced reaction. How many pounds of water are produced?

$$CH_4 + 2O_2 \rightarrow CO_2 + H_2O$$

(a) 1.5 lbs

(b) 2 lbs

(c) 2.25 lbs

(d) 56 lbs

1.78 PROBLEM 78 – THERMODYNAMICS

A refrigeration cycle uses 10 lbs/hr of R-134a. The suction pressure is at 60 PSIA and the discharge pressure is at 150 PSIA. What is the net refrigeration effect if the evaporator operates at constant pressure and there is no sub-cooling or super heating?

(a) 109 Btu/h

(b) 220 Btu/h

(c) 440 Btu/h

(d) 640 Btu/h

1.79 PROBLEM 79 – THERMODYNAMICS

What is the dew point of air at sea level and the following conditions, 88 F DB, 90% relative humidity?

(a) 77 F

(b) 80.2 F

(c) 84.7 F

(d) 88 F

1.80 PROBLEM 80 – THERMODYNAMICS

55 cubic meters per minute of air enters a cooling coil at 25 C DB and 70% relative humidity. It leaves the coil at 15 C DB and 100% relative humidity. What is the cooling provided by the cooling coil? Assume sea level.

(a) 20.4 KW

(b) 25.5 KW

(c) 34.9 KW

(d) 1200 KW

1.81 PROBLEM 81 – THERMODYNAMICS

Air enters an ideal compressor (isentropic) at 50 kPa and 45 °C. The air is compressed to 1/8th of its volume. What is the leaving temperature?

(a) 273 °C

(b) 308 °C

(c) 345 °C

(d) 458 °C

1.82 PROBLEM 82 – THERMODYNAMICS

What is the change in internal energy of 1.5 kg of argon, when it is heated from 20°C to 50 °C?

(a) 23.4 kJ

(b) 36.2 kJ

(c) 46.8 kJ

(d) 59.9 kJ

1.83 PROBLEM 83 – THERMODYNAMICS

2 cubic feet of methane is initially at 50 F and 14.7 PSIA. The gas is compressed to 1 cubic feet at a pressure of 40 PSIA. What is the change in internal energy of the gas? Assume an ideal gas and conservation of mass.

(a) 52 Btu/lbm

(b) 74 Btu/lbm

(c) 88 Btu/lbm

(d) 109 Btu/lbm

1.84 PROBLEM 84 – THERMODYNAMICS

One cubic meter of helium (45 °C & 450 kPa) is expanded to 2 cubic meters. Assume an ideal gas and no change in mass. Also no work is done and there is no change in heat. What is the new temperature when the helium is expanded?

(a) 200 K

(b) 300 K

(c) 400 K

(d) 500 K

1.85 PROBLEM 85 – THERMODYNAMICS

1 kW of heat is added to 45 kg of air for 1 hour. Air is initially at a temperature of 30 °C. What is the final temperature after 1 hour?

(a) 75 °C

(b) 92 °C

(c) 101 °C

(d) 110 °C

1.86 PROBLEM 86 – HEAT TRANSFER

Which term is not applicable to calculating the convection coefficient of a forced convection fluid, regardless of the fluid viscosity and turbulence?

(a) Nusselt Number

(b) Reynold's Number

(c) Grashof Number

(d) Prandtl Number

1.87 PROBLEM 87 – HEAT TRANSFER

Hot water enters a counterflow, plate and frame heat exchanger at 140 F at a rate of 50 gpm. Cold water is heated from 70F to 110F at a rate of 20 gpm. Given that the NTU is 1.4, what is the effectiveness of the heat exchanger?

(a) 0.6

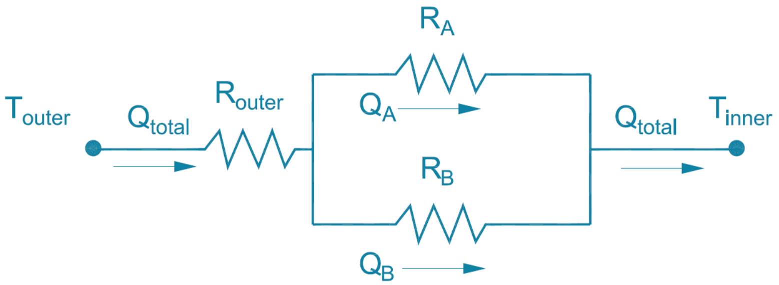
(b) 0.7

(c) 0.8

(d) 0.9

1.88 PROBLEM 88 – HEAT TRANSFER

A composite wall consists of an outer layer with a conductivity of 100 W/m-°C and an inner layer. The inner layer is comprised of an area that is 75% material A (k = 25 W/m-°C) and 25% material B (k = 45 W/m-°C). The inner layer has a thickness of 7 cm and the outer layer has a thickness of 10 cm. What is the equivalent heat transfer coefficient of the wall? Assume a difference in temperature of 45 °C and a surface area of 50 m^2.



(a) 79 W/m^2-°C

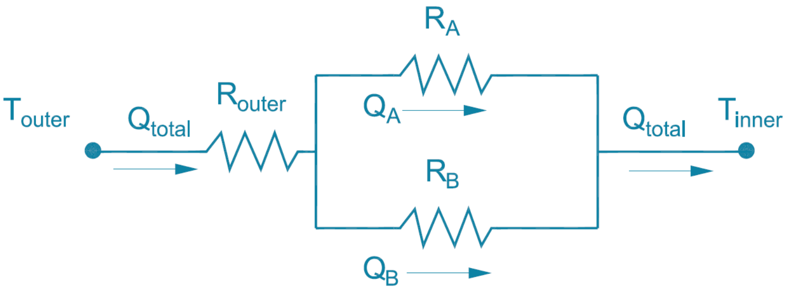
(b) 425 W/m^2-°C

(c) 145 W/m^2-°C

(d) 298 W/m^2-°C

1.89 PROBLEM 89 – HEAT TRANSFER

A composite wall consists of an outer layer with a conductivity of 55 W/m-°C and an inner layer. The inner layer is comprised of an area that is 80% material A (k = 125 W/m-°C) and 20% material B (k = 175 W/m-°C). The inner layer has a thickness of 10 cm and the outer layer has a thickness of 15 cm. What is the total heat transfer through the wall? Assume a difference in temperature of 45 °C and a surface area of 50 m^2.



(a) 50,000 W

(b) 300,000 W

(c) 650,000 W

(d) 1,520,000 W

1.90 PROBLEM 90 – HEAT TRANSFER

Which of the following statements is true about the Biot number?

(a) A larger Biot number indicates greater conductive heat transfer than convective heat transfer from a body.

(b) A larger Biot number indicates a greater conductive heat transfer resistance than convective heat transfer resistance for a body.

(c) A smaller Biot number indicates a lesser conductive heat transfer than convective heat transfer from a body.

(d) A smaller Biot number indicates a greater conductive heat transfer resistance than convective heat transfer resistance for a body.

1.91 PROBLEM 91 – HEAT TRANSFER

A material is initially at a temperature of 500 °C. The ambient air is 20 °C. The properties of the material are shown below. How long will it take the material to reach 40 °C?

$$\beta = 0.009;$$

(a) 350 seconds

(b) 1930 seconds

(c) 3230 seconds

(d) 7560 seconds

1.92 PROBLEM 92 – HEAT TRANSFER

Two parallel flat plates of equal area are separated by 50 mm. One flat plate has an emissivity of 0.2 and is at a temperature of 400 K. The other flat plate has an emissivity of 0.03 and is at a temperature of 1,000 K. The shape factor is equal to 1. What is the radiative heat transfer between the two plates? The plates are assumed to form an enclosure.

(a) 1,500 W/m^2

(b) 2,900 W/m^2

(c) 4,200 W/m^2

(d) 10,700 W/m^2

1.93 PROBLEM 93 – HEAT TRANSFER

Two surfaces of equal area are separated by 100 mm. One surface is at a temperature of 750 K and the other surface is at 250 K. Assume the surfaces are black bodies. The shape factor between the two surfaces is 0.3. Assume the surfaces are part of an enclosure, then what is the radiative heat transfer between the two surfaces?

(a) 800 W/m^2

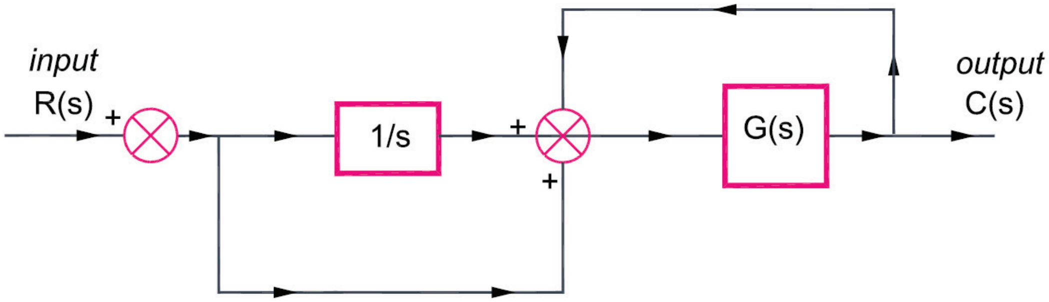
(b) 5,300 W/m^2

(c) 17,720 W/m^2

(d) 20,640 W/m^2

1.94 PROBLEM 94 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

Simplify the following control diagram and find the governing equation for the output.



$(a)\ C(s) = R(s)\left(\frac{1}{s}\right)\left(G(s)\right)$

$(b)\ C(s) = R(s)\left(\frac{1}{s}\right)\left(\frac{G(s)}{1+G(s)}\right)$

$(c)\ C(s) = R(s)\left(\frac{1}{s}+1\right)\left(\frac{G(s)}{1+G(s)}\right)$

$(d)\ C(s) = R(s)(s+1)\left(\frac{1}{1+G(s)}\right)$

1.95 PROBLEM 95 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

A thermocouple has a reference temperature of 100 °F. Which of the following statements about this thermocouple is true?

(a) An increase in temperature causes a direct decrease in the resistance of the thermocouple output.

(b) A decrease in temperature causes a direct increase in the resistance of the thermocouple output.

(c) A temperature of 100 F will cause the thermocouple to measure the greatest voltage.

(d) A change in temperature causes a change in the thermocouple voltage output.

1.96 PROBLEM 96 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

A strain gauge has an initial resistance of 300 ohms. The strain gauge factor is 2.8. A load is applied and the strain gauge now measures 299.771 ohms. What is the strain measured?

(a) 8.847×10^{-4}

(b) 2.726×10^{-4}

(c) 6.412×10^{-5}

(d) 1.989×10^{-5}

1.97 PROBLEM 97 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

Platinum is used as a resistance temperature detector. The nominal resistance of the detector is 250 ohms. The temperature changes from 0 C to 130 C. What is the percent change in resistance?

(a) 50%

(b) 75%

(c) 125%

(d) 150%

1.98 PROBLEM 98 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

Which of the following statements is true about a thermistor with a negative temperature coefficient?

(a) A thermistor reacts to an increase in temperature with an increase in voltage.

(b) A thermistor reacts to an increase in temperature with an increase in resistance.

(c) A thermistor reacts to an increase in temperature with a decrease in voltage.

(d) A thermistor reacts to an increase in temperature with a decrease in resistance.

1.99 PROBLEM 99 – MEASUREMENT, INSTRUMENTATION AND CONTROLS

Which of the following sensors most likely uses the piezoelectric effect?

(a) Temperature

(b) Humidity

(c) Vibration

(d) Enthalpy

1.100 PROBLEM 100 – MECHANICAL DESIGN AND ANALYSIS

There are four springs that support a piece of equipment that weighs 50 kg. Two of the springs have a spring constant of 20 N/m and the other two springs have a spring constant of 40 N/m. What is the vibration frequency of the four springs in parallel?

(a) 0.25 Hz

(b) 4 Hz

(c) 22 Hz

(d) 83 Hz

1.101 PROBLEM 101 – MECHANICAL DESIGN AND ANALYSIS

Given the following statistical information, determine the UCL and LCL on the X-bar chart.

$$\bar{R} = 2.7; \;\; \bar{\bar{X}} = 88.9; \;\; n = 6$$

(a) 88.9 & 88.9

(b) 90.2 & 87.6

(c) 104.7 & 78.1

(d) 122.2 & 62.7

1.102 PROBLEM 102 – MECHANICAL DESIGN AND ANALYSIS

Which of the following best describes a maximum material condition (MMC) or least material condition (LMC)?

(a) MMC is the largest shaft size fitting into smallest hole size.

(b) LMC is the largest shaft size fitting into smallest hole size.

(c) MMC is the largest shaft size fitting into largest hole size.

(d) LMC is the smallest shaft size fitting into smallest hole size.

1.103 PROBLEM 103 – MECHANICAL DESIGN AND ANALYSIS

A centrifugal fan is rated at 2,000 CFM at 2 in wg at a speed of 1190 RPM. If the fan speed is increased to 1,500 RPM, then what will be the new flow rate?

(a) 1,500 CFM

(b) 2,000 CFM

(c) 2,500 CFM

(d) 3,000 CFM

1.104 PROBLEM 104 – MECHANICAL DESIGN AND ANALYSIS

A helical torsion spring has a total number of 20 coils and a young's modulus is 100 GPa. The design load is 100 pounds and the spring has a mean diameter of 100 mm and a wire diameter of 5 mm. What is the spring constant?

(a) 0.5 N-m/rad

(b) 1.9 N-m/rad

(c) 20.7 N-m/rad

(d) 100.4 N-m/rad

1.105 PROBLEM 105 – MECHANICAL DESIGN AND ANALYSIS

A roller bearing is designed for a load of 100 N at a speed of 100 RPM for a life of 2000 hours. If the load changes to 150 N, then what will be the new life of the bearing?

(a) 517 hours

(b) 799 hours

(c) 1823 hours

(d) 8756 hours

1.106 PROBLEM 106 – MECHANICAL DESIGN AND ANALYSIS

An open air tank at sea level feeds a centrifugal pump. The pump's centerline is located 25 feet below the water level of the tank. There is a friction loss of 10 feet head from the tank to the pump suction. The vapor pressure is 0.7 psia. What is the net positive suction head available at the pump suction?

(a) 8 psi

(b) 12 psi

(c) 17 psi

(d) 21 psi

1.107 PROBLEM 107 – MECHANICAL DESIGN AND ANALYSIS

Which of the following terms is used to characterize compressibility?

(a) Young's Modulus

(b) Shear Modulus

(c) Bulk Modulus

(d) Rupture Modulus

1.108 PROBLEM 108 – MECHANICAL DESIGN AND ANALYSIS

Which of the following best describes the point at which a material will experience permanent deformation?

(a) Ultimate tensile strength

(b) Yield strength

(c) Proportional stress limit

(d) Poisson's Ratio

1.109 PROBLEM 109 – MECHANICAL DESIGN AND ANALYSIS

A square-threaded power screw is used to raise a load of 10,000 pounds. The mean thread diameter is equal to 2 inches. The screw has 4 threads per inch, which makes its pitch equal to 0.25 inches. The coefficient of friction is equal to 0.05. Hint: pitch is equal to lead for single start, screw threads. Assume no collar. What is the torque required to raise the load?

(a) 900 lb-in

(b) 1,200 lb-in

(c) 2,960 lb-in

(d) 5,990 lb-in

1.110 PROBLEM 110 – MECHANICAL DESIGN AND ANALYSIS

A bolt will be subject to a shear force of 5,000 N. If the bolt material has a shear strength of 100 MPa, then what is the required bolt diameter?

(a) .25 mm

(b) 8 mm

(c) 17 mm

(d) 32 mm

-- END SESSION 2 --

2.0 FULL EXAM SOLUTIONS

2.1 SOLUTION 1 – MATHEMATICS

Convert the following value to polar form.

$$5 + \sqrt{-49}$$

$$\textit{Remember that } \sqrt{-1} \textit{ is equal to } i$$

$$5 + \sqrt{-1}\sqrt{49}$$

$$5 + 7i$$

Now convert the complex rectangular number to polar form with your calculator.

$$5 + 7i \rightarrow 2nd \rightarrow complex \rightarrow polar \rightarrow 8.6\angle 54.5°$$

The correct answer is most nearly, **(a) $\mathbf{8.6\angle 55°}$.**

(a) $8.6\angle 55°$

(b) $5 + 7\angle 50°$

(c) $8.6\angle -55°$

(d) $5\angle -55°$

2.2 SOLUTION 2 – MATHEMATICS

Find the product of the following two vectors, a & b.

$$a = 5 + 6i;\ \ b = \sqrt{2} - 7i;$$

There are multiple ways to solve this problem, but the easiest is to use your calculator. Make sure you put parentheses around the vectors.

$$(5 + 6i)\left(\sqrt{2} - 7i\right) = 49 - 26.5i$$

$$(49 - 26.5i) \rightarrow 2nd \rightarrow complex \rightarrow polar \rightarrow 56\angle -28°$$

The correct answer is most nearly, **(c) $\mathbf{56\angle -28°}$.**

(a) $56\angle 28°$

(b) $49\angle 14°$

(c) $56\angle -28°$

(d) $49\angle -14°$

2.3 SOLUTION 3 – MATHEMATICS

Find the product of the following two vectors, a & b.

$$a = 5 + 6i;\ \ b = \sqrt{2} - 7i;$$

There are multiple ways to solve this problem, but the easiest is to use your calculator. Make sure you put parentheses around the vectors.

$$(5 + 6i)\left(\sqrt{2} - 7i\right) = 49 - 26.5i$$

The correct answer is most nearly, (d) $\mathbf{49 - 26.5}\boldsymbol{i}$.

(a) $7 - 42i$

(b) $56\angle 28°$

(c) $7 + 42i$

(d) $49 - 26.5i$

2.4 SOLUTION 4 – MATHEMATICS

Which of the following is not a solution to the below equation?

$$x^5 + 32 = 0$$

The easiest solution is shown below.

$$x^5 = -32;\ \ x = (-32)^{\frac{1}{5}}; x = \ -2;$$

The next solutions are dependent on the polar coordinates. The first solution can be rewritten in polar coordinates.

$$x = \ 2\angle 180°;$$

$$x^5 = 2^5\angle(180 + 180 + 180 + 180 + 180)° = 32\angle(900°) = 32\angle(180°)$$

The other solutions must have a magnitude of 2, but the angle must equal 180 degrees or a multiple of 180 degrees, after being multiplied by 5.

$$x = 2\angle 36°;$$

$$x^5 = (2\angle 36°)(2\angle 36°)(2\angle 36°)(2\angle 36°)(2\angle 36°) = 2^5(36 + 36 + 36 + 36 + 36) = 32\angle 180°$$

$$x = 2\angle 108°;$$

$$x^5 = (2\angle 108°)(2\angle 108°)(2\angle 108°)(2\angle 108°)(2\angle 108°) = 2^5(540°) = 32\angle 180°$$

$$x = 2\angle 252°;$$

$$x^5 = (2\angle 252°)(2\angle 252°)(2\angle 252°)(2\angle 252°)(2\angle 252°) = 2^5(1{,}260°) = 32\angle 180°$$

$$x = 2\angle 322°;$$

$$x^5 = (2\angle 322°)(2\angle 322°)(2\angle 322°)(2\angle 322°)(2\angle 322°) = 2^5(1610°) = 32\angle 180°$$

The correct answer is most nearly, **(d) 1.9 + 0.62i**.

(a) -0.62 + 1.9i

(b) -2

(c) 0.62 + 1.9i

(d) 1.9 + 0.62i

2.5 SOLUTION 5 – MATHEMATICS

The following a-b-c triangle has sides and an angle shown below. What is the length of the remaining side?

$$a = 20; B = 75°; c = 30;$$

For this problem, you need the law of cosines, which is shown in the NCEES FE Reference Handbook. Please make sure you are in degrees.

$$b^2 = a^2 + c^2 - 2ac\,cos(B)$$

$$b^2 = 20^2 + 30^2 - 2(20)(30)cos(75) = 989.4$$

$$b = 989^{.5} = 31.46$$

The correct answer is most nearly, **(c) 31**.

(a) 23

(b) 27

(c) 31

(d) 990

2.6 SOLUTION 6 – MATHEMATICS

Find the inflection point of the following equation.

$$y = e^x - 5x^2$$

The inflection point occurs where the second derivative is equal to zero.

$$y' = e^x - 10x$$

$$y'' = e^x - 10$$

Now solve for y'' = 0.

$$0 = e^x - 10$$

$$e^x = 10; \;\; x = \ln(10) = 2.3$$

Now solve for y.

$$y = e^{2.3} - 5(2.3)^2 = -16.5$$

$$Inflection\ Point \rightarrow (2.3, -16.5)$$

The correct answer is most nearly, (a) (2.3, -16.5).

(a) (2.3, -16.5)

(b) (-2.3, -16.5)

(c) (16, 2)

(d) (2.3, 0)

2.7 SOLUTION 7 – MATHEMATICS

Find the partial derivative of the following function.

$$f(x, y) = 4\sin(3x - 7y)$$

$$\frac{\partial^2 f}{\partial x^2} = ?$$

First, take the derivative of the function with respect to x.

$$\frac{\partial f}{\partial x} = 4(3)\cos(3x - 7y)$$

$$\frac{\partial^2 f}{\partial x^2} = -4(3)(3)\sin(3x - 7y)$$

$$\frac{\partial^2 f}{\partial x^2} = -36\sin(3x - 7y)$$

The correct answer is most nearly, (d)$-\mathbf{36}\,\mathbf{sin}(\mathbf{3x} - \mathbf{7y})$.

(a) $4\cos(3x - 7y)$

(b) $-12\cos(3x - 7y)$

(c) $12\sin(3x - 7y)$

(d) $-36\sin(3x - 7y)$

2.8 SOLUTION 8 – MATHEMATICS

Find the equation of the plane that is parallel to the below plane and travels through the point (-2, 2, 2).

$$10 = 10i - 2j + 2k$$

For parallel planes that travel through a point, input the point into the equation and solve for the constant.

$$10(-2) - 2(2) + 2(2)$$

$$-20 - 4 + 4 = -20$$

The equation is shown below.

$$-20 = 10i - 2j + 2k$$

The correct answer is most nearly, (b) 10i -2j + 2k = -20.

(a) 5i + 2j + 2k = 10

(b) 10i -2j - 2k = 20

(c) 5i + j + k = 5

(d) -5i + j – k = 20

2.9 SOLUTION 9 – MATHEMATICS

Find the equation for the function, y.

$$y'' + 20y' = 0$$

$$y(0) = 5; y'(0) = 2$$

$$m(m + 20) = 0$$

$$m = 0, -20$$

$$y = C_1e^{0x} + C_2e^{-20x}$$

$$y = C_1 + C_2e^{-20x}$$

Use the initial conditions to solve for the constants.

$$y'(0) = 2 = -20C_2e^0$$

$$C_2 = -0.1$$

$$y(0) = C_1 + C_2e^0 = 5$$

$$C_1 - 0.1 = 5$$

$$C_1 = 5.1$$

Now plugin the constants.

$$y = 5.1 - 0.1e^{-20x}$$

The correct answer is most nearly, **(b) 5.1 – 0.1e^{-20x}**.

(a) 4.9 + 0.1e^{10x}

(b) 5.1 – 0.1e^{-20x}

(c) 5.1 + 0.1e^{20x}

(d) 5.0 + 0.1e^{-2x}

2.10 SOLUTION 10 – PROBABILITY & STATISTICS

A manufacturing process is shown to produce satisfactory products at an 80% satisfactory rate. What is the probability that only 7 out of 10 products will be satisfactory?

For this problem you need the binomial distribution tables, because you are dealing with two options, satisfactory and unsatisfactory. Navigate to “n=10”, “x=7” and column “P=0.8”.

$$0\ thru\ 7\ sucesses = 0.3222$$

Next, navigate to “n=10”, “x=6” and column “P=0.8”.

$$0\ thru\ 6\ sucesses = 0.1209$$

Now subtract the second value from the first value to get only 7 out of 10 satisfactory results.

$$only\ 7\ sucesses \rightarrow 0.322 - 0.1209 = 0.2013$$

The correct answer is most nearly, **(b) 20%.**

2.11 SOLUTION 11 - PROBABILITY & STATISTICS

A population has a mean of 25 and a standard deviation of 5. The population is assumed to follow a normal distribution. What is the probability that a sample will be greater than 34?

For this problem, you need to determine how many standard deviations 34 is away from the mean.

$$\frac{34 - 25}{5} = 1.8\ standard\ deviations$$

Next, go to the normal distribution table and find the column that corresponds to greater than a certain standard deviation, which is R(x). Then find the row corresponding to 1.8.

$$R(1.8) = 0.0359$$

The answer is most nearly, **(a) 3.6%.**

2.12 SOLUTION 12 - PROBABILITY & STATISTICS

What is the t-value for the following scenario? A sample group of 10 widgets is selected. The sample group has a mean of 110 and a standard deviation of 8. The population mean is estimated as 100.

For this problem, use the t-value equation. The t-value is used to calculate the confidence of an estimated population mean through the use of a sample group.

$$t - value = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t - value = \frac{110 - 100}{\frac{8}{\sqrt{10}}} = 3.95$$

The correct answer is most nearly, **(c) 4.0**.

2.13 SOLUTION 13 - PROBABILITY & STATISTICS

A fair 6-sided die is rolled 9 times. What is the probability that a 4 is rolled only once?

This problem uses the binomial distribution. The two outcomes are 4 or not 4.

$$Success(roll\ a\ 4) = p = \frac{1}{6}; Fail\ (not\ 4) = q = \frac{5}{6}$$

Use the binomial distribution to find the probability of 1(x) success out of 9 (n) trials.

$$p_n(x) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$p_9(1) = \frac{9!}{1!\,(8)!} \frac{1^1}{6} \frac{5^8}{6}$$

$$p_9(1) = \frac{9!}{1!\,(8)!} \frac{1^1}{6} \frac{5^8}{6} = 0.349$$

The correct answer is most nearly, **(b) 0.35**.

(a) 0.15

(b) 0.35

(c) 0.66

(d) 0.81

2.14 SOLUTION 14 - PROBABILITY & STATISTICS

A fair, 6-sided die is rolled 8 times. What is the probability that an even number is rolled less than 4 times?

This problem uses the binomial distribution. The two outcomes are even or odd.

$$Success(roll\ even) = p = 0.5; Fail\ (roll\ odd) = 0.5$$

Use the binomial distribution to find the probability of 0, 1, 2, 3 (x) success out of 8 (n) trials.

$$p_n(x) = \frac{n!}{x!\,(n-x)!}p^x q^{n-x}$$

$$p_8(0) = \frac{8!}{0!\,(8-0)!}0.5^0 0.5^8 = 0.0039$$

$$p_8(1) = \frac{8!}{1!\,(8-1)!}0.5^1 0.5^7 = 0.03125$$

$$p_8(2) = \frac{8!}{2!\,(8-2)!}0.5^2 0.5^6 = 0.109375$$

$$p_8(3) = \frac{8!}{3!\,(8-3)!}0.5^3 0.5^5 = 0.21875$$

Finally, add up all the probabilities.

$$Probability = 0.363275$$

You can also use the cumulative binomial probability table. Navigate to column (P = 0.5). Then go to the row n=8, x=3. Since it is cumulative x =3, also includes x =2, 1 and 0.

The correct answer is most nearly, **(d) 0.36**.

(a) 0.03

(b) 0.10

(c) 0.22

(d) 0.36

2.15 SOLUTION 15 – ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly NOT true regarding engineers issuing public statements?

(a) Engineers shall issue public statements only in an objective and truthful manner.

(b) Engineers may always express publicly technical opinions.

(c) Engineers may always issue criticisms on technical matters that are paid for by interested parties.

(d) Engineers shall issue public statements along with any pertinent information and the date indicating when it was current.

The correct answer is most nearly, (c) Engineers may always issue criticisms on technical matters that are paid for by interested parties. This answer is not true, because engineers can only issue criticisms on technical matters that are paid for by interested parties, IF they also include the fact that they are paid by the interested party.

2.16 SOLUTION 16 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly true regarding public safety?

(a) Engineers may take assignments when not qualified by education or experience, as long as they disclose this information to their client.

(b) Engineers may affix their stamp to plans that have not been prepared under their direction or control.

(c) Engineers must not take assignments outside of their qualifications.

(d) Engineers may coordinate and assume responsibility for other engineers' work as long as the engineers are qualified.

The correct answer is most nearly, (c) Engineers must NOT take assignments outside of their qualifications.

2.17 SOLUTION 17 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following statements is most nearly false regarding professional reputation?

(a) Engineers in private practice can review the work of another engineer for the same client, without informing the other engineer.

(b) Engineers working for the government are entitled to review and evaluate the work of other engineers when so required by their employment duties.

(c) Engineers in sales are entitled to make engineering comparisons of represented products with products of other suppliers.

(d) Engineers may conduct an engineering review of another engineer's work, if the engineers are working for different clients.

The correct answer is most nearly, (a) Engineers in private practice can review the work of another engineer for the same client, without informing the other engineer.

2.18 SOLUTION 18 - ETHICS AND PROFESSIONAL PRACTICE

Which of the following would legally protect a design logo?

A trademark is used to protect a word, phrase, symbol, graphic design, brand name or logo. A trademark can be filed at the Federal and State level.

The correct answer is most nearly, (a) Trademark.

(a) Trademark

(b) Copyright

(c) Industry secret

(d) Patent

2.19 SOLUTION 19 – ENGINEERING ECONOMICS

A new transformer costs $5,000 and has an annual maintenance cost of $250. If the salvage value is $1000 at the end of its 20 year life, then what is the annual value of this transformer? Assume an interest rate of 5%.

For this problem, you must convert all terms to annual value. All costs are negative and all gains are positive.

$$Initial\ Cost \rightarrow AV_{initial} = -\$5{,}000 * \left(\frac{A}{P}, 5\%, 20\ yrs\right) = -\$5{,}000 *\ 0.0802 = -\$401.00$$

$$Annual\ Mainteance\ Cost \rightarrow AV_{maintenance} = -\$250$$

$$Salvage\ Value \rightarrow AV_{salvage} = +\$1{,}000 * \left(\frac{A}{F}, 5\%, 20\ yrs\right) = \$1000 * 0.0302 = \$30.20$$

$$AV = -\$401 - \$250 + \$30.20 = -\$620.80$$

The correct answer is most nearly, (d) -$620.

(a) $620

(b) $250

(c) -$250

(d) -$620

2.20 SOLUTION 20 - ENGINEERING ECONOMICS

Background: A 200 HP pump operates for 1500 hours in the year. The motor is 90% efficient and the power factor is 0.88. Energy cost is $0.23 per kilowatt-hour and monthly demand cost is $8.90/KVA.

Problem: How much does it cost to operate the pump in one year?

$$(1)\ Find\ the\ KW\ draw\ from\ the\ motor:$$

$$Convert\ pump\ brake\ horsepower\ to\ motor\ horsepower.$$

$$P_{motor[HP]} = \frac{P_{pump[BHP]}}{\varepsilon_{motor}}$$

$$P_{motor[HP]} = \frac{200\ HP}{0.90} = 222\ HP$$

$$Convert\ to\ KW\ from\ HP$$

$$222\ HP * \frac{0.7457\ KW}{HP} = 165.6\ KW$$

$$(2)\ Find\ the\ yearly\ kilowatt\text{-}hours(kwh)\ consumed.$$

$$166\ KW * 1{,}500\frac{hrs}{year} = 248{,}318\ kwh$$

$$\mathit{Find\ the\ yearly\ energy\ cost\ (\$)}$$

$$248{,}318\ kwh * \frac{\$0.23}{kwh} = \$57{,}113$$

$$\mathit{(3)\ Find\ the\ yearly\ demand\ cost\ (\$)}$$

$$(166\ KW) * \frac{1\ KVA}{0.88\ KW} * \frac{\$8.90}{KVA - month} * \frac{12\ month}{year} = \$20{,}097$$

$$\mathit{(4)\ Find\ the\ total\ electricity\ cost\ (\$)}$$

$$\text{Total Electricity Cost} = \text{Energy Cost} + \text{Demand Cost}$$

$$\$57{,}113 + \$20{,}097 = \$77{,}210$$

The correct answer is most nearly. **(C), $77,000**

2.21 SOLUTION 21 - ENGINEERING ECONOMICS

A new machine is installed in order to increase productivity. This new machine costs $160,000 and has an ongoing operating and maintenance cost of $500 per month. The new machine will save $2,000 per month and will have a salvage value of $15,000 after 10 years. If the interest rate is 4%, then what is the annual cost of the new machine?

This problem involves finding the total annual cost. First, convert all your terms to an annual value.

Initial Cost [Negative value = money lost at the beginning of the lifetime]

$$First\ navigate\ to\ your\ engineering\ economics\ tables, 4\%$$

$$Convert\ Present\ value\ (P) to\ Annual\ value\ (A)$$

$$A = P * (\frac{A}{P}, 4\%, 10)$$

$$A = -\$160{,}000 * 0.12329$$

$$A_{initial} = -\$19{,}726$$

Salvage Value [Positive value = money gained at the end of the lifetime]

$$Convert\ Future\ value\ (F) to\ Annual\ value\ (A)$$

$$A = F * (\frac{A}{F}, 4\%, 10)$$

$$A = \$15{,}000 * 0.08329$$

$$A_{salvage} = \$1249$$

Operating & Maintenance Cost [Negative value = Money lost]

$$A_{O\&M} = \frac{-\$500}{month} * 12\ months = -\$6{,}000\ per\ year$$

Savings [Positive value = Money gained]

$$A_{savings} = \frac{\$2{,}000}{month} * 12\ months = \$24{,}000\ per\ year$$

Finally, sum up all annual values.

$$A_{total} = A_{initial} + A_{salvage} + A_{O\&M} + A_{savings}$$

$$A_{total} = -\$19{,}726 + \$1249 + -\$6{,}000 + \$24{,}000$$

$$A_{total} = -\$500$$

The correct answer is most nearly**, (a)** $\$\mathbf{500}$

2.22 SOLUTION 22- ENGINEERING ECONOMICS

A new energy efficient motor will replace an existing 480 V, 3 PH, 60 HZ, 10 HP motor. The old motor had an efficiency of 90% and the new motor has an efficiency of 95%. All other properties are assumed to be the same. The horsepower value is the output power. Assume electricity costs are $0.25 per kWh and the motor runs 24 hours a day and 7 days a week. The motor runs constantly at a load of 9 HP. What is the yearly cost savings achieved by switching to the new motor?

First, find the electrical input for both motors.

$$P_{input} = \frac{P_{load}}{Efficiency}$$

$$P_{input,old} = \frac{9\ HP * 0.7457\ kW/HP}{0.9} = 7.457\ kW$$

$$P_{input,new} = \frac{9\ HP * 0.7457\ kW/HP}{0.95} = 7.065\ kW$$

Now find the difference in electrical usage.

$$Electrical\ Usage\ Savings = (7.457\ kW - 7.065\ kW) * 365 * 24 = 3{,}433\ kWh$$

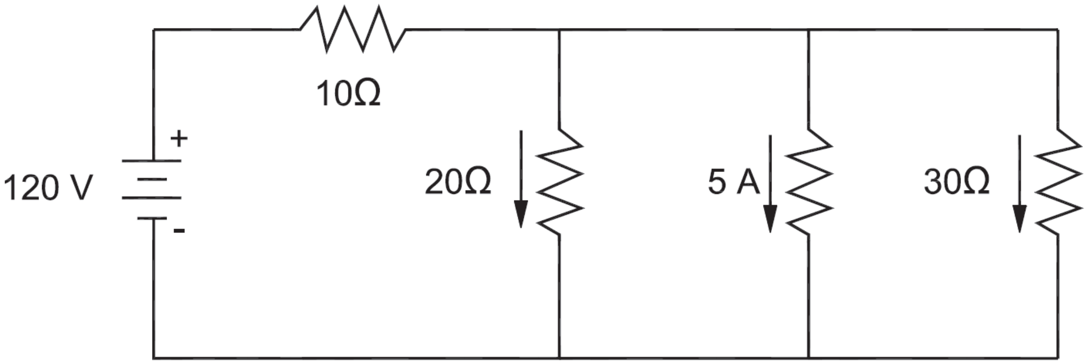
Finally, find the cost savings.

$$Cost\ Savings = 3{,}433\ kWh\ x\ 0.25 = \$858.48$$

The correct answer is most nearly, (a) $860 per year

2.23 SOLUTION 23 – ELECTRICITY & MAGNETISM

What is the current through the 10 ohm resistor?



First, create a KVL around 120 V, 10 ohm resistor and the voltage across the remaining resistors in parallel. Call this voltage V_2.

$$KVL \rightarrow 120V = I_{10} * 10 + V_2$$

$$I_{10} = \frac{120 - V_2}{10}$$

Next, create a KCL node at the intersection of the 20 ohm and 10 ohm resistor. The current into the node is equal to I_{10} (current through 10 ohm resistor). The current out of the node is equal to the current through the remaining resistors.

$$KCL \rightarrow I_{10} = \frac{V_2}{20} + 5A + \frac{V_2}{30}$$

You have two equations and two unknowns. Solve for V_2 first.

$$\frac{120 - V_2}{10} = \frac{V_2}{20} + 5A + \frac{V_2}{30}$$

$$120 - V_2 = 0.5V_2 + \frac{1}{3}V_2 + 50A$$

$$70 = 1.833V_2$$

$$V_2 = 38.19\ V$$

Now solve for I10, the current through the 10 ohm resistor.

$$I_{10} = \frac{120 - V_2}{10} = \frac{120 - 38.19\ V}{10} = 8.181\ A$$

The correct answer is most nearly, (c) 8.2A.

(a) 4.1 A

(b) 5.5 A

(c) 8.2 A

(d) 10.9 A

2.24 SOLUTION 24 – ELECTRICITY & MAGNETISM

What is the combined impedance due to a resistor and capacitor in series? Assume a 120 V, 60 Hz system. The resistor has a resistance of 20 ohms and the capacitor has a capacitance of 200 μf.

First, convert the capacitance to impedance, which has units of ohms.

$$Z_c = \frac{1}{j2\pi f C} = \frac{1}{j2\pi(60\ Hz) * 200\ x\ 10^{-6}\ F} = -13.3j\ \Omega$$

Now, add up the impedances in series.

$$Z_{combined} = 20\Omega - 13.3\mathrm{j}\Omega$$

The correct answer is most nearly, (b) 20 – 13.3j Ω.

(a) 7.3 Ω

(b) 20 - 13.3j Ω

(c) 20 + 13.3j Ω

(d) 33.3 Ω

2.25 SOLUTION 25 – ELECTRICITY & MAGNETISM

An inductor has an inductance of 100 mH at a frequency of 60 Hz. What is the impedance of the inductor?

$$Z = 2\pi f L = 2\pi(60)100x10^{-3} = 38\ ohms$$

The correct answer is most nearly, (b) 38 ohms.

(a) 4 ohms

(b) 38 ohms

(c) 47 ohms

(d) 107 ohms

2.26 SOLUTION 26 – ELECTRICITY & MAGNETISM

A 3 phase, 6 pole, synchronous generator must produce 50 Hz power. What speed should the generator be operated?

$$n = 120 * \frac{f}{P}$$

$$n = 120 * \frac{50}{6} = 1{,}000\ RPM$$

The correct answer is most nearly, **(c) 1,000 RPM**

(a) 100 RPM

(b) 750 RPM

(c) 1,000 RPM

(d) 2,000 RPM

2.27 SOLUTION 27 – ELECTRICITY & MAGNETISM

Two circular wires of the same material have the following lengths and diameters.

$$Wire\ A \rightarrow Length = L; Diameter = d;$$

$$Wire\ B \rightarrow Length = 2L; Diameter = 0.5d;$$

Which of the following statements is most nearly true?

Solve for the resistance of each wire.

$$R_A = \frac{\rho L}{A} = \frac{\rho L}{0.25\pi d^2}$$

$$R_B = \frac{\rho 2L}{A} = \frac{2\rho L}{0.25\pi(0.5d)^2} = 8\frac{\rho L}{0.25\pi(d)^2}$$

$$R_B = 8R_A$$

The correct answer is most nearly, (d) The resistance of wire B is eight times the resistance of wire A.

(a) The resistivity of wire A is twice the resistance of wire B.

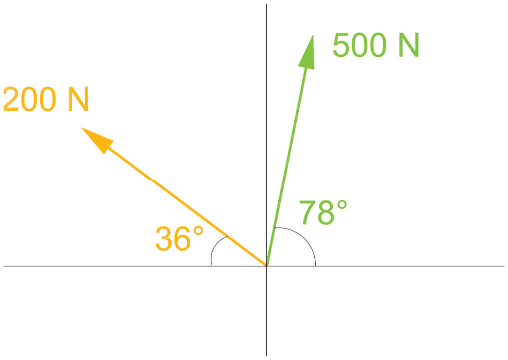
(b) The resistance of wire B is twice the resistance of wire A.

(c) The resistance of wire B is four times the resistance of wire A.

(d) The resistance of wire B is eight times the resistance of wire A.

2.28 SOLUTION 28 – STATICS

Given the two force vectors shown in the vector below, find the x and y components of the combined force vector.



First, find the x and y components of each force vector.

$$F_x = 200\cos(36) = -161.8\ N; F_y = 200\sin(36) = 117.6\ N$$

$$F_x = 500\cos(78) = 104\ N; F_y = 500\sin(78) = 489.1\ N$$

Combine the x-components together and the y-components together.

$$F_x = -161.8 + 104 = -57.8\ N\ ; F_y = 117.6 + 489.1 = 606.7\ N$$

The correct answer is most nearly, (a) F_x = -58 N; F_y = 607 N;

(a) F_x = -58 N; F_y = 607 N;

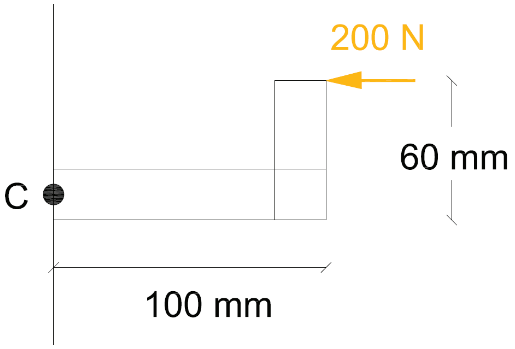
(b) F_x = -158 N; F_y = 75 N;

(c) F_x = -256 N; F_y = 444 N;

(d) F_x = -781 N; F_y = 856 N;

2.29 SOLUTION 29 – STATICS

What is the moment about point C, due to the force shown? Neglect the weight of the member.



For the moment, the force applied at a perpendicular distance to point must be calculated.

The perpendicular distance is 60 mm and the force is 200 N.

$$Moment = 200\ N * \left(60\ mm \frac{1\ m}{1000\ mm}\right) = 12\ N - m$$

The correct answer is most nearly, (a) 12 N-m.

(a) 12 N-m

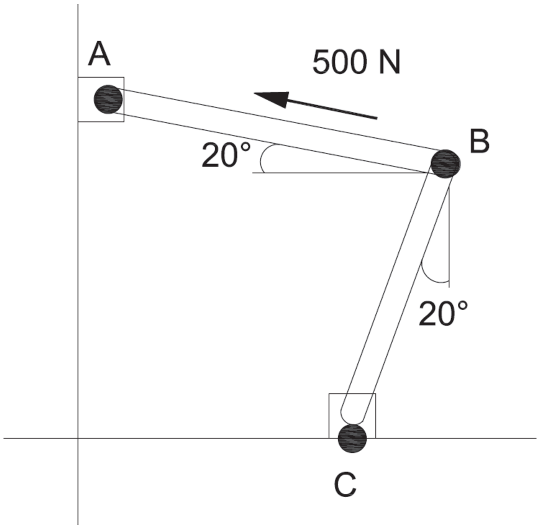
(b) 18 N-m

(c) 24 N-m

(d) 32 N-m

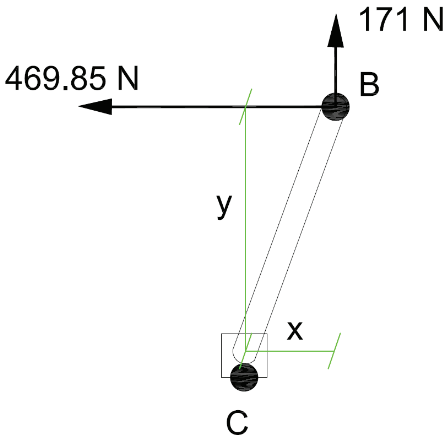
2.30 SOLUTION 30 – STATICS

Member BC = 23 mm. What is the moment about point C? Assume the two members are static.



For this problem, you can break up the 500 N into its forces in the x and y directions.

$$F_x = 500N * \cos(20) = 469.85\ N; F_y = 500N * \sin(20) = 171\ N;$$



Now, solve for the distances x and y. These are the perpendicular distances to the forces that you previously calculated.

$$x = 23mm * \sin(20) = 7.87\ mm; y = 23mm * \cos(20) = 21.61\ mm$$

Finally, solve for the moment, add up the forces that cause the member to rotate counterclockwise. Both forces cause a rotation counterclockwise.

$$Moment = 469.85\ N * \frac{21.61}{1000} m + 171N * \frac{7.87}{1000} m = 11.5\ N - m$$

The correct answer is most nearly, (b) 11.5 N-m.

(a) 1.9 N-m

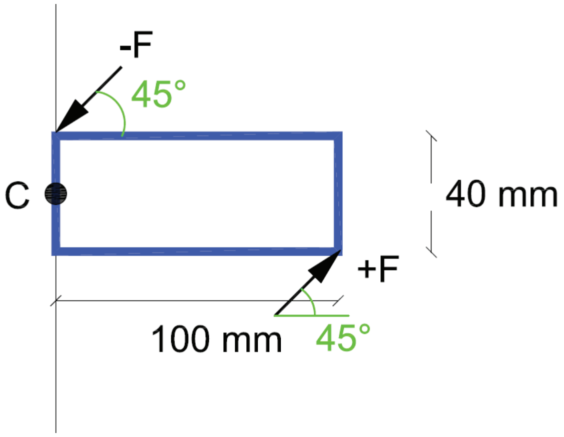
(b) 11.5 N-m

(c) 21.3 N-m

(d) 45.1 N-m

2.31 SOLUTION 31 – STATICS

One couple acts on the beam below. Assume F = 100 N, what is the couple moment about the center of the beam?



Break up the force (F = 100N) into its x and y components.

$$F_x = 100N * \cos(45) = 70.7\ N; F_y = 100N * \sin(45) = 70.7\ N$$

Now, multiply the forces by the perpendicular distance and find the moment magnitude.

$$Couple\ Moment = 70.7N * \left(\frac{40}{1000}\right) m + 70.7 * \left(\frac{100}{1000}\right) m = 9.9\ N - m$$

The correct answer is most nearly, (a) 9.9 N-m.

(a) 9.9 N-m

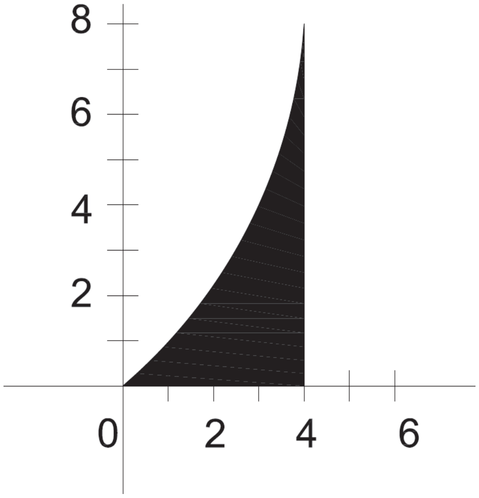
(b) 17.3 N-m

(c) 25.5 N-m

(d) 41.9 N=m

2.32 SOLUTION 32 – STATICS

Find the centroid of the following figure. F(x) = (1/2)x^2



The fast way to do this problem is to use the centroid and area moment of inertia table.

$$Generic\ Equation \rightarrow y = \left(\frac{h}{b^n}\right)x^n;\ \ Actual\ Equation \rightarrow y = \frac{1}{2}x^2; n = 2; h = 8; b = 4$$

Next, find the centroid equation.

$$x_c = \frac{n+1}{n+2}b = \frac{2+1}{2+2}4 = 3$$

$$y_c = \frac{h}{2} * \frac{n+1}{2n+1} = \frac{8}{2} * \left(\frac{2+1}{2(2)+1}\right) = 2.4$$

The correct answer is most nearly, (c) [3, 2.4]

(a) [2, 5.6]

(b) [2, 2.4]

(c) [3, 2.4]

(d) [3, 5.6]

2.33 SOLUTION 33 – STATICS

It is found that the minimum horizontal force required to move a 0.5 kg from rest is 2.5 N. What is the coefficient of static friction? Assume a flat, horizontal surface.

$$F_{static} = \mu_s F_n$$

First, calculate the normal force.

$$F_N = (0.5\ kg) * \left(9.81\frac{m}{s^2}\right) = 4.9\ N$$

Now, calculate the coefficient of static friction.

$$2.5\ N = \mu_s(4.9\ N)$$

$$\mu_s = 0.51$$

(a) 0.51

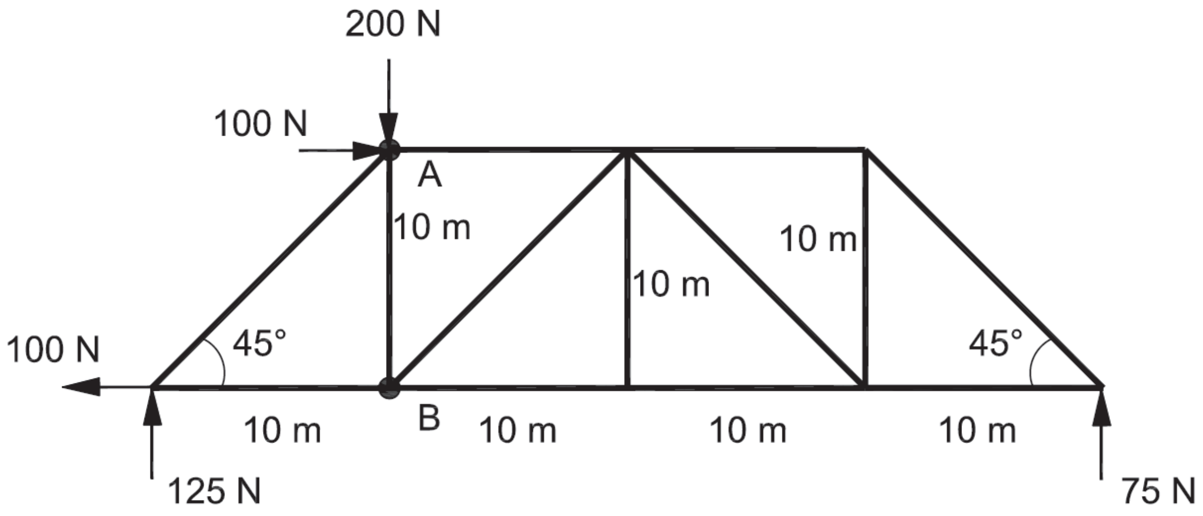
(b) 0.96

(c) 2.26

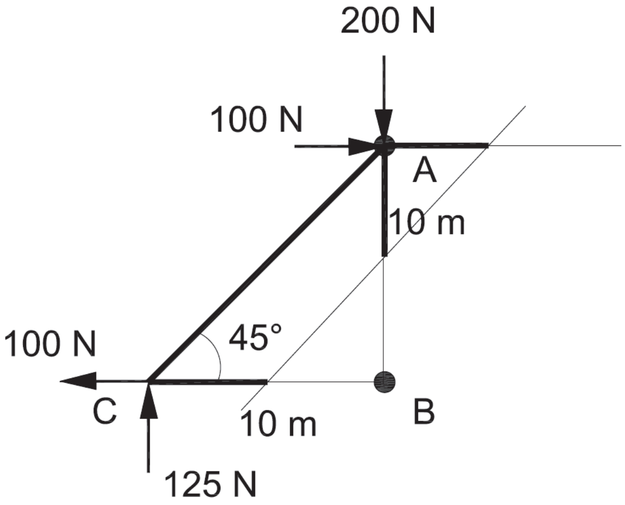
(d) 5.00

2.34 SOLUTION 34 – STATICS

What is the tension in member A-B?


200 N
100 N
A
10 m
10 m
10 m
100 N
45°
45°
10 m
B
10 m
10 m
10 m
125 N
75 N

Use the method of sections to cut through the truss, make sure you include a cut through A-B.



You can make the assumption that the truss is not moving. So now you can choose to balance forces or moment. The easiest is to balance out the forces in the y-direction, since member A-B is the only unknown member that has a force in the y-direction.

$$\sum F_y = 0 = 125N - 200N + F_{AB}$$

$$F_{AB} = 75\ N$$

The correct answer is most nearly, (d) 75 N.

(a) Unable to determine

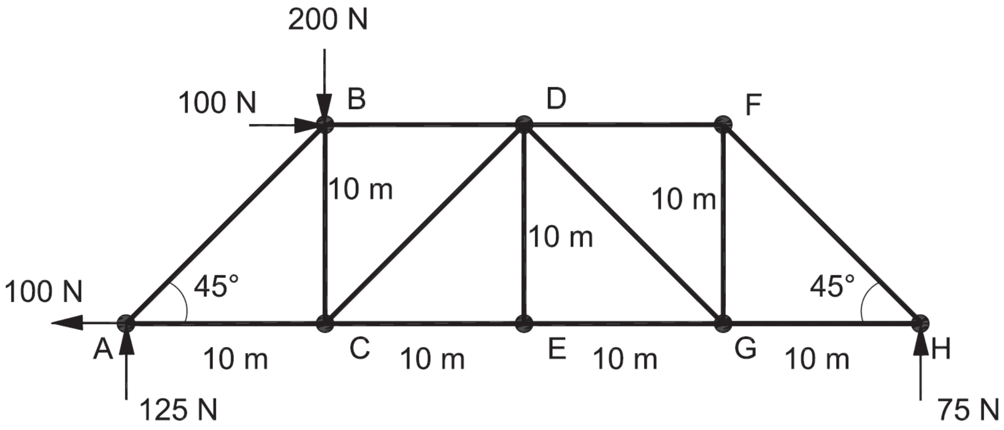
(b) 25 N

(c) 50 N

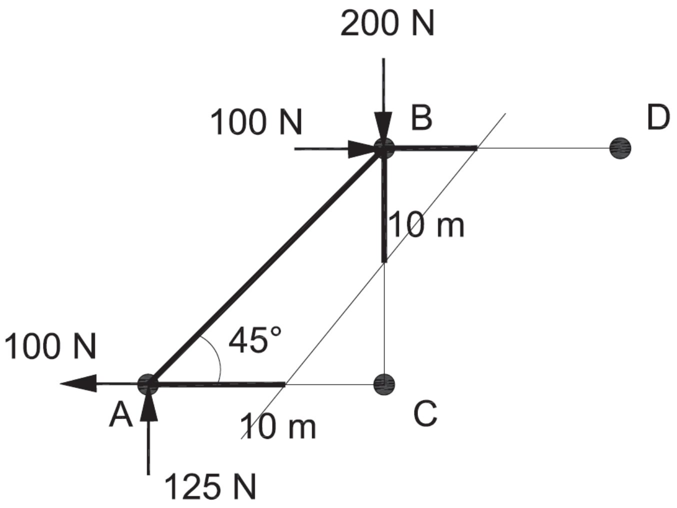
(d) 75 N

2.35 SOLUTION 35 – STATICS

What is the force in member A-C?


200 N
100 N
B
D
F
10 m
10 m
10 m
100 N
45°
45°
A
C
E
G
H
10 m
10 m
10 m
10 m
125 N
75 N

Use the method of sections to cut through the truss, make sure you include a cut through A-C.


200 N
100 N
B
D
10 m
100 N
45°
A
C
10 m
125 N

You can make the assumption that the truss is not moving. So now you can choose to balance forces or moment. The easiest is calculate the moment about point B, because it does not include the unknown forces in BD and BC.

$$\sum M_B = 0 = 100N * (10m) + 125N * (10m) - F_{AC}(10m)$$

$$F_{AC} = 225\ N$$

The correct answer is most nearly, (d) 225 N.

(a) Unable to determine

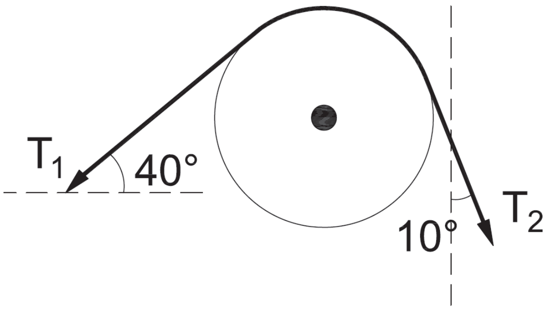
(b) 50 N

(c) 175 N

(d) 225 N

2.36 SOLUTION 36 – STATICS

The coefficient of static friction for the pulley is 0.2. The tension T1 is 100 N. What is the required tension T2 to overcome the static friction? Assume a weightless pulley.



Find the pulley equation in your FE Reference Handbook.

$$\frac{T_2}{T_1} = e^{\mu_s \theta_{wrap}}$$

$$\theta_{wrap} = 40° + 80° = 120° \rightarrow 120 * \frac{\pi}{180} = 2.09;\ \mu_s = 0.2$$

$$\frac{T_2}{T_1} = e^{0.2*2.09}$$

$$T_2 = 100N * 1.52 = 152\ N$$

The correct answer is most nearly, (d) 152 N.

(a) 115 N

(b) 123 N

(c) 135 N

(d) 152 N

2.37 SOLUTION 37 – DYNAMICS, KINEMATICS AND VIBRATIONS

The location of a particle as a function of time is described by the equation below. What is the location of the particle when the velocity is equal to zero? Time is units of seconds and distance is in units of meters.

$$x(t) = \frac{1}{3}t^3 - 4t^2 + 16t$$

First, find the derivative. The derivative is equal to the velocity.

$$v(t) = t^2 - 8t + 16$$

Solve for when velocity is equal to 0.

$$0 = t^2 - 8t + 16$$

$$0 = (t-4)(t-4)$$

$$t = 4$$

Next, solve for the location at time t = 4.

$$x(4) = \frac{1}{3}(4)^3 - 4(4)^2 + 16(4)$$

$$x(4) = 21.3\ m$$

The correct answer is most nearly, (a) 21.3 m.

(a) 21.3 m

(b) 43.1 m

(c) 88.2 m

(d) 105 m

2.38 SOLUTION 38 – DYNAMICS, KINEMATICS AND VIBRATIONS

The acceleration of a particle as a function of time is described by the equation below. What is the governing equation for velocity? V(0) = 5 m/s. Time is units of seconds and distance is in units of meters.

$$a(t) = 4t^2$$

Take the integral of the acceleration, to find the velocity equation.

$$v(t) = \left(\frac{4}{3}\right)t^3 + C$$

Use the initial value to solve for the constant C.

$$v(t) = \left(\frac{4}{3}\right)t^3 + 5$$

The correct answer is most nearly, (d) v(t) = (4/3) t^3 + 5.

(a) v(t) = 8t +5

(b) v(t) = 8t

(c) v(t) = (4/3)t^3

(d) v(t) = (4/3)t^3 + 5

2.39 Solution 39 – Dynamics, Kinematics and Vibrations

A motor is spinning at 1800 RPM. It is decelerated at a constant deceleration. The motor is at rest after 10 seconds. How many revolutions were completed in the 10 seconds?

The change in velocity is from 1800 RPM to 0 in 10 seconds. You can solve for acceleration.

$$\alpha = \frac{1800\frac{rev}{m} * \frac{1m}{60s} - 0\frac{rev}{m}}{10\ s} = 3\frac{rev}{s^2}$$

Now, solve for distance or number of revolutions.

$$\theta = V_i(t) - \frac{1}{2}at^2 = 30\frac{rev}{s^2} * (10s) - \frac{1}{2} * \left(3\frac{rev}{s^2}\right) * (10s)^2 = 150\,rev$$

The correct answer is most nearly, (a) 150 revolutions.

(a) 150 revolutions

(b) 225 revolutions

(c) 300 revolutions

(d) 475 revolutions

2.40 Solution 40 – Dynamics, Kinematics and Vibrations

A net force acts upon a 7.2 kg mass for 2 seconds. The net force varies in accordance with the following function. What is the velocity at t = 2 seconds. Assume the mass is initially at a speed of 1 m/s.

$$F(t) = 5 + t^2\ [N]$$

For this problem, you can use the impulse-momentum equation in the Dynamics section of the NCEES FE Reference Handbook.

$$\sum mv = \int Fdt$$

$$m * (\Delta v) = \int 5 + t^2$$

$$7.2\ kg * (\Delta v) = \left[5(2) + \frac{1}{3}(2)^3\right] - \left[5(0) + \frac{1}{3}(0)^3\right]$$

$$7.2\ kg * (\Delta v) = 12.67\ N - s$$

$$\Delta v = 1.76\frac{m}{s}$$

$$v_{final} = 1 + 1.76\frac{m}{s} = 2.76\frac{m}{s}$$

The correct answer is most nearly, (b) 2.76 m/s.

(a) 1.76 m/s

(b) 2.76 m/s

(c) 3.76 m/s

(d) 5.66 m/s

2.41 SOLUTION 41 – DYNAMICS, KINEMATICS AND VIBRATIONS

A force of 10 N is applied to a mass of 5 kg for 3 seconds. A force of 10 N is applied to a mass of 50 kg for 3 seconds. Which of the masses has the largest momentum?

(a) 5 kg mass

(b) 50 kg mass

(c) Momentums are equal

(d) Unable to determine

The equation for momentum is force x time. The same force and same time is applied for both masses. So the momentum must be equal for both masses. The velocity of the larger mass will be less.

The correct answer is most nearly, (c) Momentums are equal.

2.42 Solution 42 – Dynamics, Kinematics and Vibrations

A system is described as having a spring constant of 1,000 N/m and a mass of 50 kg. The viscous damping coefficient is 400 N-s/m. What is the damped frequency?

First, solve for natural frequency.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1{,}000\ N/m}{50\ kg}} = 4.47\ rad/s$$

Next, solve for the damping coefficient.

$$\zeta = c\frac{1}{2\sqrt{km}} = 400\ \frac{N-s}{m} * \frac{1}{2\sqrt{1{,}000\ N/m * 50\ kg}} = 0.894$$

Finally, use the equation from the Measurement & Instrumentation section of the NCEES Mechanical FE Reference Handbook to solve for the damped frequency.

$$\omega_d = \omega_n * \sqrt{1-\xi^2}$$

$$\omega_d = \left(4.47\frac{rad}{s}\right) * \sqrt{1-0.894^2} = 2\ rad/s$$

The correct answer is most nearly, (a) 2 rad/s.

(a) 2 rad/s

(b) 4.5 rad/s

(c) 7.8 rad/s

(d) 10 rad/s

2.43 Solution 43 – Dynamics, Kinematics and Vibrations

A system is described as having a spring constant of 1,000 N/m and a mass of 50 kg. The system is also viscously damped with a damping coefficient of 0.2. A sinusoidal force is applied to the system. What is the change in amplitude when the frequency is increased from the natural frequency to 2x the natural frequency?

The amplitude equation is in the Dynamics section of the NCEES FE Reference Handbook.

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

Initially the frequency is equal to the natural frequency, so those fractions will turn to 1.

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{(1-1)^2 + (2\zeta)^2}} = \frac{\frac{F_0}{k}}{\sqrt{0 + (2(0.2))^2}} = 2.5\frac{F_0}{k}$$

Now, complete the same equation but with the fraction equal to 2.

$$X_{reduced} = \frac{\frac{F_0}{k}}{\sqrt{(1-2^2)^2 + (2\zeta * 2)^2}} = \frac{\frac{F_0}{k}}{\sqrt{9 + (0.8)^2}} = 0.322\frac{F_0}{k}$$

The amplitude was reduced by the following factor.

$$Reduction\ Factor = \frac{X_0}{X_{reduced}} = \frac{2.5\frac{F_0}{k}}{0.322\frac{F_0}{k}} = 7.76$$

The correct answer is most nearly, (d) reduced by a factor of 7.8.

(a) Increased by a factor of 8.0.

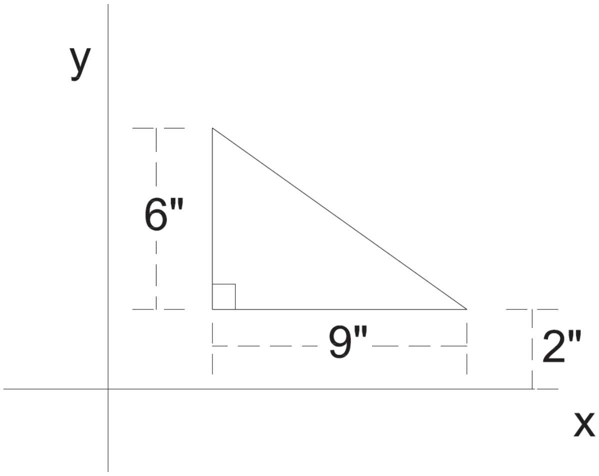
(b) Increased by a factor of 2.1.

(c) Reduced by a factor of 5.0.

(d) Reduced by a factor of 7.8.

2.44 SOLUTION 44 – DYNAMICS, KINEMATICS AND VIBRATIONS

Find the area moment of inertia of the figure below about the x-axis.



The area moment of inertia about the centroid x-axis can be found with the statics table for area moment of inertias.

$$I_{x,c} = \frac{h^3 b}{36} = \frac{(6)^3(9)}{36} = 54\ in^4$$

Next, you need to use the parallel axis theorem to move the axis from the centroid x-axis to the x-axis. This equation is in the dynamics section of the handbook, but the handbook version uses mass as opposed to area. The handbook's version of parallel axis theorem is for the mass moment of inertia, but you can derive the area moment of inertia version as shown below.

$$I_x = 54\ in^4 + Ad^2$$

$$I_x = 54\ in^4 + \frac{1}{2} 6 * 9 * 4^2 = 486\ in^4$$

The correct answer is most nearly, (d) 486 in^4.

(a) 88 in^4

(b) 162 in^4

(c) 270 in^4

(d) 486 in^4

2.45 SOLUTION 45 – DYNAMICS, KINEMATICS AND VIBRATIONS

A 6” diameter steel shaft has a length of 24”. What is the mass moment of inertia about the centerline along the length of the shaft?

First, you need to find the mass of the shaft. The density for steel is found in the Mechanics of Materials section.

$$M = \rho AL = \left(0.282\frac{lb}{in^3}\right)\left(\frac{\pi 6\ in^2}{4}\right)(24\ in) = 191\ lbs$$

Next, use the mass moment of inertia equation from the table in the Dynamics section of the FE Reference Handbook.

$$I_c = \frac{1}{2}Mr^2 = \frac{1}{2} * 191 * (3\ in)^2 = 859\ lb - in^2$$

The correct answer is most nearly, (a) 900 lb-in^2.

(a) 900 lb-in^2

(b) 3,440 lb-in^2

(c) 4,100 lb-in^2

(d) 6,270 lb-in^2

2.46 SOLUTION 46 – DYNAMICS, KINEMATICS AND VIBRATIONS

A projectile is at an initial velocity of 10 m/s at an angle of 30 degrees above the horizontal. If the projectile is initially at an elevation of 45 m, then at what time will the projectile be at an elevation of 0 m?

The y-component of the initial velocity is found through the sine of 30 degrees. Sine is equal to opposite over hypotenuse. Gravity is acting downward at constant acceleration. The position will be a function of the following equation.

$$y(t) = \left(10\frac{m}{s}\right)\sin(30)\,t - \frac{1}{2}\left(9.81\frac{m}{s^2}\right)t + C$$

You need to assign an initial y-value of 45 m, for the initial elevation. Then solve for when the location will equal 0 m.

$$y(t) = \left(10\frac{m}{s}\right)\sin(30)\,t - \frac{1}{2}\left(9.81\frac{m}{s^2}\right)t + 45\ m$$

$$0\,m = \left(10\,\frac{m}{s}\right)\sin(30)\,t - \frac{1}{2}\left(9.81\,\frac{m}{s^2}\right)t + 45\,m$$

$$0\,m = 5t - 4.905t^2 + 45\,m$$

$$-4.905t^2 + 5t + 45$$

Now solve for t.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-5 \pm \sqrt{5^2 - 4(-4.905)45}}{2(-4.905)}$$

$$t = \frac{-5 \pm 30.13}{2(-4.905)} = -2.56s \text{ } or \text{ } 3.58s$$

The correct answer is most nearly, (a) 3.6 s.

(a) 3.6 s

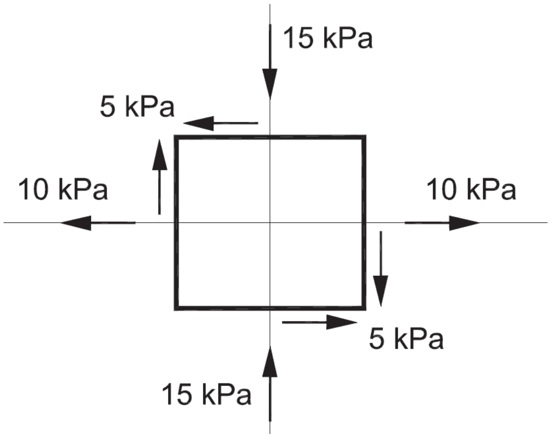
(b) 2.6 s

(c) 10.4 s

(d) 24.4 s

2.47 SOLUTION 47 – MECHANICS OF MATERIALS

What are the principal stresses?



For this problem, use Mohr's circle. This equation is located in the Mechanics of Materials section of the FE Reference Handbook.

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{10\,kPa + (-15\,kPa)}{2} = -2.5\,kPa$$

$$R = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2^2} + \tau_{xy}^2} = \sqrt{\frac{(10-(-15))^2}{2^2} + (-5)^2} = 13\ kPa$$

The principal stresses are solved with the following equation.

$$\sigma_1 = C + R;\ \ \sigma_2 = C - R;$$

$$\sigma_1 = -2.5 + 13 = 10.5\ kPa;\ \ \sigma_2 = -2.5 - 13 = -15.5\ kPa;$$

The correct answer is most nearly, (c) 10.5 & -15.5 kPa.

(a) -2.5 & 2.5 kPa

(b) 6.5 & -11.5 kPa

(c) 10.5 & -15.5 kPa

(d) 2.5 & -11.5 kPa

2.48 SOLUTION 48 – MECHANICS OF MATERIALS

A steel rod has a diameter of 150mm and a length of 10m. The rod is secured and locked in place at both ends. The temperature varies by 50 °C. What is the maximum stress developed in the rod?

You will need the steel thermal expansion coefficient from the Mechanics of Materials section in the FE Reference Handbook.

$$\alpha = 11.7\ x\ 10^{-6}/C$$

Next, solve for the change in length of the rod, if the rod was unconstrained.

$$\Delta L = \alpha L(\Delta T) = \left(11.7\ x10^{-6}\frac{}{C}\right) * (10m) * (50\ C) = 5.85\ mm$$

Next, calculate the force acting on the pipe due to the expansion. This is the force experienced within the constrained rod.

$$5.85\ mm = \frac{FL}{AE} = \frac{F(10{,}000\ mm)}{\left(\frac{1}{4}\pi(150mm)^2\right) * 200\ GPa}$$

You need to convert GPa, so that instead of meters squared on the denominator there are millimeters squared.

$$200\ GPa = 200 * 10^9 \frac{N}{m^2} * \frac{m^2}{10^6\ \ mm^2}$$

$$5.85\ mm = \frac{F(10{,}000\ mm)}{\left(\frac{1}{4}\pi(150mm)^2\right) * 200 * 10^9 * 10^{-6})}$$

$$5.85\ mm = \frac{F(10{,}000\ mm)}{\left(\frac{1}{4}\pi(150mm)^2\right) * 200 * 10^3)}$$

$$F = 2{,}067{,}560\ N$$

Now, calculate the stress.

$$\sigma = \frac{2{,}067{,}560\ N}{\frac{1}{4} * \pi * \left(\frac{150}{1000}\right)^2} = 1.17\ x\ 10^8\ Pa = 117\ MPa$$

The correct answer is most nearly, (a) 117 MPa.

(a) 117 MPa

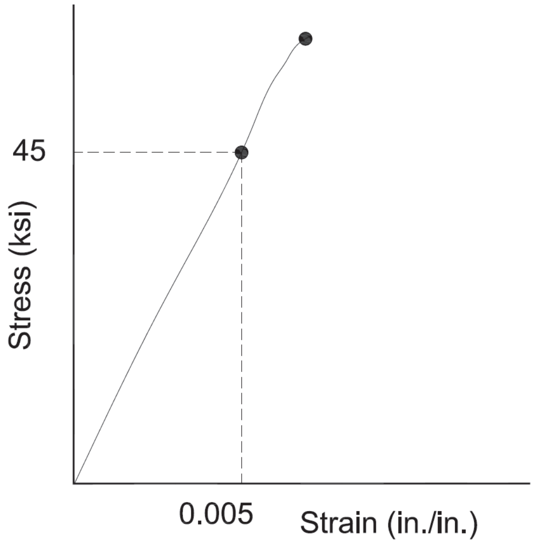
(b) 230 MPa

(c) 429 MPa

(d) 1,020 MPa

2.49 SOLUTION 49 – MECHANICS OF MATERIALS

A 10” long, 1” D shaft is put through a stress-strain test. The results are shown below. When a load of 10,000 lbf is applied to shaft in tension, the diameter is reduced by 0.1%. What is the change in length of the shaft?



Solve for the stress applied to the shaft in tension.

$$\sigma = \frac{10{,}000\ lbf}{Area} = \frac{10{,}000\ lbf}{\left(\frac{1}{4}\right) * \pi * (1in)^2} = 12.732\ ksi$$

Next, solve for the modulus of elasticity from the stress-strain graph. This is just the slope (rise over run) of the straight portion of the graph.

$$E = \frac{45}{.005} = 9000\ ksi$$

Next, solve for the lateral strain.

$$\varepsilon_{long} = \frac{\sigma}{E} = \frac{12.732\ ksi}{9000\ ksi} = 0.0014$$

$$\varepsilon_{long} = 0.0014 = \frac{\Delta L}{L}$$

$$\Delta L = 10in * 0.0014 = 0.0142\ in$$

The correct answer is most nearly, (d) 0.014".

(a) .001"

(b) .004"

(c) .009"

(d) .014"

2.50 SOLUTION 50 – MECHANICS OF MATERIALS

Which of the following rods will have the largest longitudinal change in length?

For this problem, you need to compare the multiplication of the length, coefficient of thermal expansion and change in temperature. The largest product will have the largest change in length.

(a) 25 mm diameter, 1 m long steel rod, undergoing a 50 C change in temperature.

$$\alpha_{steel} = (11.7\ x\ 10^{-6}) * (50) * (1) = 0.000585$$

(b) 50 mm diameter, 1 m long aluminum rod, undergoing a 40 C change in temperature.

$$\alpha_{aluminum} = (23.6\ x\ 10^{-6}) * (40) * (1) = 0.000944$$

(c) 25 mm diameter, 1 m long copper rod, undergoing a 50 C change in temperature.

$$\alpha_{copper} = (16.6\ x\ 10^{-6}) * (50) * (1) = 0.000830$$

(d) 50 mm diameter, 1 m long bronze rod, undergoing a 40 C change in temperature.

$$\alpha_{bronze} = (18\ x\ 10^{-6}) * (40) * (1) = 0.000720$$

The correct answer is most nearly, (b) 50 mm diameter, 1 m long aluminum rod, undergoing a 40 C change in temperature.

2.51 SOLUTION 51 – MECHANICS OF MATERIALS

A steel pipe has an inner diameter of 50 mm and an outer diameter of 63 mm. The pipe is subject to a torque of 5,000 N-m. The pipe is 10 m long. One end is fixed. The other end is free. What is the shear stress?

$$\tau = \frac{Tr}{J}$$

You need the polar moment of inertia for a pipe, from the Statics section.

$$J = \frac{\pi}{2} * \left(\left(\frac{1}{2} * \frac{63}{1000} \right)^4 - \left(\frac{1}{2} * \frac{50}{1000} \right)^4 \right)$$

$$J = \frac{\pi}{2} * ((0.0315)^4 - (0.025)^4) = 9.33\ x\ 10^{-6}\ m^4$$

$$\tau = \frac{(5{,}000\ N - m) * (r_{avg})}{9.33\ x\ 10^{-6}\ m^4} = \frac{(5{,}000\ N - m) * (0.0565/2\ m)}{9.33\ x\ 10^{-6}\ m^4} = 150\ MPa$$

The correct answer is most nearly, (a) 150 MPa.

(a) 150 MPa

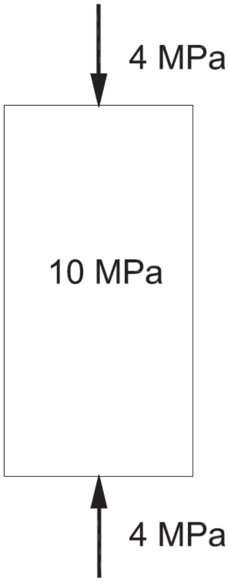
(b) 277 MPa

(c) 301 MPa

(d) 499 MPa

2.52 SOLUTION 52 – MECHANICS OF MATERIALS

A cylindrical pressure vessel has an internal pressure of 10 MPa. It is also subject to a compressive stress of 4 MPa. The pressure vessel has an inner diameter of 250 mm and a thickness of 5 mm. What is the stress in the up-down direction?



First, calculate the longitudinal (axial) stress, which is in the up-down direction, but it is tensile stress.

$$\sigma_{axial} = \frac{P_i * r}{2t}$$

$$\sigma_{axial} = \frac{10\ x\ 10^6 \frac{N}{m^2} * (125\ mm) * (\frac{1\ m}{1000\ mm})}{2\ (5\ mm) * (\frac{1\ m}{1000\ mm})} = 125\ MPa$$

Next, add up the stresses. One stress is in tension and the other is in compression, so the compression stress should be negative.

$$125\ MPa - 4\ MPa = 121\ MPa$$

The correct answer is most nearly, (c) 121 MPa.

(a) 14 MPa

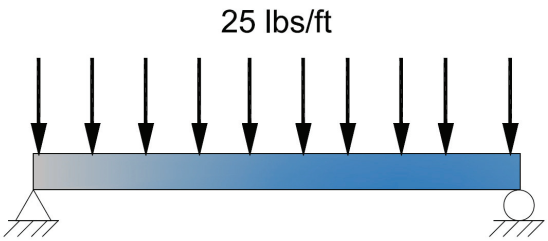
(b) 34 MPa

(c) 121 MPa

(d) 125 MPa

2.53 Solution 53 – Mechanics of Materials

What is the maximum stress in the following beam? The beam has a length of 25 feet. The beam is 6” wide by 6” thick. It is supported as shown in the figure below and is subject to the below force.



First, use the Mechanics of Materials beam diagrams to calculate the maximum moment.

$$M_{max} = \left(\frac{1}{8}\right) * wL^2 = \left(\frac{1}{8}\right) * 25 * 25^2 = 1{,}953\ lb - f$$

Next, find the maximum stress which occurs at the upper and lower surfaces of the beam (not at the center thickness of the beam). c = 3in.

$$I = \frac{1}{12}(6\ in)^4 = 108\ in^4$$

$$\sigma = \frac{Mc}{I} = \frac{1{,}935\ lb - ft * \left(\frac{12\ in}{1ft}\right)(3\ in)}{108\ in^4} = 645\ psi$$

The correct answer is most nearly, (a) 650 psi.

(a) 650 psi

(b) 1,200 psi

(c) 3,395 psi

(d) 10,240 psi

2.54 Solution 54 – Mechanics of Materials

Determine the critical buckling force of a steel column (fixed on both ends) that has the following properties.

$$L = 10m; I = 150\ cm^4$$

The critical force formula for buckling is shown in the Mechanics of Materials section.

$$F_{cr} = \frac{\pi^2 EI}{(KL)^2};\ \ K = 0.5\ for\ fixed\ on\ both\ ends; E = 200\ x\ 10^9 Pa$$

$$F_{cr} = \frac{\pi^2 (200x10^9\ Pa)(150\ cm^4 * \frac{1^4\ m}{100^4\ cm})}{(0.5 * 10\ m)^2};\ \ K = 0.5\ for\ fixed\ on\ both\ ends$$

$$F_{cr} = 118.44\ kN$$

The correct answer is most nearly, (c) 120 kN.

(a) 51 kN

(b) 95 kN

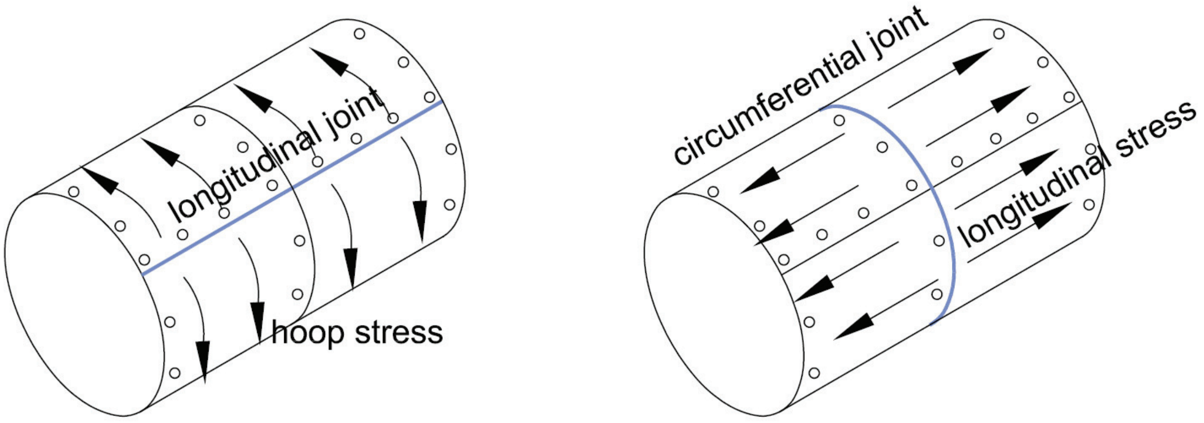
(c) 120 kN

(d) 442 kN

2.55 SOLUTION 55 – MECHANICS OF MATERIALS

Which of the following statements is true, regarding thin walled, cylindrical pressure vessels?

Hoop stress is 2x longitudinal stress. So the possible answers are (a) or (c). For the next statement, you need to determine which joint is subject to the larger hoop stress, because that is where failure will occur.



A circumferential joint is subject to longitudinal stress. A longitudinal joint is subject to the hoop stress.

(a) Hoop stress is double longitudinal stress, so failure typically occurs on longitudinal joints of the cylinder.

(b) Hoop stress is one-half longitudinal stress, so failure typically occurs on longitudinal joints of the cylinder.

(c) Hoop stress is double longitudinal stress, so failure typically occurs on circumferential joints of the cylinder.

(d) Hoop stress is one-half longitudinal stress, so failure typically occurs on circumferential joints of the cylinder.

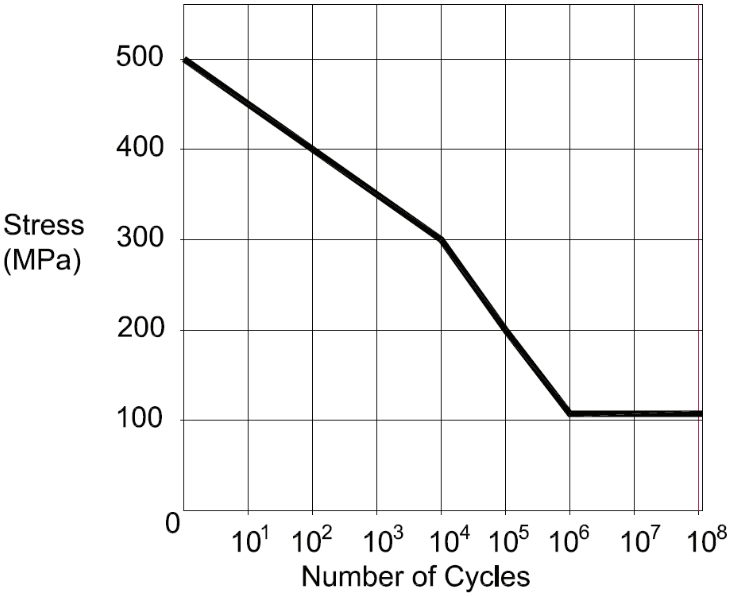
The correct answer is most nearly, (a) Hoop stress is double longitudinal stress, so failure typically occurs on longitudinal joints of the cylinder.

-- END SESSION 1 –

-- START SESSION 2 --

2.56 SOLUTION 56 – MATERIAL PROPERTIES & PROCESSING

Which of the following is most nearly the fatigue limit? Given the below fatigue failure graph.



(a) 500 MPa

(b) 300 MPa

(c) 200 MPa

(d) 100 MPa

The fatigue limit is the maximum stress that can be withstood without failure for an infinite number of cycles.

The correct answer is most nearly, (d) 100 MPa.

2.57 SOLUTION 57 – MATERIAL PROPERTIES & PROCESSING

A material will undergo a fatigue test with a maximum stress of 50 MPa and a minimum stress of -80 MPa. What will be the stress amplitude?

The stress amplitude for cyclical loading is equal to one-half the range.

$$Amplitude = \frac{1}{2}\big(50 - (-80)\big) = 65\ MPa$$

The correct answer is most nearly, (c) 65 MPa.

(a) 130 MPa

(b) 80 MPa

(c) 65 MPa

(d) 50 MPa

2.58 SOLUTION 58 – MATERIAL PROPERTIES & PROCESSING

A 4340 steel is subject to an exterior crack of length 50 mm (Y = 1.1). What is the critical value of stress?

$$K_{ic} = 46\ MPa - m^{.5}\ from\ FE\ Handbook\ under\ Materials\ Science$$

$$a = 50\ mm = \frac{50}{1000} = 0.05\ m$$

$$K_{ic} = 46MPa - m^{.5} = Y\sigma\sqrt{\pi a} = (1.1) * (\sigma)\sqrt{\pi(0.05)}$$

$$\sigma = 105\ MPa$$

The correct answer is most nearly, (d) 105 MPa.

(a) 20 MPa

(b) 59 MPa

(c) 82 MPa

(d) 105 MPa

2.59 SOLUTION 59 – MATERIAL PROPERTIES & PROCESSING

Use the iron carbide diagram in the FE Reference Handbook to determine the weight of ferrite in a 250 kg austenite (1% carbon) that has been cooled down to 700 °C.

On the graph you will be in the ferrite plus carbide region. Use the level rule to determine the percentage weight for ferrite and carbide.

$$\%\,Ferrite = \frac{6.7-1}{6.7-.02} = 0.853;\ \ \%\,Carbide = \frac{1-.02}{6.7-.02} = 0.147;$$

$$Weight\ Ferrite = \ 250 * 0.853 = 213.25\ kg$$

The correct answer is most nearly, (d) 213 kg.

(a) 85 kg

(b) 107 kg

(c) 146 kg

(d) 213 kg

2.60 SOLUTION 60 – MATERIAL PROPERTIES & PROCESSING

Which of the following materials is the anode of the following chemical reactions?

$$Pb_{(s)} + SO_4^{2-} \rightarrow PbSO_{4\ (s)} + 2e^-$$

$$PbO_{2\ (s)} + 4H^+ + SO_4^{2-} + 2e^- \rightarrow PbSO_{4\ (s)} + 2H_2O$$

(a) Pb (s)

(b) SO_4

(c) $PbSO_4$

(d) PbO_2 (s)

The anode gives up electrons. The anodic process is also called oxidation. You can remember this with LEO. Lose electrons oxidation.

The cathode gains electrons. The cathodic process is also called reduction. You can remember this with GER. Gain electrons reduction.

The Pb (s) is the anode. The PbO_2 (s) is the cathode. The SO_4 is the electrolyte that facilitates the reactions.

The correct answer is most nearly, (a) Pb (s).

2.61 SOLUTION 61 – MATERIAL PROPERTIES & PROCESSING

What is the electric potential of the following chemical reaction?

$$Ag + Fe^{3+} \rightarrow Fe^{2+} + Ag^{+}$$

(a) -0.09 V

(b) 0.09 V

(c) -1.5 V

(d) 1.5 V

First, break down the reaction into the half-cell reactions. These reactions can be found in the Chemistry section of the NCEES FE Handbook.

$$Fe^{2+} \rightarrow Fe^{3+} + e^{-}\ (-0.711)\ [Lose\ electrons\ oxidation, anode]$$

$$Ag \rightarrow Ag^{+} + e^{-} \rightarrow \quad (-0.799)\ [Gain\ electrons\ reduction,]$$

You should notice that the Fe reaction is backwards. So you need to switch the two sides and then switch the negative potential to a positive potential.

$$Fe^{3+} + e^{-} \rightarrow Fe^{2+}(0.711)\ [Lose\ electrons\ oxidation, anode]$$

$$Ag \rightarrow Ag^{+} + e^{-} \rightarrow \quad (-0.799)\ [Gain\ electrons\ reduction,]$$

Now add up the potentials from the two half reactions.

$$-0.799 + 0.711 = -0.088\ V$$

The correct answer is most nearly, (a) -0.09 V.

2.62 SOLUTION 62 – MATERIAL PROPERTIES & PROCESSING

Which of the following statements is NOT true about creep failure?

(a) Creep is more of a concern at higher temperatures than lower temperatures.

(b) Creep failure occurs at a point slightly above yield strength.

(c) Creep is more of a concern under constant stress as opposed to cyclical stress.

(d) Creep is a concern in boilers and combustion engines.

The correct answer is most nearly, (b) Creep failure occurs at a point slightly above yield strength.

In creep failure, the constant stress applied to a material is below the yield strength. However, due to the elevated temperatures and the constant stress applied over time, creep failure occurs BELOW the yield strength. This is important because you may design your system to be below the yield strength of all your materials, but it could still fail, if you do not take into account creep.

2.63 SOLUTION 63 – MATERIAL PROPERTIES & PROCESSING

Which of the following metal combinations would corrode the fastest?

In order to complete this problem, you need the Standard Oxidation Potentials for Corrosion Reaction in the *NCEES FE Reference Handbook* – Chemistry section. You should list the potentials for each metal and find the difference for each metal combination. The inverse of the values shown in the handbook are used, in order to correlate with commonly used values.

(a) Al [-1.662 V] & Cu [+0.337 V]

-[-1.662 V] + [+0.337 V] = 1.99 V

(b) Zn [-0.763 V] & Cu [+0.337 V]

-[-0.763 V] + [+0.337 V] = 1.1 V

(c) Iron [-0.440 V] & Copper [+0.337 V]

- [-0.440 V] + [+0.337 V] = 0.777 V

(d) Magnesium [-2.363] & Iron [-0.440 V]

-[-2.363] + [- 0.440 V] = 1.923 V

The combination with the largest driving force is Al & Cu with 1.99 V.

The correct answer is most nearly, (a) Al & Cu.

2.64 SOLUTION 64 – FLUID MECHANICS

A fluid has is traveling at a velocity of 8 ft/sec through a 6” pipe. The fluid has a density of 60 lb/ft^3 and has a kinematic viscosity of 1.5 x 10^{-5} ft^2/sec. What is the Reynolds number?

$$Re = \frac{V * D}{\upsilon}$$

$$V = \frac{ft}{sec}; D = diameter\ (ft); \upsilon = kinematic\ viscosity\ (\frac{ft^2}{\sec})$$

$$Re = \frac{8\frac{ft}{sec} * \frac{6\ in}{12\frac{in}{ft}}}{1.5 * 10^{-5}\frac{ft^2}{sec}}$$

$$Re = 266{,}667$$

The correct answer is (a) 266,667.

(a) 270,000

(b) 540,000

(c) 1,115,000

(d) 3,200,000

2.65 SOLUTION 65 – FLUID MECHANICS

A duct with air flowing at 0.075 lb/ft3 has a pitot tube connected to a manometer. The manometer liquid has a density of 900 kg/m^3. The pressure differential is measured as a height of 4 centimeters. The pressure differential is equal to the velocity pressure of the air in the duct. What is the velocity of air in the duct?

The first step is to convert the manometer liquid height to a pressure value.

$$Pressure = \rho g h$$

$$Pressure = 900\frac{kg}{m^3} * 9.81\frac{m}{s^2} * \frac{4}{100}m = 353.16\frac{kg}{m - s^2} = 353.16\ Pa$$

The pressure determined by the manometer is the differential pressure, which corresponds to the velocity pressure. The velocity pressure equation does not have the gravity constant, because you are in SI units.

$$Velocity\ Pressure = 353.16\ Pa = \frac{v^2}{2}$$

But first, we need to convert the density of air to SI units.

$$Density = 0.075\ \frac{lb}{ft^3} * \frac{0.453592\ kg}{1\ lb} * \frac{35.3147\ ft^3}{m^3} = 1.2\frac{kg}{m^3}$$

$$v = \sqrt{2 * 353.16\frac{kg}{m-s^2} * \frac{m^3}{1.2\ kg}} = 24.3\frac{m}{s}$$

The correct answer is most nearly, (a) 27 m/s.

(a) 27 m/s

(b) 41 m/s

(c) 57 m/s

(d) 81 m/s

2.66 SOLUTION 66 – FLUID MECHANICS

A Venturi meter has the following diameters, D_1 = 200 mm and D_2 = 100 mm. A manometer is connected between the two locations. The manometer has a delta reading of 100 mm. The density of the manometer liquid is 5,000 kg/m^3. What is the volumetric flow rate of water (density = 1,000 kg/$m^{3)}$ through the meter? C_v = 1.

$$Pressure = \rho g h$$

$$Pressure = 5{,}000\frac{kg}{m^3} * 9.81\frac{m}{s^2} * \frac{100}{1000}m = 4{,}905\frac{kg}{m-s^2} = 4{,}905\ Pa$$

Since, you are using SI units, you don't need the gravity term, but you should do dimensional analysis to confirm.

$$Q = C_v A_2 \sqrt{2\frac{\frac{-4{,}905\frac{kg}{m-s^2}}{1{,}000\ kg/m^3}}{\left(\left(\frac{A_2}{A_1}\right)^2 - 1\right)}}$$

$$Q = C_v A_2 \sqrt{2 \frac{-4.905\ m^2/s^2}{\left(\left(\frac{A_2}{A_1}\right)^2 - 1\right)}}$$

$$A_2 = \frac{\pi D_2^2}{4} = \frac{\pi (0.10\ m)^2}{4} = 0.00785\ m^2$$

$$A_1 = \frac{\pi D_1^2}{4} = \frac{\pi (0.2\ m)^2}{4} = 0.0314\ m^2$$

$$Q = 1 * 0.0113\ m^2 \sqrt{2 \frac{-4.905\ m^2/s^2}{\left(\frac{1}{16} - 1\right)}}$$

$$Q = 1 * 0.00785\ m^2 \sqrt{10.464\ m^2/s^2}$$

$$Q = 0.0254 \frac{m^3}{s}$$

The correct answer is most nearly, (d) $\mathbf{0.0254 \frac{m^3}{s}}$.

(a) $1.056 \frac{m^3}{s}$

(b) $0.194 \frac{m^3}{s}$

(c) $0.059 \frac{m^3}{s}$

(d) $0.0254 \frac{m^3}{s}$

2.67 SOLUTION 67 – FLUID MECHANICS

A centrifugal, chilled water pump is rated at 300 GPM, 50 feet head, 78% efficiency, 1100 RPM. If the pump is run at 1700 RPM, then what will be the new flow rate and pressure ratings?

Use the pump affinity laws to determine the new flow rate and pressure ratings.

$$Q_{new} = 300\ GPM * \left(\frac{1700}{1100}\right) = 463.6\ GPM$$

$$H_{new} = (50\ ft\ hd) * \left(\frac{1700}{1100}\right)^2 = 119.4\ ft\ hd$$

The correct answer is most nearly, (b) 460 GPM & 120 FT HD.

(a) 300 GPM & 120 FT HD

(b) 460 GPM & 120 FT HD

(c) 300 GPM & 50 FT HD

(d) 460 GPM & 80 FT HD

2.68 SOLUTION 68 – FLUID MECHANICS

A fan provides 2,000 m^3/h of air at a change in pressure of 500 Pa. The fan has an efficiency of 65%. What is the fan power? The air density is 1.2 kg/m^3.

$$Power = \frac{Q * \Delta P}{Efficiency}$$

$$Power = \frac{\left(2000 \frac{m^3}{h}\right) * \frac{1\ h}{3600\ s} * (500\ Pa)}{0.65} = \frac{\left(2000 \frac{m^3}{h}\right) * \frac{1\ h}{3600\ s} * (500\ N/m^2)}{0.65}$$

$$Power = \frac{\left(277.8 \frac{N-m}{S}\right)}{0.65} = 427.4 \frac{J}{s} = 427.4\ W$$

The correct answer is most nearly, (d) 430 W.

(a) 55 W

(b) 122 W

(c) 209 W

(d) 430 W

2.69 SOLUTION 69 – FLUID MECHANICS

Which of the following statements is most nearly true about net positive suction head?

The correct answer is most nearly, (c) 430 Net positive suction head is the pressure at the pump suction head that is greater than the vapor pressure of the fluid at the operating temperature. .

(a) Net positive suction head available must be less than net positive suction head required to avoid cavitation at a pump.

(b) Net positive suction head is equal to the suction pressure available.

(c) Net positive suction head is the pressure at the pump suction head that is greater than the vapor pressure of the fluid at the operating temperature.

(d) Both (a) & (c)

2.70 SOLUTION 70 – FLUID MECHANICS

An air compressor has an inlet pressure of 125 kPa. The air compressor has a pressure ratio of 5. The inlet temperature is 290 K. The isentropic efficiency of the compressor is 85%. What is the outlet temperature of the compressor? Air, k = 1.4.

Solve for outlet temperature for isentropic compression.

$$\frac{T_{out,ideal}}{T_{in}} = \left(\frac{P_{out}}{P_{in}}\right)^{\frac{k-1}{k}}$$

$$\frac{T_{out,ideal}}{290\ K} = (5)^{\frac{1.4-1}{1.4}}$$

$$T_{out,ideal} = 459$$

Next, solve for the actual outlet temperature with the isentropic efficiency equation.

$$Efficiency = \frac{T_{out,ideal} - T_{in}}{T_{out,actual} - T_{in}}$$

$$0.85 = \frac{459 - 290}{T_{out,actual} - 290}$$

$$T_{out,actual} = 489\ K$$

The correct answer is most nearly, (d) 490 K.

(a) 290 K

(b) 340 K

(c) 460 K

(d) 490 K

2.71 SOLUTION 71 – FLUID MECHANICS

A fan is rated at 120 L/s at 450 Pa. The fan is used to transport air at 1.22 kg/m^3. What is the mass of air transported in an 8 hour period?

For this problem, simply multiply the density by the volumetric flow rate to get the mass flow rate and then multiply by time.

$$\dot{m} = \rho \dot{V}$$

$$\dot{m} = 1.2\frac{kg}{m^3} * \left(120\frac{L}{s}\right) = 1.2\frac{kg}{m^3} * \left(120\frac{L}{s} * \frac{1\ m^3}{1000\ L} * \frac{3600\ s}{1\ h}\right) = 518.4\frac{kg}{h}$$

$$m = 518.4\frac{kg}{h} * 8\ h = 4{,}147\ kg$$

The correct answer is most nearly, (d) 4,200 kg.

(a) 520 kg

(b) 1,970 kg

(c) 2,950 kg

(d) 4,200 kg

2.72 SOLUTION 72 – FLUID MECHANICS

Which of the following fluids is the most turbulent? Assume each fluid has the same hydraulic diameter and the same velocity.

The turbulence is directly related to the Reynolds number. The highest Reynolds number will have the highest turbulence.

$$Re = \frac{\rho V D}{\mu}$$

Since, the velocity and diameter are the same. You just need to compare the density and absolute viscosity of each possible answer.

$$(a) = \frac{\rho}{\mu} = \frac{0.0734\ lbm/ft^3}{0.045\ lbm/(hr - ft)} = 1.6311\frac{hr}{ft^2} * \left(\frac{3600\ s}{1\ hr}\right) = 5{,}871\ \frac{s}{ft^2}$$

$$(b) = \frac{\rho}{\mu} = \frac{0.0708\ lbm/ft^3}{0.047\ lbm/(hr - ft)} = 1.506\frac{hr}{ft^2} * \left(\frac{3600\ s}{1\ hr}\right) = 5{,}421\ \frac{s}{ft^2}$$

$$(c) = \frac{\rho}{\mu} = \frac{1}{v} = \frac{1}{0.93\ x\ 10^{-5}\frac{ft^2}{s}} = 107{,}526\ \frac{s}{ft^2}$$

$$(d) = \frac{\rho}{\mu} = \frac{1}{v} = \frac{1}{0.739\ x\ 10^{-5}\frac{ft^2}{s}} = 135{,}318\ \frac{s}{ft^2}$$

The correct answer is most nearly, (d) Water at 100 °F and 1 atm.

(a) Air at 80 °F and 1 atm.

(b) Air at 100 °F and 1 atm.

(c) Water at 80 °F and 1 atm.

(d) Water at 100 °F and 1 atm.

2.73 SOLUTION 73 – FLUID MECHANICS

Which of the following factors has the least of an effect on the pressure drop due to friction in laminar flow in a pipe?

The correct answer is most nearly, (a) pipe roughness.

(a) Pipe roughness

(b) Fluid velocity

(c) Fluid dynamic viscosity

(d) Pipe diameter

2.74 SOLUTION 74 – FLUID MECHANICS

At extremely turbulent flow (Reynolds greater than 4 x 10^7), the pressure drop due to friction is independent of which term?

At extremely high Reynolds number, the Moody friction factor is no longer dependent on the Reynolds number. It is only dependent on the roughness and pipe diameter.

The correct answer is most nearly, (d) Reynolds number.

(a) Pipe roughness

(b) Pipe diameter

(c) Velocity

(d) Reynolds number

2.75 SOLUTION 75 – THERMODYNAMICS

A boiler & superheater is used to provide 100 kg/s of steam to a turbine. The condenser temperature is 50 C. The outlet of the superheater is at 500 C and 1 MPa. The isentropic efficiency of the turbine is 80%. The temperature entering the condenser is 50 C. What is the power generated by the turbine? Hint: This problem follows the Rankine Cycle.

This problem uses the Rankine Cycle and in order to complete this problem you need to do an energy balance across the turbine.

Turbine Inlet (outlet of superheater): 500 C & 1 MPa. From the superheat steam tables in the FE Reference Handbook, you will find the following values for enthalpy and entropy.

$$h_{in} = 3{,}478.5\frac{kJ}{kg}; s_{in} = 7.7622\frac{kJ}{kg-K}$$

Turbine Outlet (inlet of condenser): 50 C, but unknown pressure. You don't know the pressure, but you do know that an ideal turbine outlet will have the same entropy as the inlet. Unfortunately, there is a turbine efficiency, which means the outlet entropy will be slightly higher than the inlet entropy.

First, find the ideal values for entropy and enthalpy. The entropy value must have a temperature of 50 C. Luckily, the 50 C value has an entropy value that is in the mixed region. Next, solve for the quality.

$$s_{out} = 7.7622 = s_f + xs_{fg}$$

$$s_{out} = 7.7622 = 0.7038 + x(7.3725)$$

$$x = 0.957$$

With the quality value, solve for the ideal outgoing enthalpy value.

$$h_{out} = h_f + (0.957) * h_{fg}$$

$$h_{out} = 209.33 + (0.957) * 2382.7$$

$$h_{out} = 2{,}489.57\frac{kJ}{kg}$$

Now, use the turbine isentropic efficiency equation from your FE Reference Handbook to find the actual leaving enthalpy.

$$Efficiency = 80\% = \frac{Actual}{Ideal} = \frac{h_{in} - h_{out,actual}}{h_{in} - h_{out,ideal}} = \frac{3{,}478.5 - h_{out,actual}}{3{,}478.5 - 2{,}489.57}$$

$$h_{out,actual} = 2{,}687.5\frac{kJ}{kg}$$

The work done is found by multiplying the mass flow rate by the change in enthalpy.

$$Work = 100\frac{kg}{s} * (3478.5 - 2{,}687.4)\frac{kJ}{kg} = 79100\frac{kJ}{s} = 79{,}100\ kW = 79\ MW$$

The correct answer is most nearly, (a) 80 MW.

(a) 80 MW

(b) 300 MW

(c) 350 MW

(d) 500 MW

2.76 SOLUTION 76 – THERMODYNAMICS

Water is heated from a saturated liquid at 145 C to a vapor quality of 50%. What is the work done by the water? Assume a constant pressure process and a mass of 1 kg of water.

The equation for constant pressure in the Thermodynamics section of the FE Reference Handbook is shown below. The volume of liquid will expand as it turns into a vapor. If the pressure is constant, then the work done will be due to the change in volume.

$$Work\ (Joule\ aka\ N - m) = (mass\ kg) * \left(Pressure\frac{N}{m^2}\right) * \left(\Delta\ Specific\ Volume\frac{m^3}{kg}\right)$$

First, find the specific volume of the saturated liquid at 145 C in the steam tables of your FE Reference Handbook.

$$v_{liquid} = 0.001085\frac{m^3}{kg};$$

The specific volume of the 50% vapor is found through the mixed equation.

$$v_{50\%} = v_{liquid} + x_{quality} * (v_{vapor} - v_{liquid});$$

$$v_{50\%} = 0.001085 + 0.5 * (.4463 - .001085) = 0.22369\frac{m^3}{kg}$$

Next, you need the pressure. The pressure corresponds to the saturation pressure at 145 C.

$$Pressure = 0.4154\ MPa = 415{,}400\ Pa = 415{,}400\frac{N}{m^2}$$

$$Work\ (Joule\ aka\ N - m) = (1\ kg) * (415{,}400) * (.22369 - .001085) = 92{,}500\ J$$

The correct answer is most nearly, **(A) 92,500 J.**

(a) 92,500 J

(b) 138,750 J

(c) 225,000 J

(d) 341,000 J

2.77 SOLUTION 77 – THERMODYNAMICS

Two pounds of methane is burned with oxygen. Assume a perfectly balanced reaction. How many pounds of water are produced?

$$CH_4 + 2O_2 \rightarrow CO_2 + H_2O$$

First, figure out the moles of methane.

$$12\ (C) + 4 * 1(H) = 16 \frac{g}{mol}$$

$$2\ lbs * \left(0.454 \frac{kg}{lb}\right) * 1000 \frac{g}{kg} * \frac{mol}{16\ g} = 56.75\ mols$$

Next, you should notice that there is 1 mol of water for every mol of methane.

$$56.75\ mols\ of\ H_2O * \left(18 \frac{g}{mol}\right) = 1{,}021.5\ g * \left(\frac{1\ kg}{1000\ g}\right) * \left(\frac{1\ lb}{0.454\ kg}\right) = 2.25\ lbs\ of\ water$$

The correct answer is most nearly, (c) 2.25 lbs

(a) 1.5 lbs

(b) 2 lbs

(c) 2.25 lbs

(d) 56 lbs

2.78 SOLUTION 78 – THERMODYNAMICS

A refrigeration cycle uses 10 lbs/hr of R-134a. The suction pressure is at 60 PSIA and the discharge pressure is at 150 PSIA. What is the net refrigeration effect if the evaporator operates at constant pressure and there is no sub-cooling or super heating?

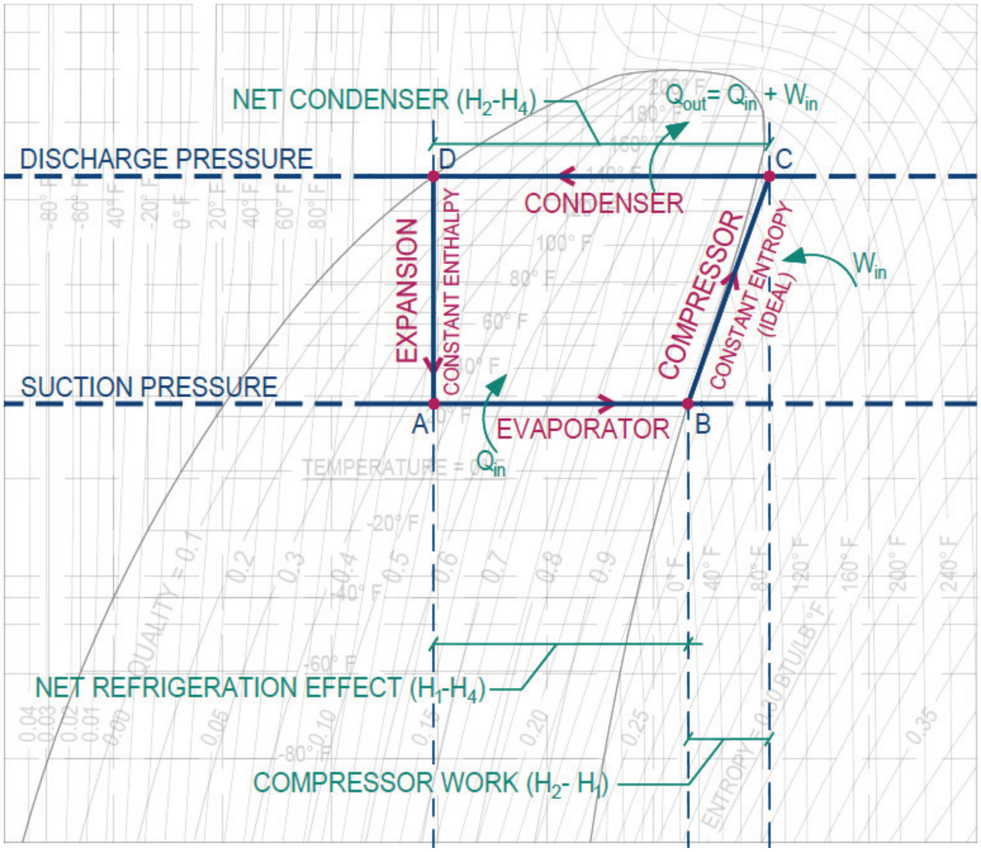
For this problem, you need to find the enthalpy entering and the enthalpy leaving the evaporator. The enthalpy entering is equal to the enthalpy of saturated liquid at 150 PSIA. The enthalpy entering is equal to the enthalpy of saturated vapor at 60 PSIA. The evaporator evaporates the refrigerant to 100% saturated vapor.

You can find the enthalpy values in the Thermodynamics section under the R-134a properties of saturated liquid and vapor tables. These tables give the properties of refrigerants as a function of temperature/pressure.

$$h_{in} = 46\frac{Btu}{lbm}\ ; h_{out} = 110\frac{Btu}{lbm}$$

Next, use the following equation to calculate heat.

$$Q = \dot{m} * (h_{out} - h_{in}) = 10\frac{lbs}{hr} * \left(110 - 46\frac{Btu}{lbm}\right) = 640\frac{Btu}{h}$$


NET CONDENSER (H_2-H_4)
Q_{out}= Q_{in} + W_{in}
DISCHARGE PRESSURE
D
C
CONDENSER
EXPANSION
CONSTANT ENTHALPY
COMPRESSOR
CONSTANT ENTROPY (IDEAL)
W_{in}
SUCTION PRESSURE
A
B
EVAPORATOR
Q_{in}
TEMPERATURE = 0° F
QUALITY = 0.1
NET REFRIGERATION EFFECT (H_1-H_4)
COMPRESSOR WORK (H_2- H_1)
ENTROPY = 0.30 BTU/LB °F

$$h_{in} = h_1; h_{out} = h_4$$

The correct answer is most nearly, (d) 640 Btu/h.

(a) 109 Btu/h

(b) 220 Btu/h

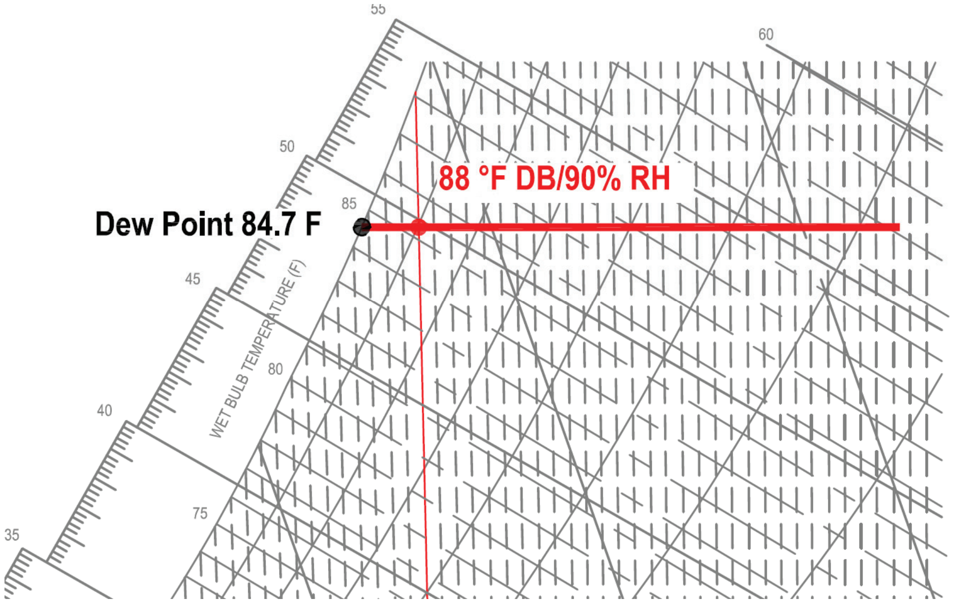
(c) 440 Btu/h

(d) 640 Btu/h

2.79 SOLUTION 79 – THERMODYNAMICS

What is the dew point of air at sea level and the following conditions, 88 F DB, 90% relative humidity?

For this problem, go to your psych chart and locate 88 F DB, 90% RH. Then take the horizontal line to the left and read the temperature on the saturation curve.



The correct answer is most nearly, (c) 84.7 F.

(a) 77 F

(b) 80.2 F

(c) 84.7 F

(d) 88 F

2.80 SOLUTION 80 – THERMODYNAMICS

55 cubic meters per minute of air enters a cooling coil at 25 C DB and 70% relative humidity. It leaves the coil at 15 C DB and 100% relative humidity. What is the cooling provided by the cooling coil? Assume sea level.

Use the metric Psychrometric chart in your FE Reference Handbook to find the enthalpy of the two points, one point entering the cooling coil and the other point leaving the cooling coil.

$$h_{in} = 60.74\frac{kJ}{kg}; h_{out} = 42.10\frac{kJ}{kg}$$

Next, find the mass flow rate and multiply it by the change in enthalpy. You also need the average specific volume from the psychrometric chart.

$$v_{in} = 0.852\frac{m^3}{kg}; v_{out} = 0.822\frac{m^3}{kg};\ v_{avg} = 0.837\frac{m^3}{kg}$$

$$Q = 55\frac{m^3}{min} * \frac{1\ min}{60s} * \frac{kg}{0.837\ m^3} * (60.74 - 42.10)\frac{kJ}{kg} = 20.41\frac{kJ}{s} 20.4\ kW$$

The correct answer is most nearly, (a) 20.4 kW

(a) 20.4 KW

(b) 25.5 KW

(c) 34.9 KW

(d) 1200 KW

2.81 SOLUTION 81 – THERMODYNAMICS

Air enters an ideal compressor (isentropic) at 50 kPa and 45 °C. The air is compressed to 1/8th of its volume. What is the leaving temperature?

Go towards the end of the FE Reference Handbook Thermodynamics section and find the k-value for air = 1.4. Also make sure you convert to absolute temperatures (kelvin).

$$T_{out} = T_{in} * \left(\frac{v_1}{v_2}\right)^{k-1}$$

$$T_{out} = (45 + 273) * \left(\frac{1}{\frac{1}{8}}\right)^{1.4-1} = 730.6\ K$$

$$T_{out} = (730.6 - 273) = 457.6\ °C$$

The correct answer is most nearly, (d) 458 °C.

(a) 273 °C

(b) 308 °C

(c) 345 °C

(d) 458 °C

2.82 SOLUTION 82 – THERMODYNAMICS

What is the change in internal energy of 1.5 kg of argon, when it is heated from 20°C to 50 °C?

For this problem, you need the heat capacity of argon. The heat capacity is located at the end of the Thermodynamics section in the FE Reference Handbook.

$$Q = mc_p(\Delta T)$$

$$Q = 1.5\,kg * (0.52\frac{kJ}{kg-K})(50-20)$$

$$Q = 23.4\,kJ$$

The correct answer is most nearly, (a) 23.4 kJ.

(a) 23.4 kJ

(b) 36.2 kJ

(c) 46.8 kJ

(d) 59.9 kJ

2.83 SOLUTION 83 – THERMODYNAMICS

2 cubic feet of methane is initially at 50 F and 14.7 PSIA. The gas is compressed to 1 cubic feet at a pressure of 40 PSIA. What is the change in internal energy of the gas? Assume an ideal gas and conservation of mass.

Frist, solve for the mass of methane with the ideal gas law.

$$m = \frac{PV}{RT}$$

Convert your pressure to the correct units. Also, get your R value for methane from the FE Reference Handbook towards the end of the Thermodynamics section.

$$P = 14.7\frac{lbf}{in^2} * \left(144\frac{in^2}{ft^2}\right) = 2116.8\frac{lbf}{ft^2};\ \ P = 40\frac{lbf}{in^2} * \left(144\frac{in^2}{ft^2}\right) = 5{,}760\frac{lbf}{ft^2};$$

$$R_{methane} = 96.35\frac{lbf-ft}{lbm-°R}$$

$$m = \frac{2116.8\frac{lbf}{ft^2} * 2\ ft^3}{96.35\frac{lbf-ft}{lbm-°R} * (50+460\ F)} = 0.0862\ lbs$$

Now, use the new volume and pressure to find the new temperature with the ideal gas law.

$$0.0862 = \frac{(5{,}760\frac{lbf}{ft^2})(1\ ft^3)}{96.35\frac{lbf-ft}{lbm-°R} * (X+460\ °F)}$$

$$X = 233.5\ °F$$

The change in internal energy is found through the below equation.

$$\Delta u = c_v(\Delta T) = 0.403\frac{Btu}{lbm-°F} * (233.5-50) = 74\frac{Btu}{lbm}$$

The change in enthalpy is found through the below equation.

$$\Delta h = c_p(\Delta T)$$

The correct answer is most nearly, (b) 74 Btu/lbm.

(a) 52 Btu/lbm

(b) 74 Btu/lbm

(c) 88 Btu/lbm

(d) 109 Btu/lbm

2.84 SOLUTION 84 – THERMODYNAMICS

One cubic meter of helium (45 °C & 450 kPa) is expanded to 2 cubic meters. Assume an ideal gas and no change in mass. Also no work is done and there is no change in heat. What is the new temperature when the helium is expanded?

$$Helium \rightarrow k = 1.67; \quad T_2 = (45 + 273) * \left(\frac{1}{2}\right)^{1.67-1} = 200\ K$$

(a) 200 K

(b) 300 K

(c) 400 K

(d) 500 K

2.85 SOLUTION 85 – THERMODYNAMICS

1 kW of heat is added to 45 kg of air for 1 hour. Air is initially at a temperature of 30 °C. What is the final temperature after 1 hour?

The heat added is 1 kW, which is equal to 1000 J/s.

$$Q = 1\frac{kJ}{s} * (1\ hours) * \left(3600\frac{s}{hr}\right) = 45\ kG * \left(1.0\frac{kJ}{kg - C}\right) * (X - 30)$$

$$X = 141\ °C$$

The correct answer is most nearly, (d) 110 C.

(a) 75 °C

(b) 92 °C

(c) 101 °C

(d) 110 °C

2.86 SOLUTION 86 – HEAT TRANSFER

Which term is not applicable to calculating the convection coefficient of a forced convection fluid, regardless of the fluid viscosity and turbulence?

(a) Nusselt Number

(b) Reynold's Number

(c) Grashof Number

(d) Prandtl Number

The correct answer is most nearly, (c) Grashof Number.

The Grashof number is the ratio of buoyancy forces to viscous forces and is used to calculate the convection coefficient for heat transfer from natural convection. Natural convection is where the fluid is moving due to temperature differences, like the temperature difference between a surface and the ambient air. Forced convection is due to the forced movement of the fluid, like water flowing in a pipe. The Nusselt Number, Reynold's Number, and Prandtl Number are all used to calculate forced convection coefficients. Prandtl and Nusselt numbers are also used in natural convection calculations.

2.87 SOLUTION 87 – HEAT TRANSFER

Hot water enters a counterflow, plate and frame heat exchanger at 140 °F at a rate of 50 gpm. Cold water is heated from 70 °F to 110 °F at a rate of 20 gpm. Given that the NTU is 1.4, what is the effectiveness of the heat exchanger?

(a) 0.6

(b) 0.7

(c) 0.8

(d) 0.9

The NTU method is used when the LMTD cannot be calculated.

Find the capacity rate ratio. Realize that the units will cancel out, so the mass flow rate can be left in gpm.

$$c_r = \frac{C_{min}}{C_{max}} = \frac{(\dot{m}c_p)_{min}}{(\dot{m}c_p)_{max}} = \frac{20gpm * 1\ Btu/lbmF}{50\ gpm * 1\ Btu/lbmF} = 0.4$$

For a counterflow heat exchanger, using the NTU method, the effectiveness is calculated as follows.

$$\varepsilon = \frac{1 - \exp[-NTU(1 - c_r)]}{1 - c_r \exp[-NTU(1 - c_r)]}\ for\ c_r < 1$$

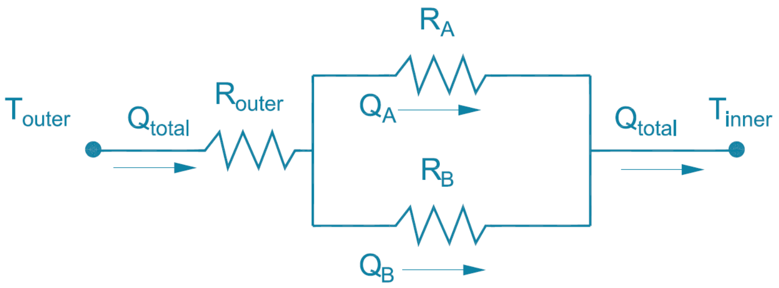
$$\varepsilon = \frac{1 - \exp[-1.4 * (1 - 0.4)]}{1 - 0.4 * \exp[-1.4 * (1 - 0.4)]} = 0.69$$

The answer is most nearly **(b) 0.7**

2.88 SOLUTION 88 – HEAT TRANSFER

A composite wall consists of an outer layer with a conductivity of 100 W/m-°C and an inner layer. The inner layer is comprised of an area that is 75% material A (k = 25 W/m-°C) and 25% material B (k = 45 W/m-°C). The inner layer has a thickness of 7 cm and the outer layer has a thickness of 10 cm. What is the equivalent heat transfer coefficient of the wall? Assume a difference in temperature of 45 °C and a surface area of 50 m².

First, create a figure to show the heat transfer graphically.



Next, you need to convert thermal conductivity to resistances.

$$R_{outer} = \frac{t}{k} = \frac{\frac{10}{100} m}{100 \frac{W}{m - °C}} = 0.001 \frac{m^2 - C}{W}$$

$$R_A = \frac{t}{k} = \frac{\frac{7}{100} m}{25 \frac{W}{m - °C}} = 0.0028 \frac{m^2 - C}{W}$$

$$R_B = \frac{t}{k} = \frac{\frac{7}{100} m}{45 \frac{W}{m - °C}} = 0.0016 \frac{m^2 - C}{W}$$

Next, combine the parallel resistances.

$$U_{parallel} = (0.75)\left(\frac{1}{0.0028} \frac{m^2 - C}{W}\right) + (0.25)\left(\frac{1}{0.0016} \frac{m^2 - C}{W}\right)$$

$$U_{parallel} = 267.9 \frac{W}{m^2 - °C} + 156.25 \frac{W}{m^2 - °C} = 424.15 \frac{W}{m^2 - °C}$$

Convert the previous U-value, to R-value. Then add the outer layer resistance.

$$R_{total} = R_{outer} + \frac{1}{424.15} \frac{m^2 - °C}{W}$$

$$R_{total} = 0.001 \frac{m^2 - C}{W} + \frac{1}{424.15} \frac{m^2 - °C}{W} = 0.0034 \frac{m^2 - °C}{W}$$

$$U_{total} = 297.8 \frac{W}{m^2 - °C}$$

(a) 79 W/m^2-°C

(b) 425 W/m^2-°C

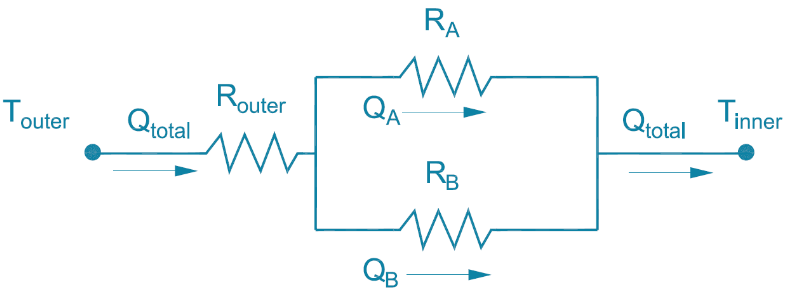
(c) 145 W/m^2-°C

(d) 298 W/m^2-°C

2.89 SOLUTION 89 – HEAT TRANSFER

A composite wall consists of an outer layer with a conductivity of 55 W/m-°C and an inner layer. The inner layer is comprised of an area that is 80% material A (k = 125 W/m-°C) and 20% material B (k = 175 W/m-°C). The inner layer has a thickness of 10 cm and the outer layer has a thickness of 15 cm. What is the total heat transfer through the wall? Assume a difference in temperature of 45 °C and a surface area of 50 m^2.

First, create a figure to show the heat transfer graphically.



Next, you need to convert thermal conductivity to resistances.

$$R_{outer} = \frac{t}{k} = \frac{\frac{15}{100}m}{55\frac{W}{m-°C}} = 0.00273\ \frac{m^2-C}{W}$$

$$R_A = \frac{t}{k} = \frac{\frac{10}{100}m}{125\frac{W}{m-°C}} = 0.0008\ \frac{m^2-C}{W}$$

$$R_B = \frac{t}{k} = \frac{\frac{10}{100}m}{175\frac{W}{m-°C}} = 0.00057\ \frac{m^2-C}{W}$$

Next, combine the parallel resistances.

$$U_{parallel} = (0.80)\left(\frac{1}{0.0008}\frac{m^2-C}{W}\right) + (0.20)\left(\frac{1}{0.00057}\frac{m^2-C}{W}\right)$$

$$U_{parallel} = 1{,}000\frac{W}{m^2-°C} + 350\frac{W}{m^2-°C} = 1{,}350\ \frac{W}{m^2-°C}$$

Convert the previous U-value, to R-value. Then add the outer layer resistance.

$$R_{total} = R_{outer} + \frac{1}{1{,}350}\frac{m^2 - °C}{W}$$

$$R_{total} = 0.00273\ \frac{m^2 - C}{W} + \frac{1}{1{,}350}\frac{m^2 - °C}{W} = 0.00347\ \frac{m^2 - °C}{W}$$

$$U_{total} = 288\ \frac{W}{m^2 - °C}$$

Finally, use the heat transfer equation.

$$Q = U_{total}A_{surface}(\Delta T)$$

$$Q = \left(288\ \frac{W}{m^2 - °C}\right)(50\ m^2)(45\ °C) = 648{,}000\ W$$

The correct answer is most nearly, (c) 650,000 W.

(a) 50,000 W

(b) 300,000 W

(c) 650,000 W

(d) 1,520,000 W

2.90 SOLUTION 90 – HEAT TRANSFER

Which of the following statements is true about the Biot number?

(a) A larger Biot number indicates greater conductive heat transfer than convective heat transfer from a body.

(b) A larger Biot number indicates a greater conductive heat transfer resistance than convective heat transfer resistance for a body.

(c) A smaller Biot number indicates a lesser conductive heat transfer than convective heat transfer from a body.

(d) A smaller Biot number indicates a greater conductive heat transfer resistance than convective heat transfer resistance for a body.

The equation for the Biot number is taken from the Heat Transfer section of the FE Reference Handbook. The top of the equation is the convective heat transfer equation and the bottom is the conductive heat transfer equation. A larger number indicates greater convective heat transfer. The opposite is also true, a larger number indicates a greater conductive resistance (opposite of heat transfer).

$$\beta i = \frac{hA_s}{\rho V c_p} = \frac{Convective\ Heat\ Transfer}{Conductive\ Heat\ Transfer} = \frac{Conductive\ Resistance}{Convective\ Resistance}$$

The correct answer is most nearly, (b) A larger Biot number indicates a greater conductive heat transfer resistance than convective heat transfer resistance for a body.

2.91 SOLUTION 91 – HEAT TRANSFER

A material is initially at a temperature of 500 °C. The ambient air is 20 °C. The properties of the material are shown below. How long will it take the material to reach 40 °C?

$$\beta = 0.009;$$

Use the following equation from the Heat Transfer section of the FE Reference Handbook.

$$T_f - T_{amb} = (T_i - T_{amb})e^{-\beta t}$$

$$40 - 20 = (500 - 28)e^{-0.009t}$$

$$e^{-0.1t} = 0.04237$$

$$\ln(0.04237) = -0.009t$$

$$t = 351\ seconds$$

The correct answer is most nearly, (a) 351 seconds.

(a) 350 seconds

(b) 1930 seconds

(c) 3230 seconds

(d) 7560 seconds

2.92 SOLUTION 92 – HEAT TRANSFER

Two parallel flat plates of equal area are separated by 50 mm. One flat plate has an emissivity of 0.2 and is at a temperature of 400 K. The other flat plate has an emissivity of 0.03 and is at a temperature of 1,000 K. The shape factor is equal to 1. What is the radiative heat transfer between the two plates? The plates are assumed to form an enclosure.

The rate of heat transfer between plate 1 and plate 2 is shown below.

$$Q_{12} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{(1-\varepsilon_1)}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{(1-\varepsilon_2)}{\varepsilon_2 A_2}}$$

$$\sigma = Stefan\ Boltzman\ Constant = 5.67\ x10^{-8}\frac{W}{m^2 - K^4}$$

$$Q_{12} = \frac{(5.67\ x10^{-8}\frac{W}{m^2 - K^4})(1000^4 - 400^4)}{\frac{(1-0.2)}{0.2A_1} + \frac{1}{A_1 1} + \frac{(1-0.03)}{0.03A_2}}$$

$$Q_{12} = \frac{(55{,}326\frac{W}{m^2})}{(4 + 1 + 32.33)A_{plate}}$$

$$Q_{12} = 1{,}482\ \frac{W}{m^2}$$

The correct answer is most nearly, (a) 1,500 W/m^2.

(a) 1,500 W/m^2

(b) 2,900 W/m^2

(c) 4,200 W/m^2

(d) 10,700 W/m^2

2.93 SOLUTION 93 – HEAT TRANSFER

Two surfaces of equal area are separated by 100 mm. One surface is at a temperature of 750 K and the other surface is at 250 K. Assume the surfaces are black bodies. The shape factor between the two surfaces is 0.3. Assume the surfaces are part of an enclosure, then what is the radiative heat transfer between the two surfaces?

The rate of heat transfer between the two surfaces is given by the equation below.

$$Q_{12} = \sigma A_{12} F_{12}(T_1^4 - T_2^4)$$

$$\sigma = Stefan\ Boltzman\ Constant = 5.67\ x10^{-8}\frac{W}{m^2 - K^4}$$

$$Q_{12} = 5.67\ x10^{-8}\frac{W}{m^2 - K^4}A_{12}0.3(750^4 - 250^4)$$

$$Q_{12} = 5315\frac{W}{m^2}$$

The correct answer is most nearly, (c) 5,300 W/m^2.

(a) 800 W/m^2

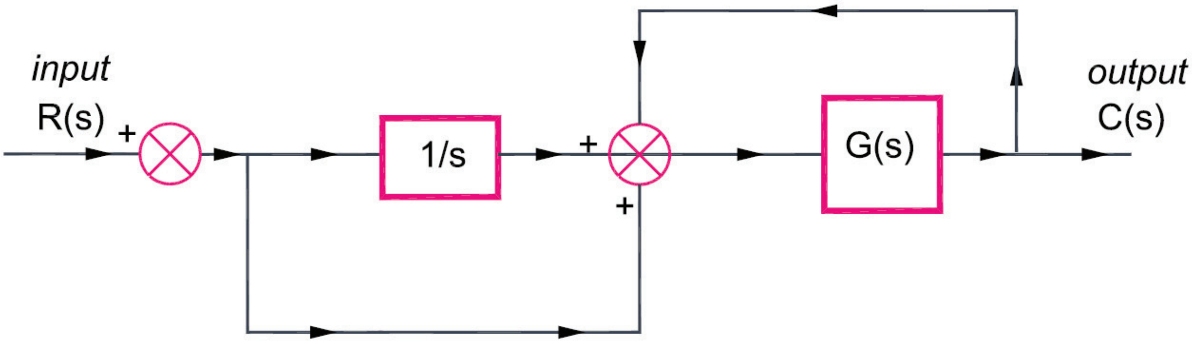
(b) 5,300 W/m^2

(c) 17,720 W/m^2

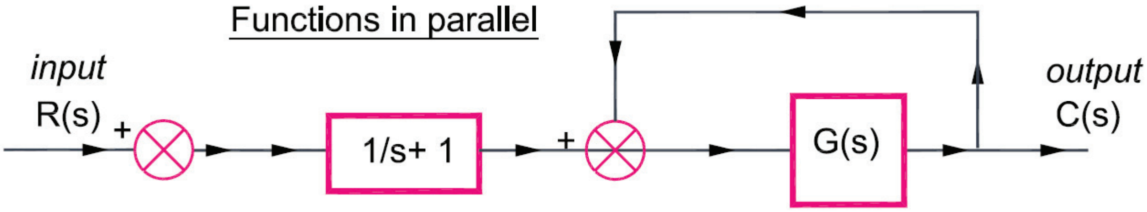
(d) 20,640 W/m^2

2.94 SOLUTION 94 – MEASUREMENTS, INSTRUMENTATION, AND CONTROLS

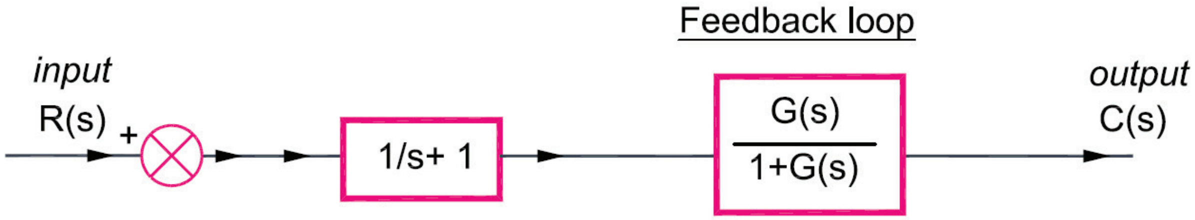
Simplify the following control diagram and find the governing equation for the output.


input
R(s)
+
1/s
+
+
G(s)
output
C(s)

First, simplify the paths in parallel.


Functions in parallel
input
R(s)
+
1/s+ 1
+
G(s)
output
C(s)

Next, simplify the feedback loop.



Now, you have your governing equation. Solve for the output.

$$\frac{C(s)}{R(s)} = \left(\frac{1}{s} + 1\right)\left(\frac{G(s)}{1 + G(s)}\right)$$

$$C(s) = R(s)\left(\frac{1}{s} + 1\right)\left(\frac{G(s)}{1 + G(s)}\right)$$

The correct answer is most nearly, (c).

$(a)\ C(s) = R(s)\left(\frac{1}{s}\right)\left(G(s)\right)$

$(b)\ C(s) = R(s)\left(\frac{1}{s}\right)\left(\frac{G(s)}{1 + G(s)}\right)$

$(c)\ C(s) = R(s)\left(\frac{1}{s} + 1\right)\left(\frac{G(s)}{1 + G(s)}\right)$

$(d)\ C(s) = R(s)(s + 1)\left(\frac{1}{1 + G(s)}\right)$

2.95 Solution 95 – Measurements, Instrumentation, and Controls

A thermocouple has a reference temperature of 100 °F. Which of the following statements about this thermocouple is true?

A thermocouple uses two dissimilar metals to produce a voltage. At the reference temperature, there is no voltage produced. As the temperature changes, the two metals change in length at different rates, which causes a voltage. A change in temperature does not affect the resistance, because resistance is not an output of the thermocouple.

The correct answer is most nearly, (d).

(a) An increase in temperature causes a direct decrease in the resistance of the thermocouple output.

(b) A decrease in temperature causes a direct increase in the resistance of the thermocouple output.

(c) A temperature of 100 F will cause the thermocouple to measure the greatest voltage.

(d) A change in temperature causes a change in the thermocouple voltage output.

2.96 SOLUTION 96 – MEASUREMENTS, INSTRUMENTATION, AND CONTROLS

A strain gauge has an initial resistance of 300 ohms. The strain gauge factor is 2.8. A load is applied and the strain gauge now measures 299.771 ohms. What is the strain measured?

$$Gauge\ Factor = \frac{\Delta R/R_0}{\varepsilon}$$

$$2.8 = \frac{(300 - 299.771)/300}{\varepsilon}$$

$$\varepsilon = 0.0002726$$

The correct answer is most nearly, (b) 2.726 x 10^{-4}

(a) 8.847 x 10^{-4}

(b) 2.726 x 10^{-4}

(c) 6.412 x 10^{-5}

(d) 1.989 x 10^{-5}

2.97 SOLUTION 97 – MEASUREMENTS, INSTRUMENTATION, AND CONTROLS

Platinum is used as a resistance temperature detector. The nominal resistance of the detector is 250 ohms. The temperature changes from 0 C to 130 C. What is the percent change in resistance?

$$\alpha = 0.00385 \frac{\Omega}{\Omega} per\ ℃$$

$$R_T = R_0(1 + 0.00385 \frac{\Omega}{\Omega}(130\ C - 0\ C))$$

$$R_T = R_0 1.5005$$

Then calculate the change in resistance.

$$\Delta R = (R_T - R_0) = R_0 1.5005 - R_0$$

$$\Delta R = R_0 0.5005$$

$$\frac{\Delta R}{R_0} = \frac{R_0 0.5005}{R_0} = 50\%$$

(a) 50%

(b) 75%

(c) 125%

(d) 150%

2.98 SOLUTION 98 – MEASUREMENTS, INSTRUMENTATION, AND CONTROLS

Which of the following statements is true about a thermistor with a negative temperature coefficient?

The FE Reference Handbook shows the following equation in the Measurements, Instrumentation and Controls section.

$$R_T = R_0 e^{\beta\left(\frac{1}{T}-\frac{1}{T_0}\right)}$$

As you can see the change in temperature results in a change in resistance. An increase in temperature causes a negative value in the exponent. That negative value causes a fractional value of the original resistance less than 1, which means the original resistance decreases.

The correct answer is most nearly, (d) A thermistor reacts to an increase in temperature with a decrease in resistance.

(a) A thermistor reacts to an increase in temperature with an increase in voltage.

(b) A thermistor reacts to an increase in temperature with an increase in resistance.

(c) A thermistor reacts to an increase in temperature with a decrease in voltage.

(d) A thermistor reacts to an increase in temperature with a decrease in resistance.

2.99 SOLUTION 99 – MEASUREMENTS, INSTRUMENTATION, AND CONTROLS

Which of the following sensors most likely uses the piezoelectric effect?

The piezoelectric effect occurs when a piezoelectric material responds to a force with a voltage output. The vibration sensor is the sensor that will most likely experience a force and thus will need the piezoelectric effect to create a corresponding voltage that can then be measured.

The correct answer is most nearly, (c) Vibration.

(a) Temperature

(b) Humidity

(c) Vibration

(d) Enthalpy

2.100 SOLUTION 100 – MECHANICAL DESIGN & ANALYSIS

There are four springs that support a piece of equipment that weighs 50 kg. Two of the springs have a spring constant of 20 N/m and the other two springs have a spring constant of 40 N/m. What is the vibration frequency of the four springs in parallel?

When you have springs in parallel, you add up the spring constants directly.

$$k_{eq} = 20 + 20 + 40 + 40 = 120\ N/m$$

Now plug-in the spring constant into the frequency equation.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{120\ N/m}{50\ kg}} = 0.25\ Hz$$

The correct answer is most nearly, (a) 0.25 Hz.

(a) 0.25 Hz

(b) 4 Hz

(c) 22 Hz

(d) 83 Hz

2.101 SOLUTION 101 – MECHANICAL DESIGN & ANALYSIS

Given the following statistical information, determine the UCL and LCL on the X-bar chart.

$$\bar{R} = 2.7;\ \bar{\bar{X}} = 88.9;\ n = 6$$

For this problem, go to the Engineering Probability and Statistics section and find the X-bar chart formulas for UCL and LCL.

$$UCL_X = \bar{\bar{X}} + A_2\bar{R}$$

$$LCL_X = \bar{\bar{X}} - A_2\bar{R}$$

So all you need to find is A_2 from the row n = 6.

$$A_2 = 0.483$$

$$UCL_X = 88.9 + (0.483)(2.7) = 90.2041$$

$$LCL_X = 88.9 - (0.483)(2.7) = 87.5959$$

The correct answer is most nearly, (b) 90.2 & 87.6

(a) 88.9 & 88.9

(b) 90.2 & 87.6

(c) 104.7 & 78.1

(d) 122.2 & 62.7

2.102 SOLUTION 102 – MECHANICAL DESIGN & ANALYSIS

Which of the following best describes a maximum material condition (MMC) or least material condition (LMC)?

The maximum material condition describes the worst case scenario for a shaft fitting into a hole. It is the largest shaft fitting into the smallest hole. The least material condition is the smallest shaft fitting into the largest hole.

The correct answer is most nearly, (a) MMC is the largest shaft size fitting into smallest hole size.

(a) MMC is the largest shaft size fitting into smallest hole size.

(b) LMC is the largest shaft size fitting into smallest hole size.

(c) MMC is the largest shaft size fitting into largest hole size.

(d) LMC is the smallest shaft size fitting into smallest hole size.

2.103 SOLUTION 103 – MECHANICAL DESIGN & ANALYSIS

A centrifugal fan is rated at 2,000 CFM at 2 in wg at a speed of 1190 RPM. If the fan speed is increased to 1,500 RPM, then what will be the new flow rate?

Use the affinity laws to calculate the new flow rate.

$$Q = 2{,}000\ CFM * \left(\frac{1500}{1190}\right) = 2{,}521\ CFM$$

The correct answer is most nearly, (c) 2,500 CFM.

(a) 1,500 CFM

(b) 2,000 CFM

(c) 2,500 CFM

(d) 3,000 CFM

2.104 SOLUTION 104 – MECHANICAL DESIGN & ANALYSIS

A helical torsion spring has a total number of 20 coils and a young’s modulus is 100 GPa. The design load is 100 pounds and the spring has a mean diameter of 100 mm and a wire diameter of 5 mm. What is the spring constant?

The equation for helical torsional springs are shown below.

$$k = \frac{d^4 E}{64DN}$$

$$k = \frac{\left(\frac{5}{1000}m\right)^4 100\ x\ 10^9\ Pa}{64\left(\frac{100}{1000}m\right)(20)} = 0.488\ N - m/rad$$

The correct answer is most nearly, **(a) 0.5 N-m/rad.**

(a) 0.5 N-m/rad

(b) 1.9 N-m/rad

(c) 20.7 N-m/rad

(d) 100.4 N-m/rad

2.105 Solution 105 – Mechanical Design & Analysis

A roller bearing is designed for a load of 100 N at a speed of 100 RPM for a life of 2000 hours. If the load changes to 150 N, then what will be the new life of the bearing?

In the Mechanical Engineering section there is an equation that relates the life of a bearing to the load and number of revolutions. a = 10/3 for roller bearings.

$$C = PL^{1/a}$$

First, solve for C, then re-solve the equation for the new life with the new load.

$$C = (100\ N)(2000)^{\frac{1}{\frac{10}{3}}} = 977.93$$

$$977.93 = (150\ N)(X)^{\frac{1}{\frac{10}{3}}}$$

$$X = 517\ hours$$

The correct answer is most nearly, (a) 517 hours.

(a) 517 hours

(b) 799 hours

(c) 1823 hours

(d) 8756 hours

2.106 Solution 106 – Mechanical Design & Analysis

An open air tank at sea level feeds a centrifugal pump. The pump's centerline is located 25 feet below the water level of the tank. There is a friction loss of 10 feet head from the tank to the pump suction. The vapor pressure is 0.7 psia. What is the net positive suction head available at the pump suction?

For this problem add the atmospheric pressure and add the elevation pressure but subtract the friction head to get the suction pressure. Then subtract the vapor pressure to get the net positive suction head.

$$NSPHA = 14.7\ PSIA\ (atmospheric\ pressure) + 25\ ft - 10\ ft - 0.7\ psi$$

$$NSPHA = 14\ psi + 15\ ft * \frac{1\ psi}{2.31\ ft\ hd} = 20.5\ psi$$

The correct answer is most nearly, (d) 21 psi.

(a) 8 psi

(b) 12 psi

(c) 17 psi

(d) 21 psi

2.107 SOLUTION 107 – MECHANICAL DESIGN & ANALYSIS

Which of the following terms is used to characterize compressibility?

The bulk modulus measures the compressibility of a material or fluid.

The correct answer is most nearly, (c) Bulk Modulus.

(a) Young's Modulus

(b) Shear Modulus

(c) Bulk Modulus

(d) Rupture Modulus

2.108 SOLUTION 108 – MECHANICAL DESIGN & ANALYSIS

Which of the following best describes the point at which a material will experience permanent deformation?

The correct answer is most nearly, (b) Yield strength.

(a) Ultimate tensile strength

(b) Yield strength

(c) Proportional stress limit

(d) Poisson's Ratio

2.109 SOLUTION 109 – MECHANICAL DESIGN & ANALYSIS

A square-threaded power screw is used to raise a load of 10,000 pounds. The mean thread diameter is equal to 2 inches. The screw has 4 threads per inch, which makes its pitch equal to 0.25 inches. The coefficient of friction is equal to 0.05. Hint: pitch is equal to lead for single start, screw threads. Assume no collar.

The pitch is the distance between screw threads. Since there are four threads per inches, then the distance between the threads is 0.25 inches. The lead is the axial movement per one revolution. For a single start, square thread, the lead is equal to the pitch.

$$T_{raise} = \frac{F * d_m}{2} * \left(\frac{l + \pi \mu d_m}{\pi d_m - \mu l}\right) + \frac{F * \mu_c d_c}{2}$$

$$T_{raise} = \frac{10{,}000\ lb * 2\ in}{2} * \left(\frac{(0.25\ in) + \pi(0.05)(2\ in)}{\pi(2\ in) - (0.05)(0.25\ in)}\right) + 0$$

$$T_{raise} = 899.1\ lb - in$$

The correct answer is most nearly, (a) 900 lb-in.

(a) 900 lb-in

(b) 1,200 lb-in

(c) 2,960 lb-in

(d) 5,990 lb-in

2.110 SOLUTION 110 – MECHANICAL DESIGN & ANALYSIS

A bolt will be subject to a shear force of 5,000 N. If the bolt material has a shear strength of 100 MPa, then what is the required bolt diameter?

The shear force equation is shown below.

$$\sigma = \frac{F}{A} = 100\ x\ 10^6\ Pa = \frac{5{,}000\ N}{\frac{\pi D^2}{4}}$$

$$D = 7.98\ mm$$

The correct answer is most nearly, (b) 8 mm.

(a) .25 mm

(b) 8 mm

(c) 17 mm

(d) 32 mm